Generalizations of some results about the regularity properties of an additive representation function

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Abstract

Let $A = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ be an infinite sequence of nonnegative integers, and let $R_{A,2}(n)$ denote the number of solutions of $a_x + a_y = n$ $(a_x, a_y \in A)$. P. Erdős, A. Sárközy and V. T. Sós proved that if $\lim_{N\to\infty} \frac{B(A,N)}{\sqrt{N}} = +\infty$ then $|\Delta_1(R_{A,2}(n))|$ cannot be bounded, where $B(A, N)$ denotes the number of blocks formed by consecutive integers in A up to N and Δ_l denotes the l-th difference. Their result was extended to $\Delta_l(R_{A,2}(n))$ for any fixed $l \geq 2$. In this paper we give further generalizations of this problem.

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1 Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers. Let $k \geq 2$ be a fixed integer and let $A = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ be an infinite sequence of nonnegative integers. For $n = 0, 1, 2, \dots$ let $R_{A,k}(n)$ denote the number of solutions of $a_{i_1} + a_{i_2} + \dots + a_{i_k} = n$, $a_{i_1} \in A, \dots, a_{i_k} \in A$, and we put

$$A(n) = \sum_{\substack{a \in A \\ a \leq n}} 1.$$

We denote the cardinality of a set H by $\#H$. Let $B(A, N)$ denote the number of blocks formed by consecutive integers in A up to N, i.e.,

$$B(A, N) = \sum_{\substack{n \leq N \\ n \in A, n-1 \notin A}} 1.$$

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If $s_0, s_1, \ldots$ is given sequence of real numbers then let $\Delta_l s_n$ denote the l-th difference of the sequence $s_0, s_1, s_2, \ldots$ defined by $\Delta_1 s_n = s_{n+1} - s_n$ and $\Delta_l s_n = \Delta_1(\Delta_{l-1} s_n)$.
In a series of papers [2], [3], [4] P. Erdős, A. Sárközy and V.T. Sós studied the regularity properties of the function $R_{A,2}(n)$. In [4] they proved the following theorem:

Theorem A *If* $\lim_{N\to\infty} \frac{B(A,N)}{\sqrt{N}} = \infty$, *then* $|\Delta_1(R_{A,2}(n))| = |R_{A,2}(n+1) - R_{A,2}(n)|$ *cannot be bounded.*

In [4] they also showed that the above result is nearly best possible:

Theorem B *For all* $\varepsilon > 0$, *there exists an infinite sequence* A *such that*

(i) $B(A,N) \gg N^{1/2-\varepsilon}$,

(ii) $R_{A,2}(n)$ *is bounded so that also* $\Delta_1 R_{A,2}(n)$ *is bounded.*

Recently, [9] A. Sárközy extended the above results the finite set of residue classes modulo a fixed m.

In [6] Theorem A was extended to any $k > 2$:

Theorem C *If* $k \geq 2$ *is an integer and* $\lim_{N\to\infty} \frac{B(A,N)}{\sqrt[k]{N}} = \infty$, *and* $l \leq k$, *then* $|\Delta_l R_{A,k}(n)|$ *cannot be bounded.*

It was shown [8] that the above result is nearly best possible.

Theorem D *For all* $\varepsilon > 0$, *there exists an infinite sequence* A *such that*

(i) $B(A,N) \gg N^{1/k-\varepsilon}$,

(ii) $R_{A,k}(n)$ *is bounded so that also* $\Delta_l R_{A,k}(n)$ *is bounded if* $l \leq k$.

In this paper we consider $R_{A,2}(n)$, thus simply write $R_{A,2}(n) = R_A(n)$. A set of positive integers A is called Sidon set if $R_A(n) \leq 2$. Let χ_A denote the characteristic function of the set A, i.e.,

$$\chi_A(n) = \begin{cases} 1, \text{ if } n \in A \\ 0, \text{ if } n \notin A. \end{cases}$$

Let $\lambda_0, \ldots, \lambda_d$ be arbitrary integers with $\left|\sum_{i=0}^d \lambda_i\right| > 0$. Let $\underline{\lambda} = (\lambda_0, \ldots, \lambda_d)$ and define the function

$$B(A, \underline{\lambda}, n) = \left|\left\{ m : m \leq n, \sum_{i=0}^{d} \lambda_i \chi_A(m-i) \neq 0 \right\}\right|.$$

In Theorems 1 and 2 we will focus on the case $\sum_{i=0}^d \lambda_i \neq 0$.

Theorem 1. *We have*

$$\limsup_{n\to\infty} \left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| \geq \limsup_{n\to\infty} \frac{|\sum_{i=0}^d \lambda_i|}{2(d+1)^2} \left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^2.$$

The next theorem shows that the above result is nearly best possible:

Theorem 2. *Let $\sum_{i=0}^{d}\lambda_i > 0$. Then for every positive integer N there exists a set A such that*

$$\limsup_{n\to\infty}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right| \le \limsup_{n\to\infty} 4\sum_{i=0}^{d}|\lambda_i|\left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^2$$

and

$$\limsup_{n\to\infty}\frac{B(A,\underline{\lambda},n)}{\sqrt{n}} \ge N.$$

Theorem 3. *Let $\sum_{i=0}^{d}\lambda_i = 0$. Then we have*

$$\limsup_{n\to\infty}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right| \ge \limsup_{n\to\infty}\frac{\sqrt{2}}{e^2\sum_{i=0}^{d}|\lambda_i|}\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}.$$

It is easy to see that if $\underline{\lambda} = (\lambda_0,\lambda_1) = (-1,1)$ then $B(A,\underline{\lambda},n) \ge B(A,n)$ thus Theorem 3 implies Theorem A. It is natural to ask whether the exponent of $\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}$ in the right hand side can be improved.

