Passivity analysis of rational LPV systems using Finsler's lemma

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Abstract— **In this paper, we show and utilize new results on the relationship between passivity, zero dynamics and stable dynamic invertibility of linear parameter-varying (LPV) systems. Furthermore, an optimization-based systematic passivity analysis procedure and a passivating output projection are proposed for asymptotically stable rational LPV systems in the linear fractional representation (LFR) form having at least as many independent output signals as input signals. The storage function is searched in a quadratic form with a symmetric rational parameter-dependent matrix. In order to form a square system and then to satisfy the Kalman-Yakubovich-Popov (KYP) properties, a parameter-dependent output projection matrix is searched in the LFR form. The nonlinear parameter dependence from the linear matrix inequality (LMI) and equality (LME) conditions provided by the KYP lemma is factorized out using the linear fractional transformation (LFT). Then, Finsler's lemma and affine annihilators are used to relax the sufficient affine parameter-dependent LMI and LME conditions. As an application example, a stable system inversion is addressed and demonstrated on a benchmark rational LPV model.**

I. Introduction

The importance of passivity of a dynamical system has been recognized in the literature [1] due its advantageous properties related to stable zero dynamics, internal stability, (vector) relative degree 1.

These system properties allow stable input-output feedback linearization of nonlinear systems [2] and global stabilization of a wide class of dynamical (possibly interconnected) systems [3]-[6]. Important passivity-based results (including stability conditions for feedback systems) are introduced by [7] for linear time-varying (LTV) systems. Asymptotically stable zero dynamics are exploited in dynamic system inversion and fault diagnosis [8, Chapter 3]. With relative degree 1, only the first derivative of the output vector is required for unknown input estimation. Several inversion-based fault diagnosis results are available by e.g. [9]-[12] for linear parameter-varying (LPV) systems with stable zero dynamics.

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To ensure the passivity property of a nonlinear dynamical system, the dissipativity inequality with a special supply function, or equivalently, the Kalman-Yakubovich-Popov (KYP) properties have to be satisfied [1]. These results are further improved in [3], where robust passivity conditions together with the robust KYP lemma is introduced for nonlinear systems with structural uncertainty. Passivity conditions for LTV systems are presented in [7].

In [13] and [14], two LMI approaches are proposed to computationally check that an affine or switched LPV system is passive. In the general case, the dynamic equation of an LPV model is not necessarily affine in the parameter variables. Therefore, the KYP properties for passivity are generally infinite-dimensional problems in the sense that the equivalent (non-linearly) parameter-dependent LMI/LME conditions for passivity should be tested in infinitely many parameter points. To find a conservative solution for these not-convex problems, different LMI-relaxation techniques are proposed in the literature. In [15], a passivity analysis procedure is proposed using the computational framework of the integral quadratic constraints (IQC), which can handle nonlinear, time-varying or uncertain components of the system dynamics.

The most closely related results in the field of LMI-based passivity analysis of nonlinear systems are published in [16]-[19]. In [16], the authors addressed local passivity analysis. Furthermore, a dissipativity-based stabilisable output feedback synthesis is proposed in [17]-[19] based on the analysis of the passivity indices using the standard dissipativity relation. In [16]-[18], a sum-of-square (SOS) relaxation technique is adopted to solve the nonlinear dissipativity relation for polynomial nonlinear systems. Whereas, in [18], [19], a polytopic approach is presented based on Finsler's lemma [20] and affine annihilators. The latter approach is capable to handle rational nonlinearities both in the system equations and in the storage function. In [16]-[19], the authors emphasize the fundamental theory of computational analysis of the passivity indices using the SOS-approach, respectively, the polytopic approach with Finsler's lemma and affine annihilators. Systematic model and storage function construction is not addressed in these references.

The polytopic method with Finsler's lemma and affine annihilators is an efficient technique to solve LMI conditions with rational parameter dependence, since the nonlinear terms are separated from the LMI, but their algebraic interdependence are implicitly injected into the LMI condition through an affine annihilator accompanied by a free matrix Lagrange multiplier (notion introduced by [20]). It is also fortunate, that systematic model construction techniques were developed for the Finsler's lemma-based control optimization problems. In [21], an efficient procedure is proposed to construct a so called maximal affine annihilator for a vector of rational functions and the linear fractional transformation (LFT, [23]) is proposed to factor out the rational terms from the system equation and to build up a differential-algebraic (or implicit) system representation, that is needed for Finsler's lemma relaxation.

Unlike in [17]-[19] where dissipativity-based output feedback synthesis is considered for a class of nonlinear (time-invariant) systems, we cope with computational output projection synthesis to ensure *strict passivity* and hence stable dynamic invertibility of rational strictly proper ($D = 0$) *LPV models*. Strict passivity of a dynamical system is a stronger property than the passivity indices-based dissipativity condition considered in [17]-[19]. In other words, strict passivity ensures the existence of a positive output feedback passivity index and a non-positive input feed-forward passivity index. The main challenge is to systematically compute a parameter-dependent storage function (PDSF) and design an appropriate (possibly parameter-dependent) output transformation, such that the system with the new output function satisfies the KYP *equality* condition [1, Eq. (2.3b)].

