On the structure of sets which have coinciding representation functions

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Abstract

For a set of nonnegative integers A denote by $R_A(n)$ the number of unordered representations of the integer n as the sum of two different terms from A. In this paper we partially describe the structure of the sets, which has coinciding representation functions.

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1 Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers. For a given set $A \subseteq \mathbb{N}$, $A = \{a_1, a_2, \dots\}$, $(0 \le a_1 < a_2 < \dots)$ the additive representation functions $R_{h,A}^{(1)}(n)$, $R_{h,A}^{(2)}(n)$ and $R_{h,A}^{(3)}(n)$ are defined in the following way:

$$R_{h,A}^{(1)}(n) = |\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1}, \dots, a_{i_h} \in A\}|,$$

$$R_{h,A}^{(2)}(n) = |\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1} \le a_{i_2} \le \dots \le a_{i_h}, a_{i_1}, \dots, a_{i_h} \in A\}|,$$

$$R_{h,A}^{(3)}(n) = |\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1} < a_{i_2} < \dots < a_{i_h}, a_{i_1}, \dots, a_{i_h} \in A\}|.$$

For the simplicity we write $R_{2,A}^{(3)}(n) = R_A(n)$. If A is finite, let $|A|$ denote the cardinality of A.

The investigation of the partitions of the set of nonnegative integers with identical representation functions was a popular topic in the last few decades [1], [3], [4], [5], [7], [9], [11], [13], [14]. It is easy to see that $R_{2,A}^{(1)}(n)$ is odd if and only if $\frac{n}{2} \in A$. It follows

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that for every positive integer n, $R^{(1)}_{2,C}(n) = R^{(1)}_{2,D}(n)$ holds if and only if $C = D$, where $C = \{c_1, c_2, \dots\}$ $(c_1 < c_2 < \dots)$ and $D = \{d_1, d_2, \dots\}$ $(d_1 < d_2 < \dots)$ are two sets of nonnegative integers. In [8] Nathanson gave a full description of the sets C and D, which has identical representation functions $R^{(1)}_{2,C}(n) = R^{(1)}_{2,D}(n)$ from a certain point on. Namely, he proved the following theorem. Let $C(z) = \sum_{c\in C} z^c$, $D(z) = \sum_{d\in D} z^d$ be the generating functions of the sets C and D respectively.

Theorem 1 (Nathanson, 1978)**.** *Let C and D be different infinite sets of nonnegative integers. Then $R^{(1)}_{2,C}(n) = R^{(1)}_{2,D}(n)$ holds from a certain point on if and only if there exist positive integers n_0, M and finite sets of nonnegative integers F_C, F_D, T with $F_C \cup F_D \subset [0, Mn_0 - 1]$, $T \subset [0, M - 1]$ such that*

$$C = F_C \cup \{lM + t : l \geq n_0, t \in T\},$$

$$D = F_D \cup \{lM + t : l \geq n_0, t \in T\},$$

$$1 - z^M | (F_C(z) - F_D(z))T(z).$$

We conjecture [6] that the above theorem of Nathanson can be generalized in the following way.

Conjecture 1 (Kiss, Rozgonyi, Sándor, 2012)**.** *For $h > 2$ let C and D be different infinite sets of nonnegative integers. Then $R^{(1)}_{h,C}(n) = R^{(1)}_{h,D}(n)$ holds from a certain point on if and only if there exist positive integers n_0, M and finite sets F_C, F_D, T with $F_C \cup F_D \subset [0, Mn_0 - 1]$, $T \subset [0, M - 1]$ such that*

$$C = F_C \cup \{lM + t : l \geq n_0, t \in T\},$$

$$D = F_D \cup \{lM + t : l \geq n_0, t \in T\},$$

$$(1 - z^M)^{h-1} | (F_C(z) - F_D(z))T(z)^{h-1}.$$

For $h = 3$ Kiss, Rozgonyi and Sándor proved [6] Conjecture 1. In the general case when $h > 3$ we proved that if the conditions of Conjecture 1 are hold then $R^{(1)}_{h,C}(n) = R^{(1)}_{h,D}(n)$ holds from a certain point on. Later Rozgonyi and Sándor in [10] proved that the above conjecture holds, when $h = p^\alpha$, where $\alpha \geq 1$ and p is a prime.

It is easy to see that for any two different sets C, $D \subset \mathbb{N}$ we have $R^{(2)}_{2,C}(n) \neq R^{(2)}_{2,D}(n)$ for some $n \in \mathbb{N}$. Let i denote the smallest index for which $c_i \neq d_i$, thus we may assume that $c_i < d_i$. It is clear that $R^{(2)}_{2,C}(c_1 + c_i) > R^{(2)}_{2,D}(c_1 + c_i)$, which implies that there exists a nonnegative integer n such that $R^{(2)}_{2,C}(n) \neq R^{(2)}_{2,D}(n)$. We pose a problem about this representation function.

Problem 1. *Determine all the sets of nonnegative integers C and D such that $R^{(2)}_{2,C}(n) = R^{(2)}_{2,D}(n)$ holds from a certain point on.*

In this paper we focus on the representation function $R_A(n)$. We partially describe the structure of the sets, which has identical representation functions. To do this we define the Hilbert cube which plays a crucial role in our results. Let $\{h_1, h_2, \dots\}$ $(h_1 < h_2 < \dots)$ be finite or infinite set of positive integers. The set

$$H(h_1, h_2, \dots) = \Big\{ \sum_i \varepsilon_i h_i : \varepsilon_i \in \{0, 1\} \Big\}$$

is called Hilbert cube. The even part of a Hilbert cube is the set

$$H_0(h_1,h_2,\dots)=\Big\{\sum_i \varepsilon_i h_i : \varepsilon_i\in\{0,1\}, 2|\sum_i \varepsilon_i\Big\},$$

and the odd part of a Hilbert cube is

$$H_1(h_1,h_2,\dots)=\Big\{\sum_i \varepsilon_i h_i : \varepsilon_i\in\{0,1\}, 2\nmid\sum_i \varepsilon_i\Big\}.$$

We say a Hilbert cube $H(h_1,h_2,\dots)$ is half non-degenerated if the representation of any integer in $H_0(h_1,h_2,\dots)$ and $H_1(h_1,h_2,\dots)$ is unique, that is $\sum_i \varepsilon_i h_i\neq\sum_i \varepsilon_i^{'} h_i$ whenever $\sum_i \varepsilon_i\equiv\sum_i \varepsilon_i^{'} \bmod 2$, where $\varepsilon_i^{'}\in\{0,1\}$.

It was studied [12] what can be told about the cardinality of the sets with identical representation functions. For the sake of completeness we present the result and the proof.

Theorem 2 (Selfridge - Straus, 1958)**.** *Let C and D be different finite sets of nonnegative integers such that for every n positive integer, $R_C(n)=R_D(n)$ holds. Then we have $|C|=|D|=2^l$ for a nonnegative integer l.*

If $0\in C$ and for $D=\{d_1,d_2,\dots\}$, $0\le d_1<d_2<\dots$ we have $R_C(m)=R_D(m)$ (sequences C and D are different), then $d_1>0$ otherwise let us suppose that $c_i=d_i$ for $i=1,2,\dots,n-1$, but $c_n<d_n$, which implies that $R_C(c_1+c_n)>R_D(c_1+c_n)$, a contradiction.

If $|C|=|D|=1$ and $0\in C$ with $R_C(n)=R_D(n)$, then we have $C=\{0\}$ and $D=\{d_1\}$. Therefore, $C=H_0(d_1)$ and $D=H_1(d_1)$.

If $|C|=|D|=2$ and $0\in C$ with $R_C(n)=R_D(n)$, then $C=\{0,c_2\}$ and $D=\{d_1,d_2\}$. In this case $1=R_C(0+c_2)$ and for $n\neq c_2$ we have $R_C(n)=0$. Moreover, $1=R_C(d_1+d_2)$ and for $n\neq d_1+d_2$ we have $R_D(n)=0$. This implies that $d_1+d_2=c_1+c_2=c_2$, that is $C=\{0,d_1+d_2\}=H_0(d_1,d_2)$ and $D=\{d_1,d_2\}=H_1(d_1,d_2)$.

If $|C|=|D|=4$ and $0\in C$ with $R_C(n)=R_D(n)$, then let $C=\{c_1,c_2,c_3,c_4\}$, $c_1=0$ and $D=\{d_1,d_2,d_3,d_4\}$, where $d_1>0$. Then we have

$$c_1+c_2<c_1+c_3<c_1+c_4, c_2+c_3<c_2+c_4<c_3+c_4$$

and

$$d_1+d_2<d_1+d_3<d_1+d_4, d_2+d_3<d_2+d_4<d_3+d_4$$

which implies that $c_1+c_2=d_1+d_2$ therefore, $c_2=d_1+d_2$ and $c_1+c_3=d_1+d_3$, thus we have $c_3=d_1+d_3$. If $c_2+c_3=d_2+d_3$, then $(d_1+d_2)+(d_1+d_3)=d_2+d_3$, that is $d_1=0$, a contradiction. Hence $c_2+c_3=d_1+d_4$, that is $(d_1+d_2)+(d_1+d_3)=d_1+d_4$. This implies that $d_4=d_1+d_2+d_3$. Finally $c_1+c_4=d_2+d_3$, that is $c_4=d_2+d_3$. Thus we have $C=\{0,d_1+d_2,d_1+d_3,d_2+d_3\}=H_0(d_1,d_2,d_3)$ and $D=\{d_1,d_2,d_3,d_1+d_2+d_3\}=H_1(d_1,d_2,d_3)$.

In the next step we prove that if the sets are even and odd part of a Hilbert cube, then the corresponding representation functions are identical.