Problem 1. *Is it true that if $\sum_{i=0}^{d}\lambda_i = 0$ then there exists a positive constant $C(\underline{\lambda})$ depends only on $\underline{\lambda}$ such that for every set of nonnegative integers A we have*

$$\limsup_{n\to\infty}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right| \ge \limsup_{n\to\infty} C(\underline{\lambda})\cdot\left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^{3/2}?$$

In the next theorem we prove that the exponent cannot exceed 3/2.

Theorem 4. *Let $\sum_{i=0}^{d}\lambda_i = 0$. For every positive integer N there exists a set $A \subset \mathbb{N}$ such that*

$$N \le \limsup_{n\to\infty}\left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right) < \infty$$

and

$$\limsup_{n\to\infty}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right| \le \limsup_{n\to\infty} 48(d+1)^4 2^{3d+7.5}\sum_{i=0}^{d}|\lambda_i|\left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^{3/2}\left(\log\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^{1/2}.$$

2 Proof of Theorem 1

Since $-\underline{\lambda} = (-\lambda_0,\dots,-\lambda_d)$ and clearly

$$\limsup_{n\to\infty}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right| = \limsup_{n\to\infty}\left|\sum_{i=0}^{d}(-\lambda_i) R_A(n-i)\right|,$$

$B(A,\underline{\lambda},n) = B(A,-\underline{\lambda},n)$, therefore

$$\limsup_{n\to\infty}\frac{|\sum_{i=0}^{d}\lambda_i|}{2(d+1)^2}\left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^2 = \limsup_{n\to\infty}\frac{|\sum_{i=0}^{d}(-\lambda_i)|}{2(d+1)^2}\left(\frac{B(A,-\underline{\lambda},n)}{\sqrt{n}}\right)^2,$$

thus we may assume that $\sum_{i=0}^{d}\lambda_i>0$. On the other hand we may suppose that

$$\limsup_{n\to\infty}\left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^2>0.$$

It follows from the definition of the lim sup that there exists a sequence $n_1,n_2,\dots$ such that

$$\lim_{j\to\infty}\frac{B(A,\underline{\lambda},n_j)}{\sqrt{n_j}}=\limsup_{n\to\infty}\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}.$$

To prove Theorem 1 we give a lower and an upper estimation for

$$\sum_{\sqrt[3]{n_j}<n\le 2n_j}\left(\sum_{i=0}^{d}\lambda_i R_A(n-i)\right). \tag{1}$$

The comparison of the two bounds will give the result. First we give an upper estimation. Clearly we have

$$\left|\sum_{\sqrt[3]{n_j}<n\le 2n_j}\left(\sum_{i=0}^{d}\lambda_i R_A(n-i)\right)\right|\le\sum_{\sqrt[3]{n_j}<n\le 2n_j}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right|$$

$$\le 2n_j\max_{\sqrt[3]{n_j}<n\le 2n_j}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right|.$$

In the next step we give a lower estimation for (1). It is clear that

$$\sum_{\sqrt[3]{n_j}<n\le 2n_j}\left(\sum_{i=0}^{d}\lambda_i R_A(n-i)\right)=\sum_{\sqrt[3]{n_j}<n\le 2n_j}(\lambda_0+\ldots+\lambda_d)R_A(n)$$

$$-\Big((\lambda_1+\ldots+\lambda_d)R_A(2n_j)+(\lambda_2+\ldots+\lambda_d)R_A(2n_j-1)+\lambda_d R_A(2n_j-d+1)\Big)$$

$$+(\lambda_1+\ldots+\lambda_d)R_A(\lfloor\sqrt[3]{n_j}\rfloor)+(\lambda_2+\ldots+\lambda_d)R_A(\lfloor\sqrt[3]{n_j}\rfloor-1)+\ldots+\lambda_d R_A(\lfloor\sqrt[3]{n_j}\rfloor-d+1).$$

Obviously,

$$R_A(m)=\#\{(a,a^{'}):a+a^{'}=m,a,a^{'}\in A\}\le 2\cdot\#\{(a,a^{'}):a+a^{'}=m,a\le a^{'},a,a^{'}\in A\}$$

$$\le 2\cdot\#\{(a:a\le m/2,a\in A\}=2A(m/2).$$

It follows that

$$\sum_{\sqrt[3]{n_j}<n\le 2n_j}\left(\sum_{i=0}^{d}\lambda_i R_A(n-i)\right)\ge(\lambda_0+\ldots+\lambda_d)\sum_{\sqrt[3]{n_j}<n\le 2n_j}R_A(n)-\left(\sum_{i=0}^{d}|\lambda_i|\right)2A(n_j)2d$$

$$\ge\left(\sum_{i=0}^{d}\lambda_i\right)\#\{(a,a^{'}):a+a^{'}=n,\sqrt[3]{n_j}<a,a^{'}\le n_j,a,a^{'}\in A\}-\left(\sum_{i=0}^{d}|\lambda_i|\right)4dA(n_j)$$

$$= \left(\sum_{i=0}^{d} \lambda_i\right) (A(n_j) - A(\sqrt[3]{n_j}))^2 - O(A(n_j)).$$