The paper is organized as follows. After two concise subsections on notations and problem formulation, the strict passivity property for LPV systems and its relation to the zero dynamics and stable dynamic invertibility is discussed in Section II. The main contributions on computational passivity analysis and passivating output selection are presented in Section III. A simulation example in Section IV highlights the applicability of the proposed method. The main theorem of Section II is given in the Appendix accompanied by an auxiliary lemma.

A. Notations, abbreviations

The dimensions of signals x (state), u (input), y (output), p (parameter) at any time instant are denoted by n_x, n_u, n_y, n_p, respectively. $V_x(x,p)$ and $V_p(x,p)$ denote the gradient (row vectors) of a scalar valued (positive definite storage) function $V : \mathbb{R}^{n_x+n_p} \to \mathbb{R}^+$ with respect to x and p.

Matrix I_m, $0_{n\times m}$ denotes the $m \times m$ identity matrix and the $n \times m$ zero matrix. Let $\mathrm{He}\{A\}$ denote $A^\top + A$, where A is a real-valued square matrix and $A^\top$ denotes its transpose. $\mathrm{Ker}(A) = \{v \in \mathbb{R}^m | Av = 0\}$ denotes the kernel space of the $n \times m$ matrix A. Operator $P \succeq 0$ and $P \preceq 0$ denotes that symmetric matrix P is positive and negative semidefinite, respectively. The lower LFR of a rational parameter-dependent matrix is denoted as follows:

$$G(p) = \mathcal{F}_l\left\{\left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22}\end{array}\right), \Delta\right\} = M_{11} + M_{12}(I - \Delta M_{22})^{-1}M_{21},$$

and is called *well-posed* if $(I - \Delta M_{22})$ is invertible for all p in a given subset $\mathcal{P}$ of $\mathbb{R}^{n_p}$.

The number of rows of rational matrices Π, Π_1, Π_2, Π_a, Π_c, ... are denoted by m, m_1, m_2, m_a, m_c, respectively. Vectors of rational functions $\pi = \Pi x$, $\pi_1 = \Pi_1 x$, $\pi_2 = \Pi_2 u$, $\pi_a = \Pi_a x$, $\pi_c = \Pi_c u$, ... are often called as sets of rational functions. The arguments of signals x, u, ..., rational matrices Π, Π_1, ..., and vectors π, π_1, ... are suppressed and only used when it is necessary. Matrices $N(p)$, $N_a(p,\dot{p})$, $N_c(p)$ of dimensions $s \times m$, $s_a \times m_a$, $s_c \times m_c$ denote affine parameter-dependent matrices, that satisfy the following algebraic equality property: $N(p)\Pi = 0$, $N_a(p,\dot{p})\Pi_a = 0$, $N_c(p)\Pi_c = 0$, for all p and $\dot{p}$, thus, they are called *annihilators* of matrices Π, Π_a, Π_c, inherently, they are also annihilators for vectors π, π_a, π_c, respectively.

When we use the notion of *rational/affine* parameter-dependent *linear* matrix inequality or equality condition, we refer to an inequality ($\succeq$,$\preceq$), respectively element-wise equality constraint of a matrix-valued expression, which is *rational/affine* in some parameter values (belonging to a bounded set) and is *linear* (more precisely affine) in the free decision variables (that are meant to be found to satisfy the condition for all admissible parameter values).

B. System class, strict passivity and problem formulation

We consider MIMO LPV systems of the form:

$$\begin{pmatrix}\dot{x}(t)\\ y(t)\end{pmatrix} = \sum_{j=1}^{K} \frac{q_{1j}(p(t))}{q_{2j}(p(t))}\begin{pmatrix}A_j & B_j\\ C_j & 0\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}, \tag{1}$$

where $x(t)$, $u(t)$, $y(t)$ and $p(t)$ are the state, input, output and the scheduling parameter signals, respectively. Scalar functions q_{1j} and q_{2j} are multivariate polynomials and $q_{11} = q_{21} = 1$. System (1) can be written in a state-space form with rational parameter-dependent matrices $A(p)$, $B(p)$, $C(p)$ (of appropriate dimensions), as follows:

$$\Sigma : \begin{cases}\dot{x} = A(p)x + B(p)u\\ y = C(p)x\end{cases}. \tag{2}$$

Assumption 1: We assume that the parameter trajectory is bounded and real-time available with a bounded time derivative, more specifically, $p(t) \in \mathcal{P}$ and $\dot{p}(t) \in \mathcal{R}$, where $\mathcal{P}$ and $\mathcal{R}$ are compact polytopic subsets of the parameter space $\mathbb{R}^{n_p}$.

Assumption 2: We assume further that $q_{2j}(p) > \varepsilon$ for all $j \geq 2$ and $p \in \mathcal{P}$ and for some $\varepsilon > 0$.