Theorem 3. *Let $H(h_1,h_2,\dots)$ be a half non-degenerated Hilbert cube.*

If $C=H_0(h_1,h_2,\dots)$ and $D=H_1(h_1,h_2,\dots)$, then for every positive integer n, $R_C(n)=R_D(n)$ holds.

It is easy to see that Theorem 3 is equivalent to Lemma 1 of Chen and Lev in [2]. First they proved the finite case $H(h_1, \dots, h_n)$ by induction on n and the infinite case was a corollary of the finite case. For the sake of completeness we give a different proof by using generating functions. Chen and Lev asked [2] whether Theorem 3 described all different sets C and D of nonnegative integers such that $R_C(n) = R_D(n)$. The following conjecture is a simple generalization of the above question formulated by Chen and Lev [2] but we use a different terminology.

Conjecture 2. *Let C and D be different infinite sets of nonnegative integers with $0 \in C$. If for every positive integer n, $R_C(n) = R_D(n)$ holds then there exist positive integers $d_{i_1}, d_{i_2}, \dots \in D$, where $d_{i_1} < d_{i_2} < \dots$ and a half non-degenerated Hilbert cube $H(d_{i_1}, d_{i_2}, \dots)$ such that*

$$C = H_0(d_{i_1}, d_{i_2}, \dots),$$

$$D = H_1(d_{i_1}, d_{i_2}, \dots).$$

We showed above that Conjecture 2 is true for the finite case $l = 0, 1, 2$. Unfortunately we could not settle the cases $l \geq 3$, which seems to be very complicated. In the next step we prove a weaker version of the above conjecture.

Theorem 4. *Let $D = \{d_1, \dots, d_{2^n}\}$, $(0 < d_1 < d_2 < \dots < d_{2^n})$ be a set of nonnegative integers, where $d_{2^k+1} \geq 4d_{2^k}$, for $k = 0, \dots, n-1$ and $d_{2^k} \leq d_1 + d_2 + d_3 + d_5 + \dots + d_{2^i+1} + \dots + d_{2^{k-1}+1}$ for $k = 2, \dots, n$. Let C be a finite set of nonnegative integers such that $0 \in C$. If for every positive integer m, $R_C(m) = R_D(m)$ holds, then*

$$C = H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{n-1}+1}),$$

and

$$D = H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{n-1}+1}).$$

For any sets of nonnegative integers A and B we define the sumset $A + B$ by

$$A + B = \{a + b : a \in A, b \in B\}.$$

In the special case $b + A$ denotes the set $\{b + a : a \in A\}$, where b is a fixed nonnegative integer. Let $q\mathbb{N}$ denote the dilate of the set $\mathbb{N}$ by the factor q, that is $q\mathbb{N}$ is the set of nonnegative integers divisible by q. Let $r_{A+B}(n)$ denote the number of solutions of the equation $a + b = n$, where $a \in A$, $b \in B$. In [2] Chen and Lev proved the following nice result.

Theorem 5. *(Chen and Lev, 2016) Let l be a positive integer. Then there exist sets C and D of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = 2^{2l} - 1 + (2^{2l+1} - 1)\mathbb{N}$ and for every positive integer n, $R_C(n) = R_D(n)$ holds.*

This theorem is an easy consequence of Theorem 3 by putting

$$H(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \dots).$$

The details can be found in the first part of the proof of Theorem 6. Chen and Lev [2] posed the following question.

Conjecture 3. *(Chen and Lev, 2016) Is it true that if C and D are different sets of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$, $m \geq 2$ and for any positive integer n, $R_C(n) = R_D(n)$ then there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$ and $m = 2^{2l+1} - 1$?*

A stronger version of this conjecture can be formulated as follows.

Conjecture 4. *Is it true that if C and D are different sets of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$, $m \geq 2$ and for every positive integer n, $R_C(n) = R_D(n)$ then there exists an integer $l \geq 1$ such that*

$$C = H_0(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \dots),$$

and

$$D = H_1(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \dots)?$$

We prove that Conjecture 2 implies Conjecture 4.

Theorem 6. *Assume that Conjecture 2 holds. Then there exist C and D different infinite sets of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$, $m \geq 2$ and for every positive integer n, $R_C(n) = R_D(n)$ if and only if there exists an integer $l \geq 1$ such that*

$$C = H_0(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \dots)$$

and

$$D = H_1(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \dots).$$

2 Proof of Theorem 2.

Proof. Applying the generating functions of the sets C and D, we get that

$$\sum_{n=1}^{\infty} R_C(n) z^n = \frac{C(z)^2 - C(z^2)}{2},$$

$$\sum_{n=1}^{\infty} R_D(n) z^n = \frac{D(z)^2 - D(z^2)}{2}.$$

It follows that

$$R_C(n) = R_D(n) \Leftrightarrow C(z)^2 - D(z)^2 = C(z^2) - D(z^2). \tag{1}$$

Let $l + 1$ be the largest exponent of the factor $(z - 1)$ in $C(z) - D(z)$ i.e.,

$$C(z) - D(z) = (z - 1)^{l+1} p(z), \tag{2}$$

where $p(z)$ is a polynomial and $p(1) \neq 0$. Writing (2) in (1) we get that

$$(C(z) + D(z))(z - 1)^{l+1} p(z) = (z^2 - 1)^{l+1} p(z^2),$$

thus we have $(C(z)+D(z))p(z) = (z+1)^{l+1}p(z^2)$. Putting $z=1$, we have $C(1)+D(1) = 2^{l+1}$, which implies that $|C|+|D| = 2^{l+1}$. On the other hand

$$\binom{|C|}{2} = \sum_m R_C(m) = \sum_m R_D(m) = \binom{|D|}{2},$$

therefore $|C| = |D|$, which completes the proof of Theorem 2. □

3 Proof of Theorem 3.

Proof. By (1) we have to prove that $C(z)^2 - D(z)^2 = C(z^2) - D(z^2)$. It is easy to see from the definition of C and D that

$$\prod_i (1 - z^{h_i}) = \sum_{i_1<\ldots<i_t} (-1)^t z^{h_{i_1}+\ldots+h_{i_t}} = C(z) - D(z).$$

On the other hand clearly we have $C(z) + D(z) = \prod_i (1 + z^{h_i})$. Thus we have

$$C(z)^2 - D(z)^2 = \Big(C(z) - D(z)\Big)\Big(C(z) + D(z)\Big) = \prod_i (1 - z^{h_i}) \cdot \prod_i (1 + z^{h_i})$$

$$= \prod_i (1 - z^{2h_i}) = C(z^2) - D(z^2).$$

The proof is completed. □

4 Proof of Theorem 4.

We prove by induction on n. In the case $n = 0$, then $C = \{0\}$ and $D = \{d_1\}$ therefore, for every positive integer m we have $R_C(m) = R_D(m) = 0$. For $n = 1$, then $C = \{0, c_2\}$ and $D = \{d_1, d_2\}$. As for every positive integer m, $R_C(m) = R_D(m)$ holds it follows that $R_D(d_1 + d_2) = 1 = R_C(d_1 + d_2)$, thus we have $C = \{0, d_1 + d_2\} = H_0(d_1, d_2)$ and $D = \{d_1, d_2\} = H_1(d_1, d_2)$. Assume that the statement of Theorem 4 holds for $n = N-1$ and we will prove it for $n = N$. Let D be a set of nonnegative integers, $D = \{d_1, \ldots, d_{2^N}\}$, where $d_{2^k+1} \geq 4d_{2^k}$, for $k = 0, \ldots, N-1$ and $d_{2^k} \leq d_1 + d_2 + d_3 + d_5 + \ldots + d_{2^i+1} + \ldots + d_{2^{k-1}+1}$ for $k = 2, \ldots, N$. If C is a set of nonnegative integers such that $0 \in C$ and for every positive integer m, $R_C(m) = R_D(m)$ holds then we have to prove that

$$C = H_0(d_1, d_2, d_3, d_5, \ldots, d_{2^k+1}, \ldots, d_{2^{N-1}+1}),$$

and

$$D = H_1(d_1, d_2, d_3, d_5, \ldots, d_{2^k+1}, \ldots, d_{2^{N-1}+1}).$$

Define the sets

$$C_1 = \{c_1, \ldots, c_{2^{N-1}}\}, \qquad C_2 = C \setminus C_1,$$

and

$$D_1 = \{d_1, \ldots, d_{2^{N-1}}\}, \qquad D_2 = D \setminus D_1.$$

We prove that for every positive integer m, we have

$$R_{C_1}(m) = R_{D_1}(m). \tag{3}$$

Since $d_{2^{N-1}} \le \frac{1}{4} d_{2^{N-1}+1}$ it follows that for any d_i, $d_j \in D_1$ we have $d_i + d_j \le \frac{1}{4} d_{2^{N-1}+1} + \frac{1}{4} d_{2^{N-1}+1} = \frac{1}{2} d_{2^{N-1}+1}$. This implies that for every $\frac{1}{2} d_{2^{N-1}+1} \le m \le d_{2^{N-1}+1}$ we have $R_D(m) = 0$, which yields $R_C(m) = 0$. As $0 \in C$, thus we have a representation $m = 0 + m$, it follows that $m \notin C$ for $\frac{1}{2} d_{2^{N-1}+1} \le m \le d_{2^{N-1}+1}$. We will show that

$$C_1 = \Big[0, \frac{1}{3} d_{2^{N-1}+1}\Big[\cap C, \qquad D_1 = \Big[0, \frac{1}{3} d_{2^{N-1}+1}\Big[\cap D.$$

We distinguish two cases. In the first case we assume that $c_{2^{N-1}+1} \le \frac{d_{2^{N-1}+1}}{2}$. Then we have

$$\binom{2^{N-1}+1}{2} \le \sum_{m < d_{2^{N-1}+1}} R_C(m) = \sum_{m < d_{2^{N-1}+1}} R_D(m) = \binom{2^{N-1}}{2}$$

which is a contradiction. In the second case we assume that $c_{2^{N-1}} > \frac{d_{2^{N-1}+1}}{2}$, which implies that $c_{2^{N-1}} \ge d_{2^{N-1}+1}$. Then we have

$$\binom{2^{N-1}-1}{2} \ge \sum_{m < d_{2^{N-1}+1}} R_C(m) = \sum_{m < d_{2^{N-1}+1}} R_D(m) = \binom{2^{N-1}}{2}$$

which is absurd. Thus we have $c_{2^{N-1}} \le \frac{1}{2} d_{2^{N-1}+1} < c_{2^{N-1}+1}$ and $d_{2^{N-1}+1} < c_{2^{N-1}+1}$.

