# Stability results for vertex Turán problems in Kneser graphs

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March 8, 2019

#### Abstract

The vertex set of the Kneser graph K(n,k) is  $V=\binom{[n]}{k}$  and two vertices are adjacent if the corresponding sets are disjoint. For any graph F, the largest size of a vertex set  $U\subseteq V$  such that K(n,k)[U] is F-free, was recently determined by Alishahi and Taherkhani, whenever n is large enough compared to k and F. In this paper, we determine the second largest size of a vertex set  $W\subseteq V$  such that K(n,k)[W] is F-free, in the case when F is an even cycle or a complete multi-partite graph. In the latter case, we actually give a more general theorem depending on the chromatic number of F.

Mathematics Subject Classification: 05C35, 05D05

Keywords: vertex Turán problems, set systems, intersection theorems

### 1 Introduction

Turán-type problems are fundamental in extremal (hyper)graph theory. For a pair H and F of graphs, they ask for the maximum number of edges that a subgraph G of the host graph H can

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have without containing the forbidden graph F. A variant of this problem is the so-called vertex Turán problem where given a host graph H and a forbidden graph F, one is interested in the maximum size of a vertex set  $U \subset V(H)$  such that the induced subgraph H[U] is F-free.

This problem has been studied in the context of several host graphs. In this paper we follow the recent work of Alishahi and Taherkhani [1], who determined the exact answer to the vertex Turán problem when H is the Kneser graph K(n,k), which is defined on the vertex set  $\binom{[n]}{k} = \{K \subseteq [n] = \{1,2,\ldots,n\} : |K| = k\}$  where two vertices K, K' are adjacent if and only if  $K \cap K' = \emptyset$ .

**Theorem 1.1** (Alishahi, Taherkhani [1]). For any graph F, let  $\chi$  denote its chromatic number and let  $\eta = \eta(F)$  denote the minimum possible size of a color class of G over all possible proper  $\chi$ -colorings of F. Then for any k there exists an integer  $n_0 = n_0(k, F)$  such that if  $n \geq n_0$  and for a family  $\mathcal{G} \subseteq {n \choose k}$  the induced subgraph  $K(n, k)[\mathcal{G}]$  is F-free, then  $|\mathcal{G}| \leq {n \choose k} - {n-\chi+1 \choose k} + \eta - 1$ . Moreover, if equality holds, then there exists a  $(\chi - 1)$ -set L such that  $|\{G \in \mathcal{G} : G \cap L = \emptyset\}| = \eta - 1$ .

Observe that the vertex Turán problem in the Kneser graph K(n,k) generalizes several intersection problems in  $\binom{[n]}{k}$ :

- If  $F = K_2$ , the graph consisting a single edge, then the vertex Turán problem asks for the maximum size of an independent set in K(n,k) or equivalently the size of a largest intersecting family  $\mathcal{F} \subseteq \binom{[n]}{k}$  (i.e.  $F \cap F' \neq \emptyset$  for all  $F, F' \in \mathcal{F}$ ). The celebrated theorem of Erdős, Ko, and Rado states that this is  $\binom{n-1}{k-1}$  if  $2k \leq n$  holds. Furthermore, for intersecting families  $\mathcal{F} \subseteq \binom{[n]}{k}$  of size  $\binom{n-1}{k-1}$  we have  $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$  provided  $n \geq 2k+1$ .
- If  $F = K_s$  for some  $s \ge 3$ , then the vertex Turán problem is equivalent to Erdős's famous matching conjecture:  $K(n,k)[\mathcal{F}]$  is  $K_s$ -free if and only if  $\mathcal{F}$  does not contain a matching of size s (s pairwise disjoint sets). Erdős conjectured that the maximum size of such a family is  $\max\{\binom{sk-1}{k}, \binom{n}{k} \binom{n-s+1}{k}\}$ .
- Gerbner, Lemons, Palmer, Patkós, and Szécsi [8] considered *l-almost intersecting* families  $\mathcal{F} \subseteq \binom{[n]}{k}$  such that for any  $F \in \mathcal{F}$  there are at most l sets in  $\mathcal{F}$  that are disjoint from F. This is equivalent to  $K(n,k)[\mathcal{F}]$  being  $K_{1,l}$ -free.
- Katona and Nagy [10] considered (s,t)-union intersecting families  $\mathcal{F} \subseteq \binom{[n]}{k}$  such that for any  $F_1, F_2, \ldots, F_s, F'_1, F'_2, \ldots, F'_t \in \mathcal{F}$  we have  $(\bigcup_{i=1}^s F_i) \cap (\bigcup_{j=1}^t F'_j) \neq \emptyset$ . This is equivalent to  $K(n,k)[\mathcal{F}]$  being  $K_{s,t}$ -free.