Matrices $A(p)$, $B(p)$ and $C(p)$ are called *well-defined* if Assumption 2 holds, namely, they have bounded norm on $\mathcal{P}$, equivalently, they admit a well-posed LFR [23].

Definition 1 (based on [6, Section 10.7]): System Σ with $n_u = n_y$ is called *strictly passive* if there exists a (possibly parameter-dependent) positive definite *storage function* $V : \mathbb{R}^{n_x+n_p} \to \mathbb{R}^+$ such that for all $x(0) \in \mathbb{R}^{n_x}$, all $t_1 \geq t_0$, all input functions u and for all parameter trajectories satisfying Assumption 1 the following inequality holds:

$$\begin{aligned}V(x(t_1), p(t_1)) - V(x(t_0), p(t_0)) \leq \\ \leq \int_{t_0}^{t_1} \big(2u^\top(t)y(t) - \alpha(\|x(t)\|)\big)\mathrm{d}t,\end{aligned} \tag{3}$$

for some positive increasing unbounded function $\alpha(\cdot)$, with $\alpha(0) = 0$. System Σ is called *passive* if (3) holds for $\alpha(\cdot) = 0$.

We consider a quadratic PDSF candidate of the form

$$V(x,p) = x^\top \mathbb{P}(p)x, \tag{4}$$

where $\mathbb{P}(p) = \mathbb{P}^\top(p)$ is positive definite and well-defined for all $p \in \mathcal{P}$, and $\mathbb{P}(\cdot)$ is also continuously differentiable in all parameter variables.

Problem 1: Having $n_y \geq n_u$, find a PDSF (4) and an output projection $\bar{y} = \mathbb{M}(p)y$, such that system Σ with the projected output $\bar{y}$ is strictly passive.

Remark 1: A strictly passive system with PDSF $V(x,p)$ of the form (4) is always asymptotically stable with Lyapunov function $V(x,p)$ if the input $u = 0$.

In order to analyze the strict passivity of a square ($n_u = n_y$) system Σ, we can check the *equivalent* KYP properties, which is recalled in the following theorem. A proof for Theorem 2 can be found in [24, Section 4.1].

Theorem 2: System Σ is strictly passive with the proper positive definite PDSF (14) and for some $\alpha(\|x\|) = \alpha_0\|x\|^2$ with $\alpha_0 > 0$ if and only if Σ has the KYP property, namely:

$$\begin{aligned} &\mathrm{He}\{\mathbb{P}(p)A(p)\} + \dot{\mathbb{P}}(p,\dot{p}) + \alpha_0 I_{n_x} \preceq 0, && \forall (p,\dot{p}) \in \mathcal{P}\times\mathcal{R}, && (5a)\\ &\mathbb{P}(p)B(p) = C^\top(p), && \forall p \in \mathcal{P}, && (5b)\end{aligned}$$

are satisfied, where $\dot{\mathbb{P}}(p,\dot{p}) = \sum_{i=1}^{n_p}(\partial\mathbb{P}/\partial p_i)\dot{p}_i$.

In Section II, we analyze the relationship of strict passivity with zero dynamics and dynamic invertibility of rational LPV systems. In Section III, we introduce a computational method for passivating output projection synthesis. The rational parameter-dependent structure for the PDSF $x^\top\mathbb{P}(p)x$ and for the output projection matrix $\mathbb{M}(p)$ are given in subsections III-A and III-B, respectively.

II. Properties of strictly passive LPV systems

The notions of this section are based on [1], [24] and [6, Chapter 10] applied for LPV systems of the form (2). In this section, we assume that Σ is a square system ($n_u = n_y$).

In the followings, we analyze a few important consequences of strict passivity, which will be exploited later for stable dynamic inversion.

Lemma 3: Assume that system Σ is passive with storage function (14) and $\mathrm{rank}(B(p)) = n_u$ for all $p \in \mathcal{P}$ and p satisfies Assumption 1. Then, $\mathrm{rank}(C(p)B(p)) = n_u, \forall p \in \mathcal{P}$.

Proof: Suppose that $\mathrm{rank}(C(p_0)B(p_0)) < n_u$ for some p_0. Then there exists a non-zero vector $v \in \mathbb{R}^{n_u}$ such that $C(p_0)B(p_0)v = 0$. From Theorem 2, we have that $C(p_0) = B^\top(p_0)\mathbb{P}(p_0)$, which implies the following identity:

$$0 = v^\top C(p_0)B(p_0)v = v^\top B^\top(p_0)\mathbb{P}(p_0)B(p_0)v$$

Since $\mathrm{rank}(B(p_0)) = n_u$ and $\mathbb{P}(p_0)$ is positive definite symmetric, $v = 0$ follows, which is a contradiction. ■

Remark 2: We say that system Σ has (vector) relative degree 1, if $C(p)B(p)$ is invertible for all $p \in \mathcal{P}$.