We get that

$$R_{C_1}(m) = \begin{cases} 0, \text{ if } m \ge d_{2^{N-1}+1} \\ R_C(m), \text{ if } m < d_{2^{N-1}+1} \end{cases},$$

and

$$R_{D_1}(m) = \begin{cases} 0, \text{ if } m \ge d_{2^{N-1}+1} \\ R_D(m), \text{ if } m < d_{2^{N-1}+1} \end{cases}.$$

It follows that for every positive integer m, $R_{C_1}(m) = R_{D_1}(m)$ so the proof of (3) is completed. By the induction hypothesis we get that

$$C_1 = H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, , d_{2^{N-2}+1})$$

and

$$D_1 = H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, , d_{2^{N-2}+1}).$$

By Theorem 4 we have $d_{2^{k-1}+1} \le d_{2^k} \le \frac{1}{4} d_{2^k+1}$ for $1 \le k \le N-1$. Then we have $d_{2^{N-i}+1} \le \frac{1}{4^{i-1}} d_{2^{N-1}+1}$ for $i = 2, \dots, N$ and $d_1 \le \frac{1}{4^N} d_{2^{N-1}+1}$. It follows that the maximal element of the set $H(d_1, d_2, d_3, d_5, d_9, \dots, d_{2^{N-2}+1})$ is $d_1 + d_2 + d_3 + d_5 + d_9 + \dots + d_{2^{N-2}+1} \le \frac{1}{4^N} d_{2^{N-1}+1} + \frac{1}{4^{N-1}} d_{2^{N-1}+1} + \dots + \frac{1}{4} d_{2^{N-1}+1} < \frac{1}{3} d_{2^{N-1}+1}$, which implies that

$$C_1 = \Big[0, \frac{1}{3} d_{2^{N-1}+1}\Big[\cap C, \qquad D_1 = \Big]0, \frac{1}{3} d_{2^{N-1}+1}\Big[\cap D. \tag{4}$$

Then we have

$$C_1 + C_1 \subset \Big[0, \frac{2}{3} d_{2^{N-1}+1}\Big[, \qquad D_1 + D_1 \subset \Big[0, \frac{2}{3} d_{2^{N-1}+1}\Big[. \tag{5}$$

On the other hand for every $d \in D_2$ we have

$$d_{2^{N-1}+1} \le d \le d_{2^N} \le d_{2^{N-1}+1} + d_{2^{N-2}+1} + \ldots + d_{2^i+1} + \ldots + d_2 + d_1$$
$$\le d_{2^{N-1}+1} + \frac{1}{4}d_{2^{N-1}+1} + \ldots + \frac{1}{4^{N-i-1}}d_{2^{N-1}+1} + \ldots + \frac{1}{4^{N-1}}d_{2^{N-1}+1} + \frac{1}{4^N}d_{2^{N-1}+1}$$
$$< \frac{4}{3}d_{2^{N-1}+1}.$$

Thus we have

$$D_1 + D_2 \subset \Big[d_{2^{N-1}+1}, \frac{5}{3}d_{2^{N-1}+1}\Big[, \tag{6}$$

and

$$D_2 + D_2 \subset \Big[2d_{2^{N-1}+1}, \frac{8}{3}d_{2^{N-1}+1}\Big[. \tag{7}$$

It follows that

$$R_C(m) = 0 \text{ for } m \ge \frac{8}{3}d_{2^{N-1}+1}. \tag{8}$$

We prove that $c_{2^{N-1}+1} = d_{2^{N-1}+1} + d_1$. Assume that

$$c_{2^{N-1}+1} < d_{2^{N-1}+1} + d_1. \tag{9}$$

Obviously, $c_{2^{N-1}+1} > d_{2^{N-1}+1}$. We have $c_{2^{N-1}+1} = c_{2^{N-1}+1} + 0$, thus $1 \le R_C(c_{2^{N-1}+1}) = R_D(c_{2^{N-1}+1})$, which implies that $c_{2^{N-1}+1} = d_i + d_j$, $i < j$, $d_i, d_j \in D$. If $j \le 2^{N-1}$, then by using the first condition in Theorem 4 we have

$$c_{2^{N-1}+1} = d_i + d_j \le 2d_{2^{N-1}} \le \frac{1}{2}d_{2^{N-1}+1},$$

which contradicts the inequality $c_{2^{N-1}+1} \ge d_{2^{N-1}+1}$. On the other hand when $j \ge 2^{N-1}+1$, we have

$$c_{2^{N-1}+1} = d_i + d_j \ge d_1 + d_{2^{N-1}+1},$$

which contradicts (9).

Assume that $c_{2^{N-1}+1} > d_{2^{N-1}+1}+d_1$. Obviously, $1 \le R_D(d_{2^{N-1}+1}+d_1) = R_C(d_{2^{N-1}+1}+d_1)$, which implies that $d_1 + d_{2^{N-1}+1} = c_i + c_j$, $i < j$, $c_i, c_j \in C$. If $j \le 2^{N-1}$, then we have

$$d_1 + d_{2^{N-1}+1} = c_i + c_j \le 2c_{2^{N-1}} \le d_{2^{N-1}+1},$$

which is absurd. On the other hand when $j \ge 2^{N-1} + 1$, we have

$$d_1 + d_{2^{N-1}+1} = c_i + c_j \ge c_{2^{N-1}+1} > d_{2^{N-1}+1} + d_1,$$

which is a contradiction.

It follows that for every $c \in C$ with $c > c_{2^{N-1}+1}$ we have $c \le \frac{5}{3}d_{2^{N-1}+1}$. Otherwise $c + c_{2^{N-1}+1} \ge \frac{8}{3}d_{2^{N-1}+1}$ and then $R_C(c + c_{2^{N-1}+1}) \ge 1$ which contradicts (8). By (4) and (8) we have

$$C_1 + C_2 \subset \Big[d_{2^{N-1}+1}, 2d_{2^{N-1}+1}\Big[, \tag{10}$$

and

$$(C_2 + C_2) \setminus \{2c_{2^N}\} \subset \Big[2d_{2^{N-1}+1}, \frac{8}{3}d_{2^{N-1}+1}\Big[. \tag{11}$$

We have to prove that $C_2 = d_{2^{N-1}+1} + H_1(d_1, d_2, d_3, d_5, \dots, d_{2^{N-2}+1}) = d_{2^{N-1}+1} + D_1$ and $D_2 = d_{2^{N-1}+1} + H_0(d_1, d_2, d_3, d_5, \dots, d_{2^{N-2}+1}) = d_{2^{N-1}+1} + C_1$.

Define the sets

$$C_{2,n} = \{c_{2^{N-1}+1}, c_{2^{N-1}+2}, \dots, c_{2^{N-1}+n}\},$$

and

$$D_{2,n} = \{d_{2^{N-1}+1}, d_{2^{N-1}+2}, \dots, d_{2^{N-1}+n}\}.$$

On the other hand define the sets

$$C_1+C_{2,n} = \{p_1, p_2, \dots\}, \ (p_1 < p_2 < \dots), \qquad C_{2,n}+C_{2,n} = \{t_1, t_2, \dots\}, \ (t_1 < t_2 < \dots),$$

and

$$D_1+D_{2,n} = \{q_1, q_2, \dots\}, \ (q_1 < q_2 < \dots), \qquad D_{2,n}+D_{2,n} = \{s_1, s_2, \dots\}, \ (s_1 < s_2 < \dots).$$

Denote by $H_0^{(n)}$ the first $2^{N-1} + n$ elements of the set

$$H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{N-1}+1}),$$

and let $H_1^{(n)}$ denote the first $2^{N-1} + n$ elements of the set

$$H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{N-1}+1}).$$

We will prove by induction on n that

$$H_0^{(n)} = C_1 \cup C_{2,n} \text{ and } H_1^{(n)} = D_1 \cup D_{2,n}$$

for $1 \le n \le 2^{N-1}$.

For $n = 1$ we have already proved that $D_{2,1} = \{d_{2^{N-1}+1}\}$ and $C_{2,1} = \{d_{2^{N-1}+1} + d_1\}$. It follows that $H_0^{(1)} = C_1 \cup C_{2,1}$ and $H_1^{(1)} = D_1 \cup D_{2,1}$.

Let us suppose that $H_0^{(n)} = C_1 \cup C_{2,n}$ and $H_1^{(n)} = D_1 \cup D_{2,n}$ and we are going to prove that $H_0^{(n+1)} = C_1 \cup C_{2,n+1}$ and $H_1^{(n+1)} = D_1 \cup D_{2,n+1}$.

In order to prove $H_0^{(n+1)} = C_1 \cup C_{2,n+1}$ and $H_1^{(n+1)} = D_1 \cup D_{2,n+1}$ we need three lemmas.

Let i be the smallest index u such that $r_{C_1+C_{2,n}}(p_u) > r_{D_1+D_{2,n}}(p_u)$. If there does not exist such i, then $p_i = +\infty$. Let j be the smallest index v such that $r_{C_1+C_{2,n}}(q_v) < r_{D_1+D_{2,n}}(q_v)$. If there does not exist such j, then $q_j = +\infty$. Let k be the smallest index w such that $R_{C_{2,n}}(t_w) > R_{D_{2,n}}(t_w)$. If there does not exist such k, then $t_k = +\infty$. Let l be the smallest index x such that $R_{C_{2,n}}(s_x) < R_{D_{2,n}}(s_x)$. If there does not exist such l, then $s_l = +\infty$. The following observations play a crucial role in the proof.