The inequaltity $\sum_{i=0}^{d} \lambda_i \chi_A(m-i) \neq 0$ implies that $[m-d, m] \cap A \neq 0$. Then we have $\{m : m \leq n, \sum_{i=0}^{d} \lambda_i \chi_A(m-i) \neq 0\} \subseteq \cup_{a \leq n, a \in A}[a, a+d]$, which implies that $B(A, \underline{\lambda}, n) \leq |\cup_{a \leq n, a \in A}[a, a+d]| \leq A(n)(d+1)$. By the definition of n_j there exists a constant c_1 such that

$$\frac{B(A, \underline{\lambda}, n_j)}{\sqrt{n_j}} > c_1 > 0.$$

It follows that $A(n_j) > \frac{c_1}{d+1}\sqrt{n_j}$ and clearly $\sqrt[3]{n_j} \geq A(\sqrt[3]{n_j})$. By using these facts we get that

$$\left(\sum_{i=0}^{d} \lambda_i\right) (A(n_j) - A(\sqrt[3]{n_j}))^2 - O(A(n_j)) = (1+o(1)) \left(\sum_{i=0}^{d} \lambda_i\right) A(n_j)^2 \geq$$

$$(1+o(1)) \left(\sum_{i=0}^{d} \lambda_i\right) \frac{B(A, \underline{\lambda}, n_j)^2}{(d+1)^2}.$$

Comparing lower and upper estimations we get that

$$2n_i \max_{\sqrt[3]{n_j} < n \leq 2n_j} \left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| \geq \sum_{\sqrt[3]{n_j} < n \leq 2n_j} \left(\sum_{i=0}^{d} \lambda_i R_A(n-i)\right)$$

$$\geq (1+o(1)) \frac{\sum_{i=0}^{d} \lambda_i}{(d+1)^2} B^2(A, \underline{\lambda}, n_j),$$

this implies that

$$\max_{\sqrt[3]{n_j} < n \leq 2n_j} \left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| \geq (1+o(1)) \frac{\sum_{i=0}^{d} \lambda_i}{2(d+1)^2} \left(\frac{B(A, \underline{\lambda}, n_j)}{\sqrt{n_j}}\right)^2. \qquad (2)$$

To complete the proof we distinguish two cases. When

$$\limsup_{n \to \infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}}\right)^2 < \infty$$

then

$$\max_{\sqrt[3]{n_j} < n \leq 2n_j} \left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| \geq (1+o(1)) \frac{\sum_{i=0}^{d} \lambda_i}{2(d+1)^2} \left(\frac{B(A, \underline{\lambda}, n_j)}{\sqrt{n_j}}\right)^2$$

$$= (1+o(1)) \frac{\sum_{i=0}^{d} \lambda_i}{2(d+1)^2} \limsup_{n \to \infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}}\right)^2,$$

which gives the result.

When

$$\limsup_{n \to \infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}}\right)^2 = \infty$$

then

$$\limsup_{j \to \infty} \left(\frac{B(A, \underline{\lambda}, n_j)}{\sqrt{n_j}}\right)^2 = \infty,$$

which implies by (2) that $\limsup_{n \to \infty} \left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| = \infty$, which gives the result.

3 Proof of Theorem 2

It is well known [5] that there exists a Sidon set S with

$$\limsup_{n\to\infty} \frac{S(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{2}},$$

where $S(n)$ denotes the number of elements of S up to n. Define the set T by removing the elements s and $s^{'}$ from S when $s - s^{'} \leq (N+1)(d+1)$. It is clear that $T(n) \geq S(n) - 2(N+1)(d+1)$ and define the set A by

$$A = T \cup (T + (d+1)) \cup (T + 2(d+1)) \cup \ldots \cup (T + N(d+1)).$$

It is easy to see that $A(n) \geq (N+1)T(n) - N$. We will prove that $B(A, \underline{\lambda}, n) \geq A(n) - d$. By the definitions of the sets T and A we get that if $a < a^{'}$, a, $a^{'} \in A$ then $a - a^{'} \geq d+1$. If

$$\sum_{i=0}^{d} \lambda_i \chi_A(m-i) \neq 0$$

then there is exactly one term, which is nonzero. Fix an index w such that $\lambda_w \neq 0$. It follows that $\sum_{i=0}^{d} \lambda_i \chi_A(a+w-i) \neq 0$ for every $a \in A$. Hence,

$$|B(A, \underline{\lambda}, n)| \geq \#\{a : a + w \leq n, a \in A\} = A(n-w) \geq A(n) - w \geq A(n) - d$$

$$\geq (N+1)T(n) - N - d \geq (N+1)S(n) - 2(N+1)^2(d+1) - N - d.$$

Thus we have

$$\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} \geq (N+1)\frac{S(n)}{\sqrt{n}} - \frac{2(N+1)^2(d+1) + N + d}{\sqrt{n}}$$

and

$$\limsup_{n\to\infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}}\right)^2 \geq \frac{(N+1)^2}{2} \geq N.$$

By the definition of A, we have

$$R_A(m) = \sum_{i=0}^{N} \sum_{j=0}^{N} \#\{(t, t^{'}) : (t + i(d+1)) + (t + j(d+1)) = m, t, t^{'} \in T\}$$

$$= \sum_{i=0}^{N} \sum_{j=0}^{N} R_T(m - (i+j)(d+1)) \leq 2(N+1)^2.$$

Then we have

$$\left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| \leq \left(\sum_{i=0}^{d} |\lambda_i|\right) \max_n R_A(n) \leq 2\left(\sum_{i=0}^{d} |\lambda_i|\right)(N+1)^2 \leq$$

$$\leq \limsup_{n\to\infty} 4 \cdot \left(\sum_{i=0}^{d} |\lambda_i|\right)\left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}}\right)^2,$$

which gives the result.