With reference to [1, Section 4], the non-singularity of $C(p)B(p)$ implies the existence of a well-defined mapping $z = T_2(p)x \in \mathbb{R}^{n_x - n_u}$, which together with $y = C(p)x$ qualify as a new set of local coordinates for Σ. The state variables of Σ in the new coordinates system are:

$$\begin{pmatrix} y \\ z \end{pmatrix} = T(p)x, \text{ with } T(p) = \begin{pmatrix} C(p) \\ T_2(p) \end{pmatrix} \in \mathbb{R}^{n_x \times n_x}. \quad (6)$$

The dynamic equation of Σ in the new coordinates has a special *normal form* [25, Eq. (9.17), Section 9.2]:

$$\Sigma_{y,z}: \begin{cases} \dot{y} = A_{yy}(p,\dot{p})\,y + A_{yz}(p,\dot{p})\,z + B_y(p)u \\ \dot{z} = A_{zy}(p,\dot{p})\,y + A_{zz}(p,\dot{p})\,z \end{cases}, \quad (7)$$

where

$$\begin{pmatrix} A_{yy} & A_{yz} \\ A_{zy} & A_{zz} \end{pmatrix} = \big(\dot{T}(p,\dot{p}) + T(p)A(p)\big)T^{-1}(p), \quad (8)$$

$$B_y(p) = C(p)B(p),\ \dot{T}(p,\dot{p}) = \textstyle\sum_{i=1}^{n_p} \partial T(p)/\partial p_i\ \dot{p}_i.$$

A possible numeric construction of the transformation matrix $T(p)$ is given in the Appendix in Lemma 6.

In this normal form, the invertibility of $B_y(p)$ makes also possible to compute the *output-zeroing input* $u^*(t)$ for system $\Sigma_{y,z}$, which can force the output vector $y(t)$ to be identically zero for $y(0) = 0$, any $z(0) \in \mathbb{R}^{n_x - n_y}$, and any parameter trajectory $p(t)$ satisfying Assumption 1. If we force $\dot{y}(t) = 0$, we can express u^* algebraically from the first equation of (7) as follows: $u^* = -B_y^{-1}(p)A_{yz}(p,\dot{p})z$.

The so-called *zero dynamics* of system Σ describes the internal behaviour of Σ, when the output zeroing input u^* is applied to it. The *zero dynamics* of system Σ in the new coordinates are characterized by

$$\dot{z} = A_{zz}(p,\dot{p})z. \quad (9)$$

The strong relation between passivity and the zero dynamics first was shown in [1] for nonlinear *time-invariant* systems, and it was generalized to a class of (time-varying) uncertain nonlinear systems in [3]. In the following theorem, we adapt these results for rational LPV systems. The proof of Theorem 4 is given in the Appendix.

Theorem 4: Assume that $p(t)$ is continuously differentiable and system Σ is strictly passive with a proper positive definite PDSF (14). Then, the zero dynamics (9) is asymptotically stable.

Note also that the stability of the zero dynamics involves the stability of the dynamic inverse as well. Based on the derivation in [25, Section 9.2] the dynamic inverse of Σ can be constructed as follows:

$$\Sigma^{-1}: \begin{cases} \dot{\hat{z}} = A_{zz}(p,\dot{p})\hat{z} + A_{zy}(p,\dot{p})y, \\ \hat{u} = B_y^{-1}(p)(\dot{y} - A_{yy}(p,\dot{p})y - A_{yz}(p,\dot{p})\hat{z}), \end{cases} \quad (10)$$

According to Theorem 4, system Σ^{-1} is an asymptotically stable dynamic inverse of system Σ.

III. Passivity analysis with convex conditions

In this section, we present our main results related to computational passivity analysis and passivating output projection synthesis for rational LPV systems.

A. Model representation and storage function candidate

Let us consider the LFT decomposition of the model matrices of Σ, as follows:

$$\begin{pmatrix} A(p) \\ C(p) \end{pmatrix} = \mathcal{F}_l\left\{ \left(\begin{array}{c|c} F_{11} & F_{13} \\ F_{21} & F_{23} \\ \hline F_{31} & F_{33} \end{array}\right), \Delta_1 \right\}, B(p) = \mathcal{F}_l\left\{ \left(\begin{array}{c|c} F_{12} & F_{14} \\ \hline F_{42} & F_{44} \end{array}\right), \Delta_2 \right\}. \quad (11)$$