Lemma 1. *Let us suppose that $H_0^{(n)} = C_1 \cup C_{2,n}$ and $H_1^{(n)} = D_1 \cup D_{2,n}$. Then we have*

(i) $min\{p_i, c_{2^{N-1}+n+1}\} = min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$,

(ii) $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} = min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$.

Proof. In the first step we prove (i). We will prove that $p_i = +\infty$ is equivalent to $q_j = +\infty$ and for $p_i = q_j = +\infty$ we have $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$. If $p_i = +\infty$, then by the definition we have $r_{C_1+C_{2,n}}(p_f) \le r_{D_1+D_{2,n}}(p_f)$ for every positive integer f, thus we have

$$r_{C_1+C_{2,n}}(m) \le r_{D_1+D_{2,n}}(m)$$

for every positive integer m. On the other hand we have

$$2^{N-1} \cdot n = \sum_m r_{C_1+C_{2,n}}(m) \le \sum_m r_{D_1+D_{2,n}}(m) = 2^{N-1} \cdot n,$$

thus for every positive integer m we have $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$, which implies that $q_j = +\infty$. If $q_j = +\infty$, the by the definition $r_{D_1+D_{2,n}}(q_g) \le r_{C_1+C_{2,n}}(q_g)$ for every positive integer g, thus we have

$$r_{C_1+C_{2,n}}(m) \ge r_{D_1+D_{2,n}}(m)$$

for every positive integer m. On the other hand we have

$$2^{N-1} \cdot n = \sum_m r_{C_1+C_{2,n}}(m) \ge \sum_m r_{D_1+D_{2,n}}(m) = 2^{N-1} \cdot n,$$

thus for every positive integer m we have $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$, which implies that $p_i = +\infty$.

We distinguish two cases.

Case 1. $p_i = +\infty$, $q_j = +\infty$, that is for every positive integer m we have $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$. We have to prove that $c_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+n+1}$. Assume that $c_{2^{N-1}+n+1} < d_1 + d_{2^{N-1}+n+1}$. Since $c_{2^{N-1}+n+1} = 0 + c_{2^{N-1}+n+1}$, where $0 \in C_1$ but $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$ it follows from (5), (6), (7) and (10) that $R_D(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1})$ and $R_C(c_{2^{N-1}+n+1}) > r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1})$.

Thus we have

$$R_D(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+n+1}),$$

which is absurd. Similarly, if $c_{2^{N-1}+n+1} > d_1 + d_{2^{N-1}+n+1}$, then $R_D(d_1 + d_{2^{N-1}+n+1}) > r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1})$ because $d_1 \in D_1$, $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$. It follows from (5), (6), (10) and (11) that $R_C(d_1 + d_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1})$. Thus we have

$$\begin{aligned} R_C(d_1 + d_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) &= r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1}) \\ &< R_D(d_1 + d_{2^{N-1}+n+1}), \end{aligned}$$

which is a contradiction.

Case 2. $p_i < +\infty$ and $q_j < +\infty$. We have two subcases.

Case 2a. $min\{p_i, c_{2^{N-1}+n+1}\} < min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$.

If $p_i \le c_{2^{N-1}+n+1}$, then obviously $p_i < d_1 + d_{2^{N-1}+n+1}$, which implies by (5), (6), (7) and (10) that $R_D(p_i) = r_{D_1+D_{2,n}}(p_i)$. By using the above facts and the definition of p_i we obtain that

$$R_C(p_i) \ge r_{C_1+C_{2,n}}(p_i) > r_{D_1+D_{2,n}}(p_i) = R_D(p_i),$$

which contradicts the fact that for every positive integer m, $R_C(m) = R_D(m)$ holds. On the other hand if $p_i > c_{2^{N-1}+n+1}$, then by the definition of p_i, $r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) \le$

$r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1})$ and since $c_{2^{N-1}+n+1} = 0 + c_{2^{N-1}+n+1}$, $0 \in C_1$ and $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$ therefore we have

$$R_C(c_{2^{N-1}+n+1}) > r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) \tag{12}$$

and the assumption $min\{p_i, c_{2^{N-1}+n+1}\} < min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$ implies that $q_j > c_{2^{N-1}+n+1}$. It follows from the definition of q_j that $r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) \le r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1})$. We get that

$$r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}). \tag{13}$$

It follows from $0 + c_{2^{N-1}+n+1} < d_1 + d_{2^{N-1}+n+1}$, $0 \in C_1$, (5), (6), (7) and (10) that $r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) = R_D(c_{2^{N-1}+n+1})$. On the other hand we obtain from (12) and (13) that

$$R_D(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+n+1}),$$

which contradicts the fact that for every positive integer m, $R_C(m) = R_D(m)$ holds.

Case 2b. $min\{p_i, c_{2^{N-1}+n+1}\} > min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$.

If $q_j \le d_1 + d_{2^{N-1}+n+1}$, then obviously $q_j < c_{2^{N-1}+n+1}$, which implies from (5), (6), (10) and (11) that $R_C(q_j) = r_{C_1+C_{2,n}}(q_j)$. By using the above facts and the definition of q_j we obtain that

$$R_C(q_j) = r_{C_1+C_{2,n}}(q_j) < r_{D_1+D_{2,n}}(q_j) \le R_D(q_j),$$

which contradicts the fact that for every positive integer m, $R_C(m) = R_D(m)$ holds.

On the other hand if $q_j > d_1 + d_{2^{N-1}+n+1}$, then by the definition of q_j, $r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) \ge r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1})$ and since $d_1 \in D_1$, $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$, we have

$$R_D(d_1 + d_{2^{N-1}+n+1}) > r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1}) \tag{14}$$

and from $min\{p_i, c_{2^{N-1}+n+1}\} > min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$ we get that $p_i > d_1 + d_{2^{N-1}+n+1}$. It follows from the definition of p_i that $r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) \le r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1})$. We obtain that

$$r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1}). \tag{15}$$

It follows from $min\{p_i, c_{2^{N-1}+n+1}\} > min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$ that $c_{2^{N-1}+n+1} > d_1 + d_{2^{N-1}+n+1}$ therefore, it follows from (5), (6), (10) and (11) that

$$r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) = R_C(d_1 + d_{2^{N-1}+n+1}). \tag{16}$$

On the other hand we obtain from (14),(15) and (16) that

$$R_D(d_1 + d_{2^{N-1}+n+1}) > r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1})$$

$$= r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) = R_C(d_1 + d_{2^{N-1}+n+1}),$$

which contradicts the fact that for every positive integer m, $R_C(m) = R_D(m)$ holds. The proof of (i) in Lemma 1 is completed.

The proof of (ii) in Lemma 1 is similar to the proof of (i). For the sake of completeness we present it.

We prove that $s_l = +\infty$ is equivalent to $t_k = +\infty$ and in this case $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ for every m.

If $t_k = +\infty$, then by the definition we have $R_{C_{2,n}}(t_f) \le R_{D_{2,n}}(t_f)$ for every positive integer f, thus for every positive integer m we have

$$R_{C_{2,n}}(m) \le R_{D_{2,n}}(m).$$

On the other hand we have

$$\binom{n}{2} = \sum_m R_{C_{2,n}}(m) \le \sum_m R_{D_{2,n}}(m) = \binom{n}{2},$$

thus we get that for every positive integer m, $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ holds. This implies that $s_l = +\infty$.
If $s_l = +\infty$, the by the definition $R_{C_{2,n}}(s_g) \ge R_{D_{2,n}}(s_g)$ for every positive integer g, thus for every positive integer m we have

$$R_{C_{2,n}}(m) \ge R_{D_{2,n}}(m).$$

On the other hand we have

$$\binom{n}{2} = \sum_m R_{C_{2,n}}(m) \ge \sum_m R_{D_{2,n}}(m) = \binom{n}{2},$$

thus we get that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ for every m, which implies that $t_k = +\infty$.
We distinguish two cases.
Case 1. $t_k = +\infty$, $s_l = +\infty$, that is $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ for every positive integer m. We have to prove that $d_{2^{N-1}+n+1} = d_1 + c_{2^{N-1}+n+1}$. Assume that $d_{2^{N-1}+n+1} > d_1 + c_{2^{N-1}+n+1}$. As $d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1} = c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$, where $c_{2^{N-1}+1} \in C_{2,n}$ and $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$ it follows that $R_C(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) > R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. On the other hand we will show that it follows from $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} > d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1} = c_{2^{N-1}+1} + c_{2^{N-1}+1}$ and (5), (6), (7), (11) that $R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. It is clear from (11) that $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} \in (C_2 + C_2) \setminus \{2c_{2^N}\} \subset \left[2d_{2^{N-1}+1}, \frac{8}{3}d_{2^{N-1}+1}\right[$. It follows from (5) and (6) that $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} \notin (D_1 + D_1) \cup (D_1 + D_2)$, which implies that $R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_2}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. On the other hand $D_2 = D_{2,n} \cup (D_2 \setminus D_{2,n})$, thus for any positive integer m we have $R_{D_2}(m) = R_{D_{2,n}}(m) + 2r_{D_2+(D_2\setminus D_{2,n})}(m) + R_{D_2\setminus D_{2,n}}(m)$. It follows that if m is a positive integer with $2d_{2^{N-1}+1} \le m < d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$ then we have $r_{D_2+(D_2\setminus D_{2,n})}(m) = 0$ and $R_{D_2\setminus D_{2,n}}(m) = 0$, which implies that $R_{D_2}(m) = R_{D_{2,n}}(m)$. Hence $R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. Therefore,

$$R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$$

$$= R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}),$$

which is absurd.