4 Proof of Theorem 3

Assume first that

$$\limsup_{n\to\infty}\frac{\sqrt{2}}{e^2\sum_{i=0}^{d}|\lambda_i|}\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}<\infty.$$

We prove by contradiction. Assume that contrary to the conclusion of Theorem 3 we have

$$\limsup_{n\to\infty}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right|<\limsup_{n\to\infty}\frac{\sqrt{2}}{e^2\sum_{i=0}^{d}|\lambda_i|}\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}. \tag{3}$$

Throughout the remaining part of the proof of Theorem 3 we use the following notations: N denotes a positive integer. We write $e^{2i\pi\alpha}=e(\alpha)$ and we put $r=e^{-1/N}$, $z=re(\alpha)$ where α is a real variable (so that a function of form $p(z)$ is a function of the real variable $\alpha: p(z)=p(re(\alpha))=P(\alpha)$). We write $f(z)=\sum_{a\in A}z^a$. (By $r<1$, this infinite series and all the other infinite series in the remaining part of the proof are absolutely convergent).

We start out from the integral $I(N)=\int\limits_0^1|f(z)(\sum_{i=0}^{d}\lambda_i z^i)|^2d\alpha$. We will give lower and upper bound for $I(N)$. The comparison of these bounds will give a contradiction.

First we will give a lower bound for $I(N)$. We write

$$f(z)\left(\sum_{i=0}^{d}\lambda_i z^i\right)=\left(\sum_{n=0}^{\infty}\chi_A(n)z^n\right)\left(\sum_{i=0}^{d}\lambda_i z^i\right)$$

$$=\sum_{n=0}^{\infty}(\lambda_0\chi_A(n)+\lambda_1\chi_A(n-1)+\ldots+\lambda_d\chi_A(n-d))z^n.$$

It is clear that if $\lambda_0\chi_A(n)+\lambda_1\chi_A(n-1)+\ldots+\lambda_d\chi_A(n-d)\neq 0$, then $(\lambda_0\chi_A(n)+\lambda_1\chi_A(n-1)+\ldots+\lambda_d\chi_A(n-d))^2\geq 1$. Thus, by the Parseval formula, we have

$$I(N)=\int_0^1\left|f(z)\left(\sum_{i=0}^{d}\lambda_i z^i\right)\right|^2d\alpha$$

$$=\int_0^1\left|\sum_{n=0}^{\infty}(\lambda_0\chi_A(n)+\lambda_1\chi_A(n-1)+\ldots+\lambda_d\chi_A(n-d))z^n\right|^2d\alpha$$

$$=\sum_{n=0}^{\infty}\left(\lambda_0\chi_A(n)+\lambda_1\chi_A(n-1)+\ldots+\lambda_d\chi_A(n-d)\right)^2r^{2n}\geq e^{-2}\sum_{\substack{n\leq N\\ \lambda_0\chi_A(n)+\lambda_1\chi_A(n-1)+\ldots+\lambda_d\chi_A(n-d)\neq 0}}1$$

$$=e^{-2}B(A,\underline{\lambda},N).$$

Now we will give an upper bound for $I(N)$. Since the sums $\sum_{i=0}^{d}|\lambda_i R_A(n-i)|$ are nonnegative integers it follows from (3) that there exists an n_0 and an $\varepsilon>0$ such that

$$\sum_{i=0}^{d}|\lambda_i R_A(n-i)|\leq\limsup_{n\to\infty}\frac{\sqrt{2}}{e^2\sum_{i=0}^{d}|\lambda_i|}\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}(1-\varepsilon). \tag{4}$$

for every $n > n_0$. On the other hand there exists an infinite sequence of real numbers $n_0 < n_1 < n_2 < \ldots < n_j < \ldots$ such that

$$\limsup_{n\to\infty} \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A,\underline{\lambda},n)}{\sqrt{n}} \sqrt{1-\varepsilon} < \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A,\underline{\lambda},n_j)}{\sqrt{n_j}}.$$

We get that

$$\limsup_{n\to\infty} \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A,\underline{\lambda},n)}{\sqrt{n}} (1-\varepsilon) < \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A,\underline{\lambda},n_j)}{\sqrt{n_j}} \sqrt{1-\varepsilon}. \qquad (5)$$

Obviously, $f^2(z) = \sum_{n=0}^{\infty} R_A(n) z^n$. By our indirect assumption, the Cauchy inequality and the Parseval formula we have

$$I(N) = \int_0^1 \left| f(z) \left(\sum_{i=0}^{d} \lambda_i z^i \right) \right|^2 d\alpha \le \left(\sum_{i=0}^{d} |\lambda_i| \right) \int_0^1 \left| f^2(z) \left(\sum_{i=0}^{d} \lambda_i z^i \right) \right| d\alpha$$

$$= \left(\sum_{i=0}^{d} |\lambda_i| \right) \int_0^1 \left| \left(\sum_{n=0}^{\infty} R_A(n) z^n \right) \left(\sum_{i=0}^{d} \lambda_i z^i \right) \right| d\alpha = \left(\sum_{i=0}^{d} |\lambda_i| \right) \int_0^1 \left| \sum_{n=0}^{\infty} \left(\sum_{i=0}^{d} \lambda_i R_A(n-i) \right) z^n \right| d\alpha$$

$$\le \left(\sum_{i=0}^{d} |\lambda_i| \right) \left(\int_0^1 \left| \sum_{n=0}^{\infty} \left(\sum_{i=0}^{d} \lambda_i R_A(n-i) \right) z^n \right|^2 d\alpha \right)^{1/2} = \left(\sum_{i=0}^{d} |\lambda_i| \right) \left(\sum_{n=0}^{\infty} (\sum_{i=0}^{d} \lambda_i R_A(n-i))^2 r^{2n} \right)^{1/2}.$$