Using (11), the system equation of Σ can be written in the following structured LFR:

$$\Sigma_{\mathcal{F}}: \begin{pmatrix} \dot{x} \\ y \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & 0 & F_{23} & 0 \\ F_{31} & 0 & F_{33} & 0 \\ 0 & F_{42} & 0 & F_{44} \end{pmatrix} \begin{pmatrix} x \\ u \\ \pi_1 \\ \pi_2 \end{pmatrix} \quad (12)$$

$$\text{with } \pi_1 = \Delta_1\eta_1, \text{ and } \pi_2 = \Delta_2\eta_2,$$

where F_{ij} are constant matrices and $\pi_1, \eta_1 \in \mathbb{R}^{m_1}$, respectively $\pi_2, \eta_2 \in \mathbb{R}^{m_2}$ are the feedback signals through the parameter-dependent blocks $\Delta_1 = \Delta_1(p)$ and $\Delta_2 = \Delta_2(p)$ corresponding to the two LFRs in (11). Eliminating vectors η_1 and η_2 from the last two equations of $\Sigma_{\mathcal{F}}$, we obtain the explicit expressions for both vectors π_1 and π_2, namely

$$\pi_1(x,p)=\Pi_1 x\in\mathbb{R}^{m_1},\ \ \pi_2(u,p)=\Pi_2 u\in\mathbb{R}^{m_2}, \tag{13}$$

where $\Pi_1=(I-\Delta_1F_{33})^{-1}\Delta_1F_{31}$, $\Pi_2=(I-\Delta_2F_{44})^{-1}\Delta_2F_{42}$.

Note that this particular structure considered for the LFR model $\Sigma_{\mathcal{F}}$ allows us to separate the state and input dependence in vectors $\pi_1(x,p)$ and $\pi_2(u,p)$. Due to Assumption 2, both LFRs in (11) are *well-posed*, namely, the inverse matrices in (13) are well-defined.

A positive definite PDSF for model $\Sigma_{\mathcal{F}}$ is searched in the following parameterized form:

$$V(x,p)=x^\top\mathbb{P}(p)x=x^\top\Pi^\top P(p)\,\Pi\,x>0, \tag{14}$$

$$\text{where } \Pi=\Pi(p)=\begin{pmatrix} I_{n_x}\\ \Pi_1\end{pmatrix}\in\mathbb{R}^{m\times n_x},\ m=n_x+m_1,$$

where the rational algebraic structure of matrix $\mathbb{P}(p)$ is determined by the rational terms of $A(p)$ and $C(p)$ through matrix Π_1 generated from the first LFR of (11). The symmetric matrix $P(p)$ in (14) is allowed to be an affine function of p, namely $P(p)=P_0+\sum_{i=1}^{n_p}p_i\,P_i\in\mathbb{R}^{m\times m}$, where P_i are constant free indefinite symmetric matrix variables.

B. Passivating output projection synthesis

Assume that Σ has at least as many output signals as input signals ($n_u\leq n_y$). In order to form a square system, a rational parameter-dependent output projection is searched in a special parameterized LFR form:

$$\bar{y}=\mathbb{M}^\top(p)\,y,\ \text{ where } \mathbb{M}(p)=M(p)\Pi_c(p)\in\mathbb{R}^{n_y\times n_u}. \tag{15}$$

In (15), $M(p)\in\mathbb{R}^{n_y\times m_c}$ is an affine matrix with free coefficient values, $\Pi_c(p)$ is a fixed rational matrix generated from the system's LFR. The value of $\Pi_c(p)$ is given in (22).

Remark 3: The second KYP condition (5b) for a preliminarily structured rational matrix $\mathbb{P}(p)$ is fairly strict, not to mention the fact that the structure of matrix $\mathbb{P}(p)$ composed of the rational terms of matrix $A(p)$ and $C(p)$ and it does not contain the rational terms of matrix $B(p)$. However, if $\mathbb{M}(p)$ has a similar structure to matrix $B(p)$, system Σ with the modified output vector is more likely to satisfy the relaxed KYP property

$$\mathbb{P}(p)B(p)=C^\top(p)\mathbb{M}(p). \tag{16}$$

In this section, we give a computationally more tractable sufficient condition for (16) by using affine annihilators.

Remark 4: For the case when $n_u>n_y$, we consider the dual problem with $(A^\top(p),C^\top(p),B^\top(p))$, then, the output projection for the dual system constitutes a passivating input blending transformation.

In Theorem 2, we formulated the sufficient and necessary conditions for the strict passivity of a square system. Also note that (in)equalities (5) are nonlinear parameter-dependent. Therefore, in its original formulation the KYP properties (5) should be checked in infinitely many parameter points. In the next theorem, we give sufficient but convex conditions for strict passivity.

Theorem 5: System Σ in representation $\Sigma_{\mathcal{F}}$ with output $\bar{y}$ is strictly passive if the following affine parameter-dependent linear conditions are satisfied for all $p\in\mathcal{P}$ and $\dot{p}\in\mathcal{R}$:

$$P(p)+\mathrm{He}\{LN(p)\}-\widehat{\alpha}_0I_b\succeq 0, \tag{17a}$$

$$R(p,\dot{p};P)+\mathrm{He}\{L_aN_a(p,\dot{p})\}+\alpha_0I_a\preceq 0, \tag{17b}$$

$$P(p)B_c-C_c^\top M(p)+N^\top(p)L_{c,1}^\top+L_{c,2}N_c(p)=0, \tag{17c}$$

where $P(p)$, $M(p)$ are free affine matrix functions, $P(p)$ is symmetric, L, L_a, $L_{c,1}$ and $L_{c,2}$ are free full matrix variables (with the appropriate dimensions), $N(p)\in\mathbb{R}^{s\times m}$, $N_a(p,\dot{p})\in\mathbb{R}^{s_a\times m_a}$ and $N_c(p)\in\mathbb{R}^{s_c\times m_c}$ are affine annihilators for the rational matrices Π, Π_a and Π_c. In (17c), B_c and C_c are constant matrices and their values are given in (21). The values of Π_a, Π_c and $R(p,\dot{p};P)$ are given in (19), (22) and (18). The upper-left $n_x\times n_x$ block of matrices I_b and I_a is the identity matrix, the other values of I_b and I_a are zeros.