Similarly, if $d_1 + c_{2^{N-1}+n+1} > d_{2^{N-1}+n+1}$, then it follows that $R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) > R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$ because $d_{2^{N-1}+1} \in D_{2,n}$ and $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$. It follows from (5), (7), (10) and (11) and $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} < d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1} = c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$ that $R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$.

Thus we have

$$R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$$
$$= R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) < R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}),$$

which is a contradiction.

Case 2. $t_k < +\infty$ and $s_l < +\infty$. We have two subcases.

Case 2a. $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} < min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$. If $t_k \le c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$, then obviously $t_k < d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$, by using (5), (6), (7) and (11) this implies that $R_D(t_k) = R_{D_{2,n}}(t_k)$. Applying the definition of t_k we obtain that

$$R_C(t_k) \ge R_{C_{2,n}}(t_k) > R_{D_{2,n}}(t_k) = R_D(t_k),$$

which contradicts the fact that $R_C(t_k) = R_D(t_k)$.

On the other hand if $t_k > c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$, then by the definition of t_k we have $R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) \le R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$, moreover since $c_{2^{N-1}+1} \in C_{2,n}$ and $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$ we obtain that $R_C(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) > R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. It follows from $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} < min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$ that $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} < s_l$ and we get from the definition of s_l that $R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) \le R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. Then we have $R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. It follows from $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} = min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} < min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\} \le d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$ and from (5), (6), (7) and (11) that $R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. Hence

$$R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$$
$$= R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}),$$

which contradicts the fact that $R_C(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$.

Case 2b. $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} > min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$.
If $s_l \le d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$, then obviously $s_l < c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$. It follows from the definition of s_l, (5), (7), (10) and (11) that $R_{C_{2,n}}(s_l) = R_C(s_l)$.

By using the definition of s_l we obtain that

$$R_C(s_l) = R_{C_{2,n}}(s_l) < R_{D_{2,n}}(s_l) \le R_D(s_l),$$

which contradicts the fact that $R_C(s_l) = R_D(s_l)$.

On the other hand if $s_l > d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$, then by the definition of s_l, $R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) \ge R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$ and since $d_{2^{N-1}+1} \in D_2$, $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$, we have $R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) > R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$. From $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} > min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$ we get $t_k > d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$ and it follows from the definition of t_k that $R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) \le R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$. We get that $R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$.

It follows from $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} < c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$ and (5), (7), (10), (11) that $R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$. Thus we have

$$R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$$
$$= R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) < R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}),$$

which contradicts the fact that $R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$. The proof of (ii) in Lemma 1. is completed. □

Let

$$H = H(d_1, d_2, d_3, d_5, \ldots, d_{2^k+1}, \ldots, , d_{2^{N-1}+1})$$

and

$$H_0 = H_0(d_1, d_2, d_3, d_5, \ldots, d_{2^k+1}, \ldots, , d_{2^{N-1}+1})$$
$$H_1 = H_1(d_1, d_2, d_3, d_5, \ldots, d_{2^k+1}, \ldots, , d_{2^{N-1}+1})$$

If $R_{H_0}(m) > 0$ or $R_{H_1}(m) > 0$ then

$$m = \delta_0 d_1 + \sum_{i=1}^{N} \delta_i d_{2^{i-1}+1},$$

where δ_0, $\delta_i \in \{0, 1, 2\}$. It follows from $d_2 \geq 4d_1$, $d_{2^k+1} \geq 4d_{2^{k-1}+1}$, $(k = 1, \ldots, N-1)$ that when

$$m' = \delta_0' d_1 + \sum_{i=1}^{N} \delta_i' d_{2^{i-1}+1},$$

where δ_0', $\delta_i' \in \{0, 1, 2\}$ and $(\delta_0, \ldots, \delta_N) \neq (\delta_0', \ldots, \delta_N')$ then $m \neq m'$. On the other hand if

$$m = \delta_0 d_1 + \sum_{i=1}^{N} \delta_i d_{2^{i-1}+1},$$

where δ_0, $\delta_i \in \{0, 1, 2\}$ then $m = k + k'$ with

$$k = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1},$$

where ε_0, $\varepsilon_i \in \{0, 1\}$ and

$$k' = \varepsilon_0' d_1 + \sum_{i=1}^{N} \varepsilon_i' d_{2^{i-1}+1},$$

where ε_0', $\varepsilon_i' \in \{0, 1\}$ if only if $\delta_0 = \varepsilon_0 + \varepsilon_0'$ and $\delta_i = \varepsilon_i + \varepsilon_i'$, $1 \leq i \leq N$.

Let $H_{0,n}$ and $H_{1,n}$ denote the $2^{N-1}+n$th elements of H_0 and H_1 respectively. It follows from $d_2 \geq 4d_1$, $d_{2^k+1} \geq 4d_{2^{k-1}+1}$, $(k = 1, \ldots, N-1)$ that when

$$H_{0,n} = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1},$$

where ε_0, $\varepsilon_i \in \{0, 1\}$, then

$$H_{1,n} = (1 - \varepsilon_0) d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1}.$$

In the next step we prove the following lemma.

Lemma 2. *Let us suppose that* $H_0^{(n)} = C_1 \cup C_{2,n}$ *and* $H_1^{(n)} = D_1 \cup D_{2,n}$ *holds for some* $1 \leq n < 2^{N-1}$ *. Let* $H_{0,n+1} = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1}$*. If* $\varepsilon_0 = 0$ *and* $H_{0,n+1} = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$*, where* $1 \leq i_1 < i_2 < \ldots < i_t < N$ *then we have* *(i)* $q_j = H_{0,n+1}$,

(ii) $p_i > q_j$.

If $t > 1$ *then*

(iii) $s_l = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$,

(iv) $t_k > s_l$,

if $t = 1$ *then*

(v) $t_k = s_l = +\infty$.

Proof. We prove (i) and (ii) simultaneuosly. It is enough to show that if $m < H_{0,n+1}$ then $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$ and $r_{D_1+D_{2,n}}(H_{0,n+1}) > r_{C_1+C_{2,n}}(H_{0,n+1})$. If $m < d_{2^{N-1}+1}$ then it follows from (6) and (10) that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m) = 0$. If $d_{2^{N-1}+1} \le m < H_{0,n+1}$ then by using (5), (7), (11) and $H_{1,n+1} = H_{0,n+1} + d_1$ it follows that $R_{H_0}(m) = r_{C_1+C_{2,n}}(m)$ and $R_{H_1}(m) = r_{D_1+D_{2,n}}(m)$. It follows from $R_{H_0}(m) = R_{H_1}(m)$ that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$.

By using (5), (7), (11) and $H_{0,n+1} < H_{1,n+1}$ we get that $R_{H_1}(H_{0,n+1}) = r_{D_1+D_{2,n}}(H_{0,n+1})$. Since $H_{0,n+1} = 0 + H_{0,n+1}$, where $0, H_{0,n+1} \in H_0$ and $H_{0,n+1} \notin C_{2,n}$ we have $R_{H_0}(H_{0,n+1}) > r_{C_1+C_{2,n}}(H_{0,n+1})$. It follows from $R_{H_0}(H_{0,n+1}) = R_{H_1}(H_{0,n+1})$ that $r_{D_1+D_{2,n}}(H_{0,n+1}) > R_{C_1+C_{2,n}}(H_{0,n+1})$. This proves (i) and (ii).

We prove (iii) and (iv) simultaneuosly. Let $M = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$. It is enough to show that if $m < M$ then $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ and $R_{C_{2,n}}(M) < R_{D_{2,n}}(M)$.

When $m < 2d_{2^{N-1}+1}$ then by using (7) and (11) $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$. Let $2d_{2^{N-1}+1} \le m < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$ and write $m = h + h^{'}$, with $h, h^{'} \in H_0$. By using (5) and (10) we get that $h, h^{'} \in H_0 \setminus C_1$. Since $h \ge d_{2^{N-1}+1}$ we have $h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1}$, thus $h, h^{'} \in C_{2,n}$, which yields $R_{H_0}(m) = R_{C_{2,n}}(m)$. On the other hand write $m = h + h^{'}$, with $h, h^{'} \in H_1$. By using (5) and (6) we get that $h, h^{'} \in H_1 \setminus D_1$. Since $h \ge d_{2^{N-1}+1}$ we have $h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1} < H_{1,n+1}$, thus $h, h^{'} \in D_{2,n}$, which yields $R_{H_1}(m) = R_{D_{2,n}}(m)$. It follows from $R_{H_0}(m) = R_{H_1}(m)$ that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$.

Assume that $d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1} \le m < 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$. We can assume that

$$m = \delta_0 d_1 + \sum_{i=1}^{u} d_{2^{x_j-1}+1} + \sum_{i=1}^{v} 2d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1},$$

where $\delta_0 \in \{0, 1, 2\}$ and $1 \le x_1 < x_2 < \ldots < x_u < i_1$ and $1 \le y_1 < y_2 < \ldots < y_v < i_1$, where $x_\alpha \ne y_\beta$ are integers, otherwise $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$. Since $H_{0,n+1} = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$ then t is odd, thus we can assume that $\delta_0 + u + t$ is even otherwise $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$.

We distinguish three cases.

Case 1. $\delta_0 = 0$. Then u is odd. If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \dots, z_w\} \subset \{x_1, \dots, x_u\} \neq \emptyset$ and $w + v + t + 1$ is even. To choose the set $\{z_1, \dots, z_w\}$ we have 2^{u-1} possibilities, thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 2^{u-1}$.