In view of (4), (5) and the lower bound for $I(n_j)$ we

$$e^{-2} B(A,\underline{\lambda},n_j) < I(n_j) < \left(\sum_{i=0}^{d} |\lambda_i| \right) \left(\sum_{n=0}^{\infty} \left(\sum_{i=0}^{d} \lambda_i R_A(n-i) \right)^2 r^{2n} \right)^{1/2}$$

$$\le \left(\sum_{i=0}^{d} |\lambda_i| \right) \left(\sum_{n=0}^{n_0} \left(\sum_{i=0}^{d} \lambda_i R_A(n-i) \right)^2 r^{2n} + \sum_{n=n_0+1}^{\infty} \left(\frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A,\underline{\lambda},n_j)}{\sqrt{n_j}} \sqrt{1-\varepsilon} \right)^2 r^{2n} \right)^{1/2}$$

$$< \left(\sum_{i=0}^{d} |\lambda_i| \right) \left(c_2 + \sum_{n=0}^{\infty} \left(\frac{2}{e^4 (\sum_{i=0}^{d} |\lambda_i|)^2} \frac{B^2(A,\underline{\lambda},n_j)}{n_j} (1-\varepsilon) \right) r^{2n} \right)^{1/2},$$

where c_2 is a constant. Taking the square of both sides we get that

$$e^{-4} B^2(A,\underline{\lambda},n_j) < \left(\sum_{i=0}^{d} |\lambda_i| \right)^2 \left(c_2 + \frac{2}{e^4 (\sum_{i=0}^{d} |\lambda_i|)^2} \frac{B^2(A,\underline{\lambda},n_j)}{n_j} (1-\varepsilon) \sum_{n=0}^{\infty} r^{2n} \right). \qquad (6)$$

It is easy to see that

$$1 - e^{-x} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots > x - \frac{x^2}{2!} = x(1 - \frac{x}{2}) > \frac{x}{x+1}$$

for $0 < x < 1$. Applying this observation, where $r = e^{-1/n_j}$ we have

$$\sum_{n=0}^{\infty} r^{2n} = \frac{1}{1-r^2} = \frac{1}{1-e^{-\frac{2}{n_j}}}$$
$$< \frac{n_j}{2} + 1.$$

In view of (6) we obtain that

$$e^{-4}B^2(A,\underline{\lambda},n_j) < \left(\sum_{i=0}^{d}|\lambda_i|\right)^2 \left(c_2 + \frac{2}{e^4(\sum_{i=0}^{d}|\lambda_i|)^2}\frac{B^2(A,\underline{\lambda},n_j)}{n_j}(1-\varepsilon)\left(\frac{n_j}{2}+1\right)\right)$$
$$< c_3 + e^{-4}B^2(A,\underline{\lambda},n_j)(1-\varepsilon),$$

where c_3 is an absolute constant and it follows that

$$B^2(A,\underline{\lambda},n_j) < c_3e^4 + B^2(A,\underline{\lambda},n_j)(1-\varepsilon),$$

or in other words

$$B^2(A,\underline{\lambda},n_j) < \frac{c_3e^4}{\varepsilon},$$

which is a contradiction if n_j is large enough because $\lim_{j\to\infty} B(A,\underline{\lambda},n_j) = \infty$. This proves the result in the first case.
Assume now that

$$\limsup_{n\to\infty} \frac{\sqrt{2}}{e^2\sum_{i=0}^{d}|\lambda_i|}\frac{B(A,\underline{\lambda},n)}{\sqrt{n}} = \infty.$$

Then there exists a sequence $n_1 < n_2 < \dots$ such that

$$\limsup_{j\to\infty} \frac{B(A,\underline{\lambda},n_j)}{\sqrt{n_j}} = \infty.$$

We prove by contradiction. Suppose that

$$\limsup_{n\to\infty} \left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right| < \infty.$$

Then there exists a positive constant c_4 such that $|\sum_{i=0}^{d}\lambda_i R_A(n-i)| < c_4$ for every n. It follows that

$$e^{-2}B(A,\underline{\lambda},n_j) < I(n_j) < \left(\sum_{i=0}^{d}|\lambda_i|\right)\left(\sum_{n=0}^{\infty}\left(\sum_{i=0}^{d}\lambda_i R_A(n-i)\right)^2 r^{2n}\right)^{1/2} < \left(c_4\sum_{n=0}^{\infty}r^{2n}\right)^{1/2} < c_5\sqrt{n_j},$$

thus we have

$$\frac{B(A,\underline{\lambda},n_j)}{\sqrt{n_j}} < c_5e^2,$$

where c_5 is a positive constant, which contradicts our assumption.

5 Proof of Theorem 4

We argue as Sárközy in [9]. In the first step we will prove the following lemma:

Lemma 1. *There exists a set* $C_M \subset [0, M(d+1)-1]$ *for which* $|R_{C_M}(n) - R_{C_M}(n-1)| \le 12\sqrt{M(d+1)\log M(d+1)}$ *for every nonnegative integer* n *and* $B(C_M, \underline{\lambda}, M(d+1)-1) \ge \frac{M}{2^{d+2}}$ *if* M *is large enough.*

Proof of Lemma 1 To prove the lemma we use the probabilistic method due to Erdős and Rényi. There is an excellent summary about this method in books [1] and [5]. Let $\mathbb{P}(E)$ denote the probability of an event E in a probability space and let $\mathbb{E}(X)$ denote the expectation of a random variable X. Let us define a random set C with $\mathbb{P}(n \in C) = \frac{1}{2}$ for every $0 \le n \le M(d+1)-1$. In the first step we show that

$$\mathbb{P}\left(\max_n |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)}\right) < \frac{1}{2}.$$

Define the indicator random variable

$$\varrho_C(n) = \begin{cases} 1, \text{ if } n \in C \\ 0, \text{ if } n \notin C. \end{cases}$$