Proof: Due to Finsler's lemma presented by [20], LMI (17a) directly implies the strict positive definiteness of $\mathbb{P}(p)$ of the PDSF (14).

In order to give a sufficient condition in the form of a convex affine LMI for the KYP property (5a), we give a quadratic LFR decomposition for the left hand side of (5a), as follows:

$$\mathrm{He}\{\mathbb{P}(p)A(p)\}+\dot{\mathbb{P}}(p,\dot{p})=\Pi_a^\top R(p,\dot{p};P)\Pi_a\preceq 0, \tag{18}$$

$$\text{with } R(p,\dot{p};P)=\mathrm{He}\big\{E_a^\top P(p)A_a\big\}+E_a^\top\dot{P}(\dot{p})E_a,$$

$$\dot{P}(\dot{p})=\textstyle\sum_{i=1}^{n_p}P_i\,\dot{p}_i.$$

Matrices A_a, E_a and Π_a are given below.

$$A_a=\begin{pmatrix} F_{11} & F_{13} & 0 & 0 & 0\\ 0 & 0 & I_{m_1} & I_{m_1} & I_{m_1}\end{pmatrix},\ E_a=\begin{pmatrix} I_{n_x} & 0 & 0\ 0\ 0\\ 0 & I_{m_1} & 0\ 0\ 0\end{pmatrix},$$

$$\Pi_a=\begin{pmatrix} \Pi\\ \Pi_1F_{11}\\ \Pi_1F_{13}\Pi_1\\ \sum_{i=1}^{n_p}(\partial\Pi_1/\partial p_i)\dot{p}_i\end{pmatrix}\in\mathbb{R}^{m_a\times n_x}, \tag{19}$$

and $m_a=n_x+4m_1$. Applying Finsler's lemma, a sufficient condition for (18) can be given by LMI (17b) if $N_a(p,\dot{p})$ is an annihilator for Π_a.

Finally, let us rewrite the KYP equality (5b) with the projected output vector $\bar{y}=\mathbb{M}^\top(p)C(p)x$ as follows:

$$0=\Pi^\top\big(P(p)B_c-C_c^\top M(p)\big)\Pi_c, \tag{20}$$

$$=\Pi^\top\big(P(p)B_c-C_c^\top M(p)+N^\top(p)L_{c,1}^\top+L_{c,2}N_c(p)\big)\Pi_c,$$

$$\text{where } B_c=\begin{pmatrix} F_{12} & F_{14} & 0 & 0\\ 0 & 0 & I_{m_1} & I_{m_1}\end{pmatrix},\ C_c=\begin{pmatrix} F_{21} & F_{23}\end{pmatrix}, \tag{21}$$

$$\Pi_c=\begin{pmatrix} I_{n_u}\\ \Pi_2\\ \Pi_1F_{12}\\ \Pi_1F_{14}\Pi_2\end{pmatrix}\in\mathbb{R}^{m_c\times n_u}, \tag{22}$$

and $m_c=n_u+2m_1+m_2$. On the other hand, multiplying (17c) from the left hand side by Π and from the right hand side by Π_c we obtain (16). ■

Remark 5: In order to solve the nonlinear PDLME (16), the equality conditions between the coefficients of the identical rational terms in each element of the matrix identity have to be extracted, which requires computationally demanding symbolic operations. However, the computationally more tractable PDLME $P(p)B_c-C_c^\top M(p)=0$ is only a sufficient condition for (16). In order to make it less conservative, we use again affine annihilators, which introduce new degrees of freedom into (17c).

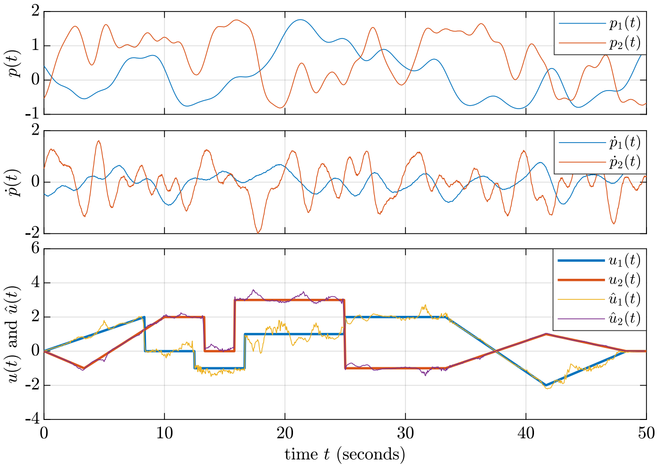

Fig. 1. Dynamic inversion for system in Example 1.