On the other hand if $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_1$ then it follows from (5) and (6) that $h, h^{'} \in H_0 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \dots, z_w\} \subset \{x_1, \dots, x_u\} \neq \emptyset$ and $w + v + t + 1$ is odd. To choose the set $\{z_1, \dots, z_w\}$ we have 2^{u-1} possibilities, thus we have $R_{C_{2,n}}(m) = R_{H_1}(m) - 2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$ it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$.

Case 2. $\delta_0 = 1$. Then u is even. If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0, 1\}$ and $\{z_1, \dots, z_w\} \subset \{x_1, \dots, x_u\}$ and $\varepsilon_0 + w + v + t + 1$ is even. When $u = 0$ then by a suitable ε_0 there is only one possibility for $h^{'}$ thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 1$. When $u > 0$ to choose the pairs $(\varepsilon_0, \{z_1, \dots, z_w\})$ we have $2 \cdot 2^{u-1} = 2^u$ possibilities, thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 2^u$.

On the other hand if $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_1$ then it follows from (5) and (6) that $h, h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0, 1\}$ and $\{z_1, \dots, z_w\} \subset \{x_1, \dots, x_u\}$ and $\varepsilon_0 + w + v + t + 1$ is odd. When $u = 0$ then by a suitable ε_0 there is only one possibility for $h^{'}$ thus we have $R_{D_{2,n}}(m) = R_{H_1}(m) - 1$. When $u > 0$ to choose the pairs $(\varepsilon_0, \{z_1, \dots, z_w\})$ we have $2 \cdot 2^{u-1} = 2^u$ possibilities, thus we have $R_{D_{2,n}}(m) = R_{H_1}(m) - 2^u$. As $R_{H_0}(m) = R_{H_1}(m)$ it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$.

Case 3. $\delta_0 = 2$. Then u is odd. If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \dots, z_w\} \subset \{x_1, \dots, x_u\} \neq \emptyset$ and $1 + w + v + t + 1$ is even. To choose the set $\{z_1, \dots, z_w\}$ we have 2^{u-1} possibilities, thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 2^{u-1}$.

On the other hand if $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_1$ then it follows from (5) and (6) that $h, h^{'} \in H_0 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \dots, z_w\} \subset \{x_1, \dots, x_u\} \neq \emptyset$ and $1 + w + v + t + 1$ is odd. To choose the set $\{z_1, \dots, z_w\}$ we have 2^{u-1} possibilities, thus we have $R_{C_{2,n}}(m) = R_{H_1}(m) - 2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$ it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$.

Let $M = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$. Now we prove $R_{D_{2,n}}(M) = R_{H_1}(M)$. Assume that $M = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1} = h + h^{'}$, where $h, h^{'} \in H_1$ with $h < h^{'}$. Then it follows from (5) and (6) that $h, h^{'} \in H_1 \setminus D_1$. It follows that

$$h^{'} = d_{2^{i_1-1}+1} + d_{2^{z_1-1}+1} + \dots + d_{2^{z_w-1}+1} + d_{2^{N-1}},$$

where $\{z_1, \dots, z_w\} \subset \{i_2, \dots, i_t\}$. Thus we have $h^{'} \le d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1} < H_{1,n+1}$, which implies that $R_{D_{2,n}}(M) = R_{H_1}(M)$. On the other hand

$$M = (d_{2^{i_1-1}+1} + d_{2^{N-1}+1}) + (d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}),$$

where $d_{2^{i_1-1}+1} + d_{2^{N-1}+1} \in C_{2,n}$ and $d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} \notin C_{2,n}$, and $d_{2^{i_1-1}+1} + d_{2^{N-1}+1} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$ because $t \ge 2$. This gives $R_{H_0}(M) > R_{C_{2,n}}(M)$. It follows from $R_{H_0}(M) = R_{H_1}(M)$ that $R_{D_{2,n}}(M) > R_{C_{2,n}}(M)$.

Assume that $t = 1$, that is $H_{0,n+1} = d_{2^{i_1-1}+1} + d_{2^{N-1}+1}$. The previous argument shows that for $m < 2d_{2^{i_1-1}+1} + 2d_{2^{N-1}+1}$ we have $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$. Moreover, if $m \ge 2d_{2^{i_1-1}} + 2d_{2^{N-1}+1} = 2H_{0,n+1}$ and $R_{C_{2,n}}(m) \neq 0$ or $R_{D_{2,n}}(m) \neq 0$ then

$$m = \delta_0 d_1 + \sum_{u=1}^{s} \delta_u d_{2^{j_u-1}+1} + 2d_{2^{N-1}+1},$$

where $\delta_0 \in \{0, 1, 2\}$, $\delta_u \in \{1, 2\}$, $1 \le j_1 < j_2 < \dots < j_s < N$ and $j_s \ge i_1$. If $m = h + h^{'}$, where $h, h^{'} \in H_0$ or $h, h^{'} \in H_1$, $h < h^{'}$ then

$$h^{'} = \varepsilon_0 d_1 + \sum_{l=1}^{r} d_{2^{h_l-1}+1} + d_{2^{N-1}+1},$$

where $1 \le h_1 < j_2 < \dots < h_r$ and $h_r = j_s \ge i_1$. Hence we have $h^{'} \ge H_{0,n+1}$. It follows that $h^{'} \notin C_{2,n}$ and $h^{'} \notin D_{2,n}$, which implies that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$. This proves that $s_l = t_k = +\infty$. □

Lemma 3. *For $1 \le n < 2^{N-1}$ let $H_0^{(n)} = C_1 \cup C_{2,n}$ and $H_1^{(n)} = D_1 \cup D_{2,n}$. Let $H_{0,n+1} = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1}$. If $\varepsilon_0 = 1$ and $H_{0,n+1} = d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$, where $1 \le i_1 < i_2 < \dots < i_t < N$ then we have*

(i) $p_i = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$,

(ii) $q_j > p_i$,

(iii) $t_k = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$,

(iv) $s_l > t_k$.

Proof. We prove (i) and (ii) simultaneuosly. It is enough to show that for $\varepsilon_0 = 1$ and $H_{0,n+1} = d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$ if $m < 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$ then $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$ and $r_{D_1+D_{2,n}}(K) < r_{C_1+C_{2,n}}(K)$, where $K = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$. If $m < d_{2^{N-1}+1}$ then it follows from (6) and (10) that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m) = 0$. Assume that $d_{2^{N-1}+1} \le m < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$. If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then it follows from (5) and (11) that $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. Since $h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} < H_{0,n+1}$, thus $h^{'} \in C_{2,n}$, which yields $R_{H_0}(m) = r_{C_1+C_{2,n}}(m)$.

On the other hand write $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_1$ then it follows from (5) and (7) that $h \in D_1$ and $h^{'} \in H_1 \setminus D_1$. Since $h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{1,n+1}$, thus $h^{'} \in D_{2,n}$. As $R_{H_0}(m) = R_{H_0}(m)$, which yields $r_{D_1+D_{2,n}}(m) = r_{C_1+C_{2,n}}(m)$.

Suppose that $d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} \le m < 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$. Then we may assume that m can be written in the form

$$m = \delta_0 d_1 + \sum_{i=1}^{u} d_{2^{x_j-1}+1} + \sum_{i=1}^{v} 2d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\delta_0 \in \{0, 1, 2\}$ and $1 \le x_1 < x_2 < \ldots < x_u < i_1$ and $1 \le y_1 < y_2 < \ldots < y_v < i_1$, where $x_\alpha \ne y_\beta$ are integers, otherwise $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m) = 0$.

Since $H_{0,n+1} = d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$ then t is even, thus $\delta_0 + u + 2v + t + 1$ is even, which implies that $\delta_0 + u$ is odd.

We distinguish three cases.

Case 1. $\delta_0 = 0$. Then u is odd. If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then it follows from (5) and (10) that $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \ne \emptyset$ and $w + v + t + 1$ is even. To choose the set $\{z_1, \ldots, z_w\}$ we have 2^{u-1} possibilities, thus we have $r_{C_1+C_{2,n}}(m) = R_{H_0}(m) - 2^{u-1}$.

On the other hand if $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_1$ then it follows from (5) and (7) that $h \in D_1$ and $h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \ne \emptyset$ and $w + v + t + 1$ is odd. To choose the set $\{z_1, \ldots, z_w\}$ we have 2^{u-1} possibilities, thus we have $r_{D_1+D_{2,n}}(m) = R_{H_1}(m) - 2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$ it follows that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$.

Case 2. $\delta_0 = 1$. Then $1 + u + 2v + t + 1$ is even, which implies that u is even.

If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0,1\}$ and $\{z_1,\dots,z_w\} \subset \{x_1,\dots,x_u\}$ and $\varepsilon_0+w+v+t+1$ is even. When $u=0$ then $\{z_1,\dots,z_w\}=\emptyset$ and by a suitable ε_0 there is only one possibility for $h^{'}$ that is $r_{C_1+C_{2,n}}(m) = R_{H_0}(m)-1$. When $u>0$ even to choose the pairs $(\varepsilon_0,\{z_1,\dots,z_w\})$ we have $2\cdot 2^{u-1}=2^u$ possibilities, thus we have $r_{C_{2,n}}(m) = R_{H_0}(m)-2^u$.

On the other hand if $m=h+h^{'}$, with $h<h^{'}$ and $h,h^{'} \in H_0$ then $h \in D_1$ and $h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0,1\}$ and $\{z_1,\dots,z_w\} \subset \{x_1,\dots,x_u\}$ and $\varepsilon_0+w+v+t+1$ is odd. When $u=0$ then $\{z_1,\dots,z_w\}=\emptyset$ and by a suitable ε_0 there is only one possibility for $h^{'}$ that is $r_{D_1+D_{2,n}}(m) = R_{H_1}(m)-1$. When $u>0$ even to choose the set of pairs $\{z_1,\dots,z_w\}$ we have $2\cdot 2^{u-1}=2^u$ possibilities, thus we have $r_{D_1+D_{2,n}}(m) = R_{H_1}(m)-2^u$. As $R_{H_0}(m) = R_{H_1}(m)$ it follows that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$.