It is clear that

$$R_C(n) = 2\sum_{k<n/2} \varrho_C(k)\varrho_C(n-k) + \varrho_C(n/2)$$

is the sum of independent indicator random variables. Define the random variable ζ_i by $\zeta_i = \varrho_C(i)\varrho_C(n-i)$. Then we have

$$R_C(n) = 2X_n + Y_n,$$

where $X_n = \zeta_0 + \ldots + \zeta_{\lfloor \frac{n-1}{2} \rfloor}$ and $Y_n = \varrho_C(n/2)$.
Case 1. Assume that $0 \le n \le M(d+1)-1$. Obviously, $\mathbb{P}(\zeta_i = 0) = \frac{3}{4}$ and $\mathbb{P}(\zeta_i = 1) = \frac{1}{4}$ and

$$\mathbb{E}(X_n) = \frac{\lfloor \frac{n+1}{2} \rfloor}{4}.$$

As $Y_n \le 1$, it is easy to see that the following events satisfy the following relations

$$\begin{aligned}
&\left\{\max_{0\le n\le M(d+1)-1} |R_C(n)-R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)}\right\}\\
\subseteq\ &\left\{\max_{0\le n\le M(d+1)-1} \left|R_C(n)-\frac{n}{4}+R_C(n-1)-\frac{n-1}{4}\right| > 10\sqrt{M(d+1)\log M(d+1)}\right\}\\
\subseteq\ &\left\{\max_{0\le n\le M(d+1)-1} \left(\left|R_C(n)-\frac{n}{4}\right|+\left|R_C(n-1)-\frac{n-1}{4}\right|\right) > 10\sqrt{M(d+1)\log M(d+1)}\right\}\\
\subseteq\ &\left\{\max_{0\le n\le M(d+1)-1} \left|R_C(n)-\frac{n}{4}\right| > 5\sqrt{M(d+1)\log M(d+1)}\right\}\\
=\ &\left\{\max_{0\le n\le M(d+1)-1} \left|2X_n+Y_n-\frac{n}{4}\right| > 5\sqrt{M(d+1)\log M(d+1)}\right\}\\
\subseteq\ &\left\{\max_{0\le n\le M(d+1)-1} \left|2X_n-\frac{n}{4}\right| > 4\sqrt{M(d+1)\log M(d+1)}\right\}\\
=\ &\left\{\max_{0\le n\le M(d+1)-1} \left|X_n-\frac{n}{8}\right| > 2\sqrt{M(d+1)\log M(d+1)}\right\}\\
\subseteq\ &\left\{\max_{0\le n\le M(d+1)-1} \left|X_n-\frac{\lfloor\frac{n+1}{2}\rfloor}{4}\right| > \sqrt{M(d+1)\log M(d+1)}\right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbb{P}\left(\max_{0\le n\le M(d+1)-1} |R_C(n)-R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)}\right)\\
&\le \mathbb{P}\left(\max_{0\le n\le M(d+1)-1} \left|X_n-\frac{\lfloor\frac{n+1}{2}\rfloor}{4}\right| > \sqrt{M(d+1)\log M(d+1)}\right)\\
&\le \sum_{n=0}^{M(d+1)-1} \mathbb{P}\left(\left|X_n-\frac{\lfloor\frac{n+1}{2}\rfloor}{4}\right| > \sqrt{M(d+1)\log M(d+1)}\right).
\end{aligned}$$

It follows from the Chernoff type bound [1], Corollary A 1.7. that if the random variable X is of binomial distribution with parameters m and p then for $a>0$ we have

$$\mathbb{P}(|X-mp|>a) \le 2e^{-2a^2/m}. \tag{7}$$

Applying (7) to $\lfloor\frac{n+1}{2}\rfloor$ and $p=\frac{1}{4}$ we have

$$\begin{aligned}
&\mathbb{P}\left(\left|X_n-\frac{\lfloor\frac{n+1}{2}\rfloor}{4}\right| > \sqrt{M(d+1)\log M(d+1)}\right) < 2\cdot\exp\left(\frac{-2M(d+1)\log M(d+1)}{\lfloor\frac{n+1}{2}\rfloor}\right)\\
&\le 2e^{-4\frac{M(d+1)\log M(d+1)}{M(d+1)}} = 2e^{-4\log M(d+1)} = \frac{2}{(M(d+1))^4} < \frac{1}{4M(d+1)}.
\end{aligned} \tag{8}$$

It follows that

$$\mathbb{P}(\{\max_{0\le n\le M(d+1)-1} |R_C(n)-R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)}\}) \tag{9}$$

$$< \frac{M(d+1)}{4M(d+1)} = \frac{1}{4}.$$

Case 2. Assume that $M(d+1) \le n \le 2M(d+1)-2$.

Obviously, $\mathbb{P}(\zeta_i = 0) = \frac{3}{4}$ and $\mathbb{P}(\zeta_i = 1) = \frac{1}{4}$ when $n - M(d+1) < i < \frac{n}{2}$, and if $0 \le i \le n - M(d+1)$ then $\zeta_i = 0$. Clearly we have

$$\mathbb{E}(X_n) = \frac{\lfloor \frac{2M(d+1)-1-n}{2} \rfloor}{4}.$$

As $Y_n \le 1$, it is easy to see that the following relations holds among the events