Remark 6: Due to the fact that matrices $P(p)$, $M(p)$, $R(p, \dot{p}; P)$ and annihilators $N(p)$, $N_a(p, \dot{p})$, $N_c(p)$ are affine matrices in p and $\dot{p}$, it is enough to check the feasibility of (17) only in the corner points of polytopes $\mathcal{P}$ and $\mathcal{R}$.

The computed (strict) passivating output projection (15) provides an asymptotically stable zero dynamics and vector relative degree 1 for system Σ. Therefore, we can given an asymptotically stable inverse dynamics for Σ, which can reconstruct the input signal u applied to Σ from the projected output signal $\bar{y}$ and of its time-derivative.

Applying the systematic symbolic model generation tools proposed by [21], [22], the procedure for passivating output projection design can be summarized as follows:

1) Compute an LFR for both $\begin{pmatrix} A(p) \\ C(p) \end{pmatrix}$ and $C(p)$, e.g. by using the direct LFT implementation of the LFR-Toolbox.
2) Generate the maximal annihilators $N(p)$, $N_a(p, \dot{p})$ and $N_c(p)$ for vectors Πx, $\Pi_a x$, $\Pi_c u$ as proposed by [21].
3) Solve the affine parameter-dependent feasibility problem (17) over the corner points of $\mathcal{P} \times \mathcal{R}$.

IV. Computational Example

In this section, we illustrate the operations of the proposed passivity analysis procedure through an illustrative LPV system model. The results were computed in the Matlab environment. For LFR modeling, we used the object oriented LFT-realization implemented in the `sym2lfr` function of the Enhanced LFR-Toolbox for Matlab (LFR-Toolbox) [26]. To model and solve LMI problems, YALMIP [27] with Mosek solver [28] was used.

Example 1: Consider the following rational LPV system with 2 inputs and 3 outputs and with two time-varying parameters p_1 and p_2:

$$\left(\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & 0 \end{array}\right) = \left(\begin{array}{cccc|cc} -p_2^2 - 1 & 5 & 0 & 0 & \frac{p_1+1}{p_2^2+1} + 1 & 1 \\ 0 & p_1 - 4 & 0 & 0 & p_1 & 0 \\ \frac{1}{10} & 0 & \frac{-5p_1-9}{p_1+2} & 0 & 0 & 0 \\ 0 & \frac{p_1}{p_2-5} & 0 & -1 & 0 & 4 - 2p_2 \\ \hline 1 & \frac{1}{p_2-5} & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$
$$\mathcal{P} = [-1, 2] \times [-1, 2], \ \mathcal{R} = [-1, 1] \times [-3, 3]. \tag{23}$$

The rational structure of the PDSF $V(x, p)$ and of matrix $\mathbb{M}(p)$ are determined by rational matrices

$$\Pi_1 = \begin{pmatrix} 0 & p_1 & 0 & 0 \\ 0 & 0 & \frac{p_1}{p_1+2} & 0 \\ 0 & \frac{p_1 p_2}{p_2-5} - p_1 & 0 & 0 \\ p_2^2 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 0 \\ 0 & \frac{p_2}{p_2-5} & 0 & 0 \end{pmatrix}, \Pi_2 = \begin{pmatrix} p_1 - \frac{p_1 p_2^2}{p_2^2+1} & 0 \\ p_1 & 0 \\ \frac{p_2^2}{p_2^2+1} & 0 \\ \frac{p_2}{p_2^2+1} & 0 \\ 0 & -p_2 \end{pmatrix}.$$

and by affine matrices $P(p) \in \mathbb{R}^{10\times 10}$ and $M(p) \in \mathbb{R}^{10\times 14}$.

The values of $M(p)$, $P(p)$ alongside with $\Pi \in \mathbb{R}^{10\times 4}$, $\Pi_a \in \mathbb{R}^{24\times 4}$, $\Pi_c \in \mathbb{R}^{14\times 2}$, with their annihilators $N(p) \in \mathbb{R}^{7\times 10}$, $N_a(p, \dot{p}) \in \mathbb{R}^{29\times 24}$, $N_c(p) \in \mathbb{R}^{16\times 14}$ and other symbolic/numeric variables are available on-line at [29].

The feasibility problem (17) includes a 10×10 PD-LMI (17b), a 24×24 PD-LMI (17b), and 10×14 PDLME (17c), which were evaluated in 4, 16, and 4 corner points, respectively. Number of free decision variables in $P(p)$, in $M(p)$ and in the matrix Lagrange multipliers L, L_a, $L_{c,1}$ and $L_{c,2}$ is 1315. The LMI computations last 3 seconds on a PC with Intel Core i7-4710MQ CPU at 2.50 GHz. and 16 GB of RAM.