Case 3. $\delta_0 = 2$. Then $2+u+2v+t+1$ is even, thus u is odd. If $m=h+h^{'}$, with $h<h^{'}$ and $h,h^{'} \in H_0$ then $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1,\dots,z_w\} \subset \{x_1,\dots,x_u\} \neq \emptyset$ and $1+w+v+t+1$ is even. To choose the set $\{z_1,\dots,z_w\}$ we have 2^{u-1} possibilities, thus we have $r_{C_1+C_{2,n}}(m) = R_{H_0}(m)-2^{u-1}$.

On the other hand if $m=h+h^{'}$, with $h<h^{'}$ and $h,h^{'} \in H_1$ then $h \in D_1$, $h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1,\dots,z_w\} \subset \{x_1,\dots,x_u\} \neq \emptyset$ and $1+w+v+t+1$ is odd. To choose the set $\{z_1,\dots,z_w\}$ we have 2^{u-1} possibilities, thus we have $R_{D_1+D_{2,n}}(m) = R_{H_1}(m)-2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$ it follows that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$.

If $K = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = h+h^{'}$, with $h<h^{'}$ and $h,h^{'} \in H_0$ then it follows from (5) and (10) that $h \in C_1$, $h^{'} \in H_0 \setminus C_1$ and $h^{'}$ can be written in the form

$$h^{'} = d_{2^{i_1-1}+1} + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1,\dots,z_w\} \subset \{i_2,\dots,i_t\} \neq \emptyset$. Thus we have

$$h^{'} \le d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$$

$$< d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1},$$

thus we have $h^{'} \in C_{2,n}$ and $R_{H_0}(K) = r_{C_1+C_{2,n}}(K)$.

In the last step we prove $r_{C_1+C_{2,n}}(K) > r_{D_1+D_{2,n}}(K)$. It is clear that $K = d_{2^{i_1-1}+1} + (d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}) = d_{2^{i_1-1}+1} + H_{1,n+1}$, where $d_{2^{i_1-1}+1}, H_{1,n+1} \in$

H_1. Since $H_{1,n+1} \notin D_{2,n}$ then we have $R_{H_1}(K) > r_{D_1+D_{2,n}}(K)$ It follows from $R_{H_0}(K) = R_{H_1}(K)$ that $r_{D_1+D_{2,n}}(K) < r_{C_1+C_{2,n}}(K)$.

We will prove (iii) and (iv) simultaneuosly. Let $L = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$. We have to prove that if $m < L$ then $R_{D_{2,n}}(m) = R_{C_{2,n}}(m)$ and $R_{D_{2,n}}(L) < R_{C_{2,n}}(L)$. If $m < 2d_{2^{N-1}+1}$ the by using (7) and (11) we get that $R_{D_{2,n}}(m) = R_{C_{2,n}}(m) = 0$. Assume that $2d_{2^{N-1}+1} \le m < L = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$. If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. This implies that $h \ge d_{2^{N-1}+1}$ and $h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} < H_{0,n+1}$. It follows that $h, h^{'} \in C_{2,n}$, which yields $R_{H_0}(m) = R_{C_{2,n}}(m)$. If $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_1$ then it follows from (5) and (6) that $h, h^{'} \in H_1 \setminus D_1$. Since $h \ge d_{2^{N-1}+1}$ and $h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{1,n+1}$. It follows that $h, h^{'} \in D_{2,n}$, which yields $R_{H_1}(m) = R_{D_{2,n}}(m)$. As $R_{H_0}(m) = R_{H_1}(m)$ it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$.

If $L = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1} = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_0$ then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It follows that $h > d_{2^{N-1}+1}$ and

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{z_1-1}+1} + \ldots + d_{2^{z_w-1}+1} + d_{2^{N-1}} < H_{0,n+1},$$

thus we have $h, h^{'} \in C_{2,n}$, which implies that $R_{H_0}(L) = R_{C_{2,n}}(L)$. On the other hand

$$L = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$$

$$= d_{2^{N-1}+1} + (d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1})$$

$$= d_{2^{N-1}+1} + H_{1,n+1}.$$

Note that $H_{1,n+1}, d_{2^{N-1}+1} \in H_1$ and $H_{1,n+1} \notin D_{2,n}$ which gives $R_{H_1}(L) > R_{C_{2,n}}(L)$. It follows from $R_{H_0}(L) = R_{H_1}(L)$ that $R_{D_{2,n}}(L) > R_{C_{2,n}}(L)$. □

Now we are ready to prove that $H_0^{(n)} = C_1 \cup C_{2,n}$ and $H_1^{(n)} = D_1 \cup D_{2,n}$ holds for every $1 \le n \le 2^{N-1}$. We prove by induction on n that $C_1 \cup C_{2,n} = H_0^{(n)}$ and $D_1 \cup D_{2,n} = H_1^{(n)}$. We have already proved $C_1 \cup C_{2,1} = H_0^{(1)}$ and $D_1 \cup D_{2,1} = H_1^{(1)}$.

Assume that $C_1 \cup C_{2,n} = H_0^{(n)}$ and $D_1 \cup D_{2,n} = H_1^{(n)}$ holds for some $1 \le n < 2^{N-1}$. We will prove that $C_1 \cup C_{2,n+1} = H_0^{(n+1)}$ and $D_1 \cup D_{2,n+1} = H_1^{(n+1)}$ holds, i.e., $c_{2^{N-1}+n+1} = H_{0,n+1}$ and $d_{2^{N-1}+n+1} = H_{1,n+1}$. Let $H_{0,n+1} = \varepsilon_0 d_1 + \sum_{j=1}^{t} d_{2^{i_j-1}+1} + d_{2^{N-1}+1}$, where $\varepsilon_0 \in \{0, 1\}$, $(1 \le i_1 < \ldots < i_t < N)$.

Case 1. $\varepsilon_0 = 0$, $t = 1$. We know from Lemma 1 that $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} = min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$ and from Lemma 2 that $t_k = s_l = +\infty$. These facts imply that $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} = d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$. On the other hand we know that $c_{2^{N-1}+1} = d_{2^{N-1}+1} + d_1$, thus we have $c_{2^{N-1}+n+1} + d_1 = d_{2^{N-1}+n+1}$, and then $d_{2^{N-1}+n+1} > c_{2^{N-1}+n+1}$. It follows from Lemma 1 that $min\{p_i, c_{2^{N-1}+n+1}\} = min\{q_j, d_{2^{N-1}+n+1}\}$ and from Lemma 2 that $p_i > q_j = H_{0,n+1}$. Then we have $c_{2^{N-1}+n+1} = q_j = H_{0,n+1}$ and $d_{2^{N-1}+n+1} = c_{2^{N-1}+n+1} + d_1 = H_{0,n+1} + d_1 = H_{1,n+1}$.

Case 2. $\varepsilon_0 = 0$, $t > 1$. Applying Lemma 2 we get that $p_i > q_j$, thus from Lemma 1 we have $min\{q_j, d_1 + d_{2^{N-1}+n+1}\} = min\{p_i, c_{2^{N-1}+n+1}\} = c_{2^{N-1}+n+1}$. On the other hand, it follows from Lemma 2 that $s_l < t_k$ thus by Lemma 1 we have $min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\} = min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} = c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$.

Assume that $c_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+n+1}$. Then we have $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+1} + d_1 + d_{2^{N-1}+n+1} = 2d_1 + d_{2^{N-1}+1} + d_{2^{N-1}+n+1} > d_{2^{N-1}+1} + d_{2^{N-1}+n+1} \geq min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$ which is a contradiction. It follows from Lemma 2 that $c_{2^{N-1}+n+1} = q_j = H_{0,n+1}$ and $c_{2^{N-1}+1}+c_{2^{N-1}+n+1} = d_1+d_{2^{N-1}+1}+d_{2^{i_1-1}+1}+d_{2^{i_2-1}+1}+\ldots+d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{1,n+1} + d_{2^{N-1}+1} = min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\} = min\{2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$. Since $d_1 + d_{2^{N-1}+1} + d_{2^{i_1-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} < 2d_{2^{i_1-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$, it follows that $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} = H_{1,n+1} + d_{2^{N-1}+1}$, thus we have $d_{2^{N-1}+n+1} = H_{1,n+1}$.

Case 3. $\varepsilon_0 = 1$. Applying Lemma 3 we get that $q_j > p_i$, thus from Lemma 1 we have $min\{p_i, c_{2^{N-1}+n+1}\} = min\{q_j, d_1 + d_{2^{N-1}+n+1}\} = d_1 + d_{2^{N-1}+n+1}$. On the other hand, it follows from Lemma 3 that $s_l > t_k$ thus by Lemma 1 we have $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} = min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\} = d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$.

Assume that $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} = d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$. Then we have $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1}$, thus we have $d_{2^{N-1}+n+1} = d_1 + c_{2^{N-1}+n+1}$. It follows that $d_1+d_{2^{N-1}+1} = 2d_1+c_{2^{N-1}+n+1} = min\{p_i, c_{2^{N-1}+n+1}\}$, which is a contradiction because $d_1 > 0$. Then we have $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} = t_k = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$. It follows that $d_{2^{N-1}+n+1} = d_{2^{i_1-1}+1}+d_{2^{i_2-1}+1}+\ldots+d_{2^{i_t-1}+1}+d_{2^{N-1}+1} = H_{1,n+1}$. Applying Lemma 1 and Lemma 3 we get that $d_1 + d_{2^{N-1}+n+1} = H_{0,n+1} = d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1}+\ldots+d_{2^{i_t-1}+1}+d_{2^{N-1}+1} = min\{p_i, c_{2^{N-1}+n+1}\} = min\{2d_{2^{i_1-1}+1}+d_{2^{i_2-1}+1}+\ldots+d_{2^{i_t-1}+1} + d_{2^{N-1}+1}, c_{2^{N-1}+n+1}\} = c_{2^{N-1}+n+1}$ thus we have $c_{2^{N-1}+n+1} = H_{0,n+1}$. The proof of Theorem 4 is completed.