$$\begin{aligned}
&\left\{ \max_{M(d+1)\le n\le 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)} \right\} \\
\subseteq &\left\{ \max_{M(d+1)\le n\le 2M(d+1)-2} \left| R_C(n) - \frac{M(d+1)-\frac{n}{2}}{4} + R_C(n-1) - \frac{M(d+1)-\frac{n-1}{2}}{4} \right| \right. \\
&\left. >10\sqrt{M(d+1)\log M(d+1)} \right\} \\
\subseteq &\left\{ \max_{M(d+1)\le n\le 2M(d+1)-2} \left(\left| R_C(n) - \frac{M(d+1)-\frac{n}{2}}{4} \right| + \left| R_C(n-1) - \frac{M(d+1)-\frac{n-1}{2}}{4} \right| \right) \right. \\
&\left. >10\sqrt{M(d+1)\log M(d+1)} \right\} \\
\subseteq &\left\{ \max_{M(d+1)-1\le n\le 2M(d+1)-2} \left| R_C(n) - \frac{M(d+1)-\frac{n}{2}}{4} \right| > 5\sqrt{M(d+1)\log M(d+1)} \right\} \\
= &\left\{ \max_{M(d+1)-1\le n\le 2M(d+1)-2} \left| 2X_n + Y_n - \frac{2M(d+1)-n}{4} \right| > 5\sqrt{M(d+1)\log M(d+1)} \right\} \\
\subseteq &\left\{ \max_{M(d+1)-1\le n\le 2M(d+1)-2} \left| 2X_n - \frac{2M(d+1)-n}{4} \right| > 4\sqrt{M(d+1)\log M(d+1)} \right\} \\
= &\left\{ \max_{M(d+1)-1\le n\le 2M(d+1)-2} \left| X_n - \frac{2M(d+1)-n}{8} \right| > 2\sqrt{M(d+1)\log M(d+1)} \right\} \\
\subseteq &\left\{ \max_{M(d+1)-1\le n\le 2M(d+1)-2} \left| X_n - \frac{\lfloor \frac{2M(d+1)-1-n}{2} \rfloor}{4} \right| > \sqrt{M(d+1)\log M(d+1)} \right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbb{P}\left(\max_{M(d+1)-1\le n\le 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)} \right) \\
\le\, &\mathbb{P}\left(\max_{M(d+1)-1\le n\le 2M(d+1)-1} \left| X_n - \frac{\lfloor \frac{2M(d+1)-1-n}{2} \rfloor}{4} \right| > \sqrt{M(d+1)\log M(d+1)} \right) \\
\le &\sum_{n=M(d+1)-1}^{2M(d+1)-2} \mathbb{P}\left(\left| X_n - \frac{\lfloor \frac{2M(d+1)-1-n}{2} \rfloor}{4} \right| > \sqrt{M(d+1)\log M(d+1)} \right).
\end{aligned}$$

Applying (7) for $m = \frac{\lfloor \frac{2M(d+1)-1-n}{2} \rfloor}{4}$ and $p = \frac{1}{4}$ we have for $M(d+1) \le n \le 2M(d+1)-2$

$$\mathbb{P}\left(\left|X_n - \frac{\lfloor \frac{2M(d+1)-1-n}{2} \rfloor}{4}\right| > \sqrt{M(d+1)\log M(d+1)}\right) < 2 \cdot \exp\left(\frac{-2M(d+1)\log M(d+1)}{\lfloor \frac{2M(d+1)-1-n}{2} \rfloor}\right)$$

$$< 2e^{-4\frac{M(d+1)\log M(d+1)}{M(d+1)}} = 2e^{-4\log M(d+1)} = \frac{2}{(M(d+1))^4} < \frac{1}{4M(d+1)}$$

and by (8) we have

$$\mathbb{P}\left(\left|X_{M(d+1)-1} - \frac{\lfloor \frac{n+1}{2} \rfloor}{4}\right| > \sqrt{M(d+1)\log M(d+1)}\right) < \frac{1}{4M(d+1)}.$$

It follows that

$$\mathbb{P}\left(\max_{M(d+1)\le n\le 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)}\right) \tag{10}$$

$$< \frac{M(d+1)}{4M(d+1)} = \frac{1}{4}.$$

By (9) and (10) we get that

$$\mathbb{P}\left(\max_{0\le n\le 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)}\right) < \frac{1}{2}. \tag{11}$$

In the next step we show that

$$\mathbb{P}\left(B(C, \underline{\lambda}, M(d+1)-1) < \frac{M}{2^{d+2}}\right) < \frac{1}{2}.$$

It is clear that the following events $E_1, \dots, E_M$ are independent:

$$E_1 = \left\{\sum_{i=0}^{d} \lambda_i \varrho_C(d-i) \neq 0\right\},$$

$$E_2 = \left\{\sum_{i=0}^{d} \lambda_i \varrho_C(d+1+d-i) \neq 0\right\},$$

$$\vdots$$

$$E_M = \left\{\sum_{i=0}^{d} \lambda_i \varrho_C((m-1)(d+1)+d-i) \neq 0\right\}.$$

Obviously, $\mathbb{P}(E_i) = \mathbb{P}(E_j)$, where $1 \le i, j \le M$. Let $p = \mathbb{P}(E_1)$. It is clear that there exists an index u such that $\lambda_u \neq 0$. Thus we have

$$p \ge \mathbb{P}(\varrho_C(0) = 0, \varrho_C(1) = 0, \dots, \varrho_C(u-1) = 0, \varrho_C(u) = 1, \varrho_C(u+1) = 0, \dots, \varrho_C(d) = 0)$$

$$= \frac{1}{2^{d+1}}.$$

Define the random variable Z as the number of occurrence of the events E_j. It is easy to see that Z is of binomial distribution with parameters M and p. Applying the Chernoff bound (7) we get that

$$\mathbb{P}\left(|Z - Mp| > \frac{Mp}{2}\right) < 2e^{\frac{-2(Mp/2)^2}{M}} < 2e^{-\frac{M}{2}\cdot 2^{-2d-2}} < \frac{1}{2}$$

if M is large enough. On the other hand, we have

$$\frac{1}{2} > \mathbb{P}\left(|Z - Mp| > \frac{Mp}{2}\right) \geq \mathbb{P}\left(Z < \frac{Mp}{2}\right) \geq \mathbb{P}\left(Z < \frac{M}{2^{d+2}}\right).$$