The computed matrices $\mathbb{M}(p)$, $\mathbb{P}(p)$ satisfy the KYP equality (5b) with a 10^{-10} tolerance, namely for some $p^{(i)} \in \mathcal{P}$ (including the corner points) the absolute value of the worst nonzero element of $\mathbb{P}(p^{(i)})B(p^{(i)}) - C^\top(p^{(i)})\mathbb{M}(p^{(i)})$ was less than 10^{-10}.

Using $\mathbb{M}^\top(p)y$ as the new output vector, a stable dynamic inverse was computed as presented Eq. (10) of Section II. The results of the dynamic inversion of system Σ are illustrated in Figure 1. Note that the estimation errors $u - \widehat{u}$ are affected by the numeric approximation of the time derivatives of signals y and p.

V. Conclusions

We have have shown that a strictly passive LPV system has relative degree 1 and asymptotically stable zero dynamics. We proposed an efficient systematic procedure for the passivity analysis of rational LPV systems in the LFR form with parameter-dependent storage functions. In order to relax the KYP equality condition, a rational parameter-dependent output projection is co-designed through LMI computations, which also allows the handling of non-square systems. The LPV system equation is given in a structured LFR form, from which we generate the rational algebraic structure of the PDSF and of the output projection matrix. The main contributions in the field of dissipativity analysis of nonlinear systems is that we gave an automatic procedure to generate a fixed rational structure for the PDSF candidate with free coefficient variables, and then systematically formulate sufficient convex LMI conditions for (strict) passivity analysis and passivating output projection synthesis. As presented in Theorem 4, we proposed a new way to use maximal affine annihilators with matrix Lagrange multipliers to solve a matrix-valued equality condition (e.g. the KYP equality condition (17c)).

Appendix

Lemma 6: Assume that system Σ is strictly passive. Then, there exists an invertible state transformation (6), such that the system equation in the new coordinates has the normal form (7).

Proof: By assumption, $B_y(p) = C(p)B(p)$ is non-singular, therefore, both matrices $C(p)$ and $B(p)$ are full-rank matrices for all $p \in \mathcal{P}$. Due to the fact that $B(p)$ is well-defined on the compact set $\mathcal{P}$ (in a well-posed LFR form), there exists a well-defined matrix $T_2(p)$ in a well-posed LFR form [30] with $\mathrm{rank}(T_2(p)) = n_x - n_u$ and with a bounded norm for all $p \in \mathcal{P}$, such that $T_2(p)B(p) = 0$, namely, $T_2^\top(p)$ is a basis for the kernel space $\mathrm{Ker}(B^\top(p))$ of $B^\top(p)$. The non-singularity of matrix $C(p)B(p)$ implies that the rows of $C(p)$ are are not in $\mathrm{Ker}(B^\top(p))$, additionally, $C(p) \in \mathbb{R}^{n_u \times n_x}$ contains linearly independent rows. Therefore, the square matrix $T(p) = \left(\begin{smallmatrix} C(p) \\ T_2(p) \end{smallmatrix}\right)$ is invertible $\forall p \in \mathcal{P}$. If we apply the state transformation $\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) = T(p)x$ to Σ, we obtain its partitioned equivalent $\Sigma_{y,z}$. ∎

In the proof of Theorem 4, we followed the derivations of [24, Section 5.1].

Proof of Theorem 4: Due to the fact that the quadratic PDSF (4) is continuously differentiable with respect to t, inequality (3) can be written in a differential form as follows:

$$\dot{V} + \alpha(\|x\|) \leq y^\top u + u^\top y. \tag{24}$$

Assume that $x(0) \in \mathrm{Ker}\{C(p(0))\}$ and inequality (24) holds. Applying the output zeroing input $u^*(t)$, $y = 0$ implies $\dot{V} < 0$ along the solutions $x(t) \in \mathrm{Ker}(C(p(t)))$ of the forced dynamics

$$\dot{x} = A(p)x + B(p)u^*, \text{ with } x(0) \in \mathrm{Ker}\{C(p(0))\}. \tag{25}$$

Due to the quadratic form (14) of the PDSF, $\dot{V} < 0$ implies the asymptotic stability of (25).

As it is shown in Lemma 6, mapping $z = T_2(p)x$ can be chosen such that the parameter-dependent (non-singular) transformation matrix $T(p)$ is a well-defined rational function of p, and hence it is also continuously differentiable in p_i [23, Section 7.1.14] and $\dot{T}(p,\dot{p})$ has a bounded norm for all $p \in \mathcal{P}$ and all $\dot{p} \in \mathcal{R}$. Then, according to [31, Section 9.1], the state transformation $T(p)$ in (6) preserves the internal stability of system $\Sigma_{y,z}$ if additionally p is continuously differentiable.

Finally, the asymptotic stability of (25) implies the asymptotic stability of the zero dynamics (9). ∎

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