Theorem 1.1 leads into several directions. One can try to determine the smallest value of the threshold  $n_0(k, G)$ . Alishahi and Taherkhani [1] improved the upper bound on  $n_0$  for l-almost intersecting and (s, t)-union intersecting families. Erdős's matching conjecture is known to hold

if  $n \ge (2s+1)k - s$ . This is due to Frankl [6] and he also showed [5] that the conjecture is true if k = 3.

Another direction is to determine the "second largest" family with  $K(n,k)[\mathcal{F}]$  being G-free. In the case of  $F = K_2$  this means that we are looking for the largest intersecting family  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $\bigcap_{F \in \mathcal{F}} F = \emptyset$ . This is the following famous result of Hilton and Milner.

**Theorem 1.2** (Hilton, Milner [9]). If  $\mathcal{F} \subseteq \binom{[n]}{k}$  is an intersecting family with  $n \geq 2k+1$  and  $\bigcap_{F \in \mathcal{F}} F = \emptyset$ , then  $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ .

In the case of  $F = K_{s,t}$  extremal families are not intersecting, so to describe the condition of being "second largest" precisely, we introduce the following parameter.

**Definition 1.3.** For a family  $\mathcal{F}$  and integer  $t \geq 2$  let  $\ell_t(\mathcal{F})$  denote the minimum number m such that one can remove m sets from  $\mathcal{F}$  with the resulting family not containing t pairwise disjoint sets. We will write  $\ell(\mathcal{F})$  instead of  $\ell_2(\mathcal{F})$ . Note that this is the minimum number of sets one needs to remove from  $\mathcal{F}$  in order to obtain an intersecting family.

Observe that if  $s \leq t$ , then for any family  $\mathcal{F}$  with  $\ell(\mathcal{F}) \leq s - 1$  the induced subgraph  $K(n,k)[\mathcal{F}]$  is  $K_{s,t}$ -free. In [1], the following asymptotic stability result was proved.

**Theorem 1.4** (Alishahi, Taherkhani [1]). For any integers  $s \le t$  and k, and positive real number  $\beta$ , there exists an  $n_0 = n_0(k, s, t, \beta)$  such that the following holds for  $n \ge n_0$ . If for  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $\ell(\mathcal{F}) \ge s$ , the induced subgraph  $K(n, k)[\mathcal{F}]$  is  $K_{s,t}$ -free, then  $|\mathcal{F}| \le (s + \beta)(\binom{n-1}{k-1} - \binom{n-k-1}{k-1})$  holds.

Note that the above bound is asymptotically optimal as shown by any family  $\mathcal{F}_{s,t} = \{F \in \binom{[n]}{k} : 1 \in F, F \cap S \neq \emptyset\} \cup \{H_1, H_2, \dots, H_s\} \cup \{F'_1, F'_2, \dots, F'_{t-1}\}$ , where  $S = [2, sk + 1], H_i = [(i-1)k+2, ik+1]$  for all  $i = 1, 2, \dots, s$  and  $F'_1, F'_2, \dots, F'_{t-1}$  are distinct sets containing 1 and disjoint with S.

We improve Theorem 1.4 to obtain the following precise stability result for families  $\mathcal{F}$  for which  $K(n,k)[\mathcal{F}]$  is  $K_{s,t}$ -free.

**Theorem 1.5.** For any  $2 \le s \le t$  and k there exists  $n_0 = n_0(s,t,k)$  such that the following holds for  $n \ge n_0$ . If  $\mathcal{F} \subseteq \binom{[n]}{k}$  is a family with  $\ell(\mathcal{F}) \ge s$  and  $K(n,k)[\mathcal{F}]$  is  $K_{s,t}$ -free, then we have  $|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-sk-1}{k-1} + s + t - 1$ . Moreover, equality holds if and only if  $\mathcal{F}$  is isomorphic to some  $\mathcal{F}_{s,t}$ .

Using Theorem 1.5, we obtain a general stability result for the case when F is a complete multi-partite graph. We consider the family  $\mathcal{F}_{s_1,s_2,\dots,s_{r+1}}$  that consists of  $s_{r+1}$  pairwise disjoint k-subsets  $F_1, F_2, \dots, F_{s_{r+1}}$  of [n] that do not meet [r] and those k-subsets of [n] that either (i) intersect [r-1] or (ii) contain r and meet  $\bigcup_{j=1}^{s_r+1} F_j$  and (iii)  $s_r-1$  other k-sets containing r. Clearly, if  $s_1 \geq s_2 \geq \dots \geq s_r \geq s_{r+1}$  holds, then  $K(n,k)[\mathcal{F}_{s_1,s_2,\dots,s_{r+1}}]$  is  $K_{s_1,s_2,\dots,s_{r+1}}$ -free and its size is  $\binom{n}{k} - \binom{n-r+1}{k} + \binom{n-r}{k-1} - \binom{n-s_{r+1}k-r}{k-1} + s_