Hence,

$$\mathbb{P}\left(B(C, \underline{\lambda}, 2M(d+1) - 2) < \frac{M}{2^{d+2}}\right) < \frac{1}{2}. \tag{12}$$

Let $\mathcal{E}$ and $\mathcal{F}$ be the events

$$\mathcal{E} = \left\{\max_{0 \leq n \leq 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)\log M(d+1)}\right\},$$

$$\mathcal{F} = \left\{B(C, \underline{\lambda}, M(d+1) - 1) < \frac{M}{2^{d+2}}\right\}.$$

It follows from (11) and (12) that

$$\mathbb{P}\left(\mathcal{E} \cup \mathcal{F}\right) < 1,$$

then

$$\mathbb{P}\left(\overline{\mathcal{E}} \cap \overline{\mathcal{F}}\right) > 0,$$

therefore there exists a suitable set C_M if M is large enough, which completes the proof of Lemma 1.

We are ready to prove Theorem 4. It is well known [5] that there exists a Sidon set S with

$$\limsup_{n\to\infty} \frac{S(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{2}},$$

where $S(n)$ is the number of elements of S up to n. Let $s, s' \in S$ and assume that $s > s'$. Define $S_M = S \setminus \{s, s' \in S : s - s' \leq 2M(d+1)\}$ and let $A = C_M + S_M$, where C_M is the set from the lemma.

$$\left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| = \left|\sum_{i=0}^{d} \lambda_i \#\{(a, a') : a + a' = n - i, a, a' \in A\}\right|$$

$$= \left|\sum_{i=0}^{d} \lambda_i \#\{(s, c, s', c') : s + c + s' + c' = n - i, s, s' \in S_M, c, c' \in C_M\}\right|$$

$$= \left|\sum_{i=0}^{d} \sum_{j=0}^{2M(d+1)} \lambda_i \#\{(s, c, s', c') : c + c' = j, s + s' = n - i - j, s, s' \in S_M, c, c' \in C_M\}\right|$$

$$= \left| \sum_{i=0}^{d} \sum_{j=0}^{2M(d+1)} \lambda_i R_{C_M}(j) R_{S_M}(n-i-j) \right|$$

$$= \left| \sum_{j=0}^{2M(d+1)} \sum_{i=0}^{d} \lambda_i R_{C_M}(j) R_{S_M}(n-i-j) \right| = \left| \sum_{k=0}^{2M(d+1)+d} \sum_{i=0}^{d} \lambda_i R_{C_M}(k-i) R_{S_M}(n-k) \right|$$

$$\left| \sum_{k=0}^{2M(d+1)+d} R_{S_M}(n-k) \sum_{i=0}^{d} \lambda_i R_{C_M}(k-i) \right| \le \sum_{k=0}^{2M(d+1)+d} R_{S_M}(n-k) \left| \sum_{i=0}^{d} \lambda_i R_{C_M}(k-i) \right|$$

$$\le 2(M+1)(d+1)2 \cdot \max_k \left| \sum_{i=0}^{d} \lambda_i R_{C_M}(k-i) \right|.$$

In the next step we give an upper estimation to $|\sum_{i=0}^{d} \lambda_i R_{C_M}(k-i)|$. We have

$|\lambda_0 R_{C_M}(k) + \ldots + \lambda_d R_{C_M}(k-d)|$
$= |\lambda_0(R_{C_M}(k) - R_{C_M}(k-1)) + (\lambda_0 + \lambda_1)(R_{C_M}(k-1) - R_{C_M}(k-2)) + \ldots$
$+ (\lambda_0 + \lambda_1 + \ldots + \lambda_{d-1})(R_{C_M}(k-d+1) - R_{C_M}(k-d)) + (\lambda_0 + \lambda_1 + \ldots + \lambda_d) R_{C_M}(k-d)|.$

Since $\sum_{i=0}^{d} \lambda_i = 0$, the last term in the previous sum is zero. Then we have

$$|\lambda_0 R_{C_M}(k) + \ldots + \lambda_d R_{C_M}(k-d)| \le d \left(\sum_{i=0}^{d} |\lambda_i| \right) \max_t |R_{C_M}(t) - R_{C_M}(t-1)| \le$$

$$12d \sum_{i=0}^{d} |\lambda_i| \sqrt{M(d+1) \log M(d+1)}.$$

Then we have

$$\left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \le 48d \sum_{i=0}^{d} |\lambda_i| (M(d+1))^{3/2} (\log M(d+1))^{1/2}.$$

We give a lower estimation for

$$\limsup_{n\to\infty} \frac{B(A, \underline{\lambda}, n)}{\sqrt{n}}.$$

If $0 \le v \le M(d+1) - 1$ and $\sum_{i=0}^{d} \lambda_i \chi_{C_M}(v-i) \ne 0$ then $\sum_{i=0}^{d} \lambda_i \chi_A(s+v-i) \ne 0$ for every $s \in S_M$. Then we have

$$B(A, \underline{\lambda}, n) \ge (S_M(N) - 1) B(C_M, \underline{\lambda}, (M+1) - 1).$$

Thus we have

$$\limsup_{n\to\infty} \frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} \ge \frac{M}{2^{d+2.5}}.$$

It follows that

$$\limsup_{n\to\infty}\left|\sum_{i=0}^{d}\lambda_i R_A(n-i)\right| \le 48d\sum_{i=0}^{d}|\lambda_i|\left((M(d+1))^3\log M(d+1)\right)^{1/2}$$

$$\le \limsup_{n\to\infty} 48(d+1)^4 2^{3d+7.5}\sum_{i=0}^{d}|\lambda_i|\left(\left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^3\log\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^{1/2},$$

if M is large enough. The proof of Theorem 4 is completed.

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