5 Proof of Theorem 6.

First we prove that for $H = H(1, 2, 4, 8, \ldots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \ldots)$, $C = H_0$ and $D = H_1$ we have $C \cup D = \mathbb{N}$, $C \cap D = 2^{2l} - 1 + (2^{2l+1} - 1)\mathbb{N}$ and $R_C(m) = R_D(m)$. It is easy to see that for $H' = H(h_1, h_2, \ldots, h_{2l+1}) = H(1, 2, 4, 8, \ldots, 2^{2l-1}, 2^{2l} - 1)$, $C' = H_0'$ and $D' = H_1'$ we have $C' \cup D' = [0, 2^{2l+1} - 2]$ and $C' \cap D' = \{2^{2l}-1\}$ because $2^{2l}-1 = h_{2l+1} = h_1+h_2+\cdots+h_{2l} = 1+2+4+\cdots+2^{2l-1}$. On the other hand for $H'' = H(2^{2l+1}-1, 2(2^{2l+1}-1), 4(2^{2l+1}-1), 8(2^{2l+1}-1), \ldots)$, $C'' = H_0''$ and $D'' = H_1''$ we have $C'' \cup D'' = (2^{2l+1} - 1)\mathbb{N}$ and $C'' \cap D'' = \emptyset$, which implies that $C \cup D = \mathbb{N}$ and $C \cap D = 2^{2l} - 1 + (2^{2l+1} - 1)\mathbb{N}$, moreover by Theorem 3 for every positive integer m we have $R_C(m) = R_D(m)$.

On the other hand let us suppose that for some sets C and D we have $C \cup D = \mathbb{N}$ and $C \cap D = r + m\mathbb{N}$. By Conjecture 2 we may assume that for some Hilbert cube $H(h_1, h_2, \ldots)$ we have $C = H_0$ and $D = H_1$. We have to prove the existence of integer l such that $h_i = 2^{i-1}$ for $1 \leq i \leq 2l$, $h_{2l+1} = 2^{2l} - 1$ and $h_{2l+2+j} = 2^j(2^{2l+1} - 1)$ for $j = 0, 1, \ldots$. We may suppose that $h_1 = 1$ and $h_2 = 2$. Consider the Hilbert cube $H(1, 2, 4, \ldots, 2^u, h_{u+2}, \ldots)$, where $h_{u+2} \neq 2^{u+1}$. Denote by $v = h_{u+2}$. We will prove that $v = 2^{u+1} - 1$. Assume that $v > 2^{u+1}$. Then it is clear that $2^{u+1} \notin H$, because $1 + 2 + \ldots + 2^u = 2^{u+1} - 1 < 2^{u+1}$. Thus we have $v < 2^{u+1}$, i.e., $v \leq 2^{u+1} - 1$. Assume that $v \leq 2^{u+1} - 2$. Considering v as a one term sum it follows that $v \in D$. On the other hand if $v = \sum_{i=0}^{u} \lambda_i 2^i$, $\lambda_i \in \{0, 1\}$ then $\sum_{i=0}^{u} \lambda_i$ must be even otherwise v would have two different representations from D. It follows that $v \in C$ and that $v + 1 = h_1 + h_{u+2} \in C$. On the other hand if we have a representation $v + 1 = \sum_{i=0}^{u} \delta_i 2^i$, $\delta_i \in \{0, 1\}$ then $\sum_{i=0}^{u} \delta_i$

must be odd otherwise v would have two different representations from C. This implies that $v+1 \in D$ thus we have $v,\ v+1 \in C \cap D$. It follows that $C \cap D = \{v, v+1, \dots\}$ is an arithmetic progression with difference 1. This implies that the generating function of the sets C and D are of the form

$$C(z) = p(z) + \frac{z^v}{1-z},$$

where $p(z)$ is a polynomial and

$$D(z) = q(z) + \frac{z^v}{1-z},$$

where $q(z)$ is a polynomial and

$$p(z) + q(z) = 1 + z + z^2 + \ldots + z^{v-1} = \frac{1-z^v}{1-z}.$$

Since $R_C(n) = R_D(n)$ then we have $C^2(z) - D^2(z) = C(z^2) - D(z^2)$. It follows that

$$\Big(p(z) + \frac{z^v}{1-z}\Big)^2 - \Big(q(z) + \frac{z^v}{1-z}\Big)^2 = p(z^2) + \frac{z^{2v}}{1-z^2} - q(z^2) - \frac{z^{2v}}{1-z^2},$$

which implies that

$$p^2(z) - q^2(z) + \frac{2z^v}{1-z}(p(z) - q(z)) = p(z^2) - q(z^2).$$

Thus we have

$$(p(z) - q(z)) \cdot \frac{1+z^v}{1-z} = p(z^2) - q(z^2).$$

We get that

$$(p(z) - q(z)) \cdot (1 + z^v) = (p(z^2) - q(z^2)) \cdot (1-z).$$

The leading coefficient in one side is -1 and the other side is 1 which is a contradiction. Thus we get that $v = 2^{u+1} - 1$. It follows that the Hilbert cube is the form $H(1, 2, 4, 8, \dots, 2^u, 2^{u+1}-1, \dots)$. As $h_{u+2} = 2^{u+1}-1 = 1+2+\ldots+2^u = h_1+h_2+\cdots+h_{u+1}$ and $2^{u+1}-1$ as a one term sum contained in D, thus $u+1$ must be even i.e., $u+1 = 2l$. It follows that there exists an integer l such that $h_i = 2^{i-1}$ for $1 \le i \le 2l$ and $h_{2l+1} = 2^{2l}-1$. It follows that $2^{2l} - 1 \in C \cap D$ and $r = 2^{2l} - 1$.

We will prove by induction on j that $h_{2l+2+j} = 2^j(2^{2l+1} - 1)$ for $j = 0, 1, \dots$. For $j = 0$ take the Hilbert cube of the form $H(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l} - 1, h_{2l+2}, \dots)$. Denote by $w = h_{2l+2}$. We prove that $w = 2^{2l+1} - 1$. Assume that $w > 2^{2l+1} - 1$. Since $1 + 2 + \ldots + 2^{2l-1} + 2^{2l} - 1 < 2^{2l+1} - 1$ it follows that $2^{2l+1} - 1 \notin H = C \cup D$, which is impossible, therefore $w \le 2^{2l+1} - 1$. Assume that $w \le 2^{2l+1} - 3$. We will show that $w \in C \cap D$. Obviously w is a one-term sum contained in D. Since w has a representation from $H(h_1, \dots, h_{2l+1})$, w must be an element of C otherwise w would have two different representations from D which is absurd. In the next step we will prove that $w+1 \in C \cap D$. Obviously $w + 1 = h_1 + h_{2l+2}$ as a two terms sum contained in C. Since $w + 1$ has a representation from the Hilbert cube $H(h_1, \dots, h_{2l+1})$ and $w + 1 \le 2^{2l+1} - 2$ we have $w + 1 \in D$. It follows that $w,\ w + 1 \in C \cap D$, which is impossible. It follows that the only possible values of w are $w = 2^{2l+1} - 2$, or $w = 2^{2l+1} - 1$. Assume that $w = 2^{2l+1} - 2$. Then it is clear that $w \in D$. On the other hand $2^{2l} - 2 = 1 + 2 + \ldots + 2^{2l-1} + 2^{2l} - 1 = h_1 + h_2 + \cdots + h_{2l+1}$, where in the right hand side there is $2l + 1$ terms, which is absurd. Thus we have $w = 2^{2l+1} - 1$. In this case $2^{2l} - 1$, $(2^{2l} - 1) + (2^{2l+1} - 1) \in C \cap D$, $(C \cap D) \cap \{1, 2, \ldots, 2^{2l+1} - 1\} = \{2^{2l} - 1\}$. It follows that $m \mid 2^{2l+1} - 1$. If $m \le \frac{2^{2l+1}-1}{2}$ then $(C \cap D) \cap \{1, 2, \ldots, 2^{2l+1} - 1\} \ne \{2^{2l} - 1\}$, a contradiction. Then we have $r = 2^{2l} - 1$ and $m = 2^{2l+1} - 1$.

In the induction step we assume that for some k we know $h_{2l+2+j} = 2^j(2^{2l+1} - 1)$ holds for $j = 0, 1, \ldots, k$ and we prove $h_{2l+2+k+1} = 2^{k+1}(2^{2l+1} - 1)$. Let $H^{(k)} = H(1, 2, 4, 8, \ldots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \ldots, 2^k(2^{2l+1} - 1))$, $C^{(k)} = H_0^{(k)}$ and $D^{(k)} = H_0^{(k)}$. Then $C^{(k)} \cap D^{(k)} = \{2^{2l} - 1 + i(2^{2l+1} - 1) : i = 0, 1, \ldots, 2^k - 1$. If $C = H_0(1, 2, 4, 8, \ldots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \ldots, 2^k(2^{2l+1} - 1), h_{2l+2+k+1}, \ldots)$, then $C \cap D = \{e_1, e_2, \ldots\}$, where $e_i = 2^{2l} - 1 + (i - 1)(2^{2l+1} - 1)$ for $i = 1, 2, \ldots$ and $e_{2^{k+1}+1} = 2^{2l} - 1 + 2^{k+1}(2^{2l+1} - 1)$. On the other hand $e_{2^{k+1}+1} = 2^{2l} - 1 + h_{2l+1+k+1}$, which yields $h_{2l+1+k+1} = 2^{k+1}(2^{2l+1} - 1)$, which completes the proof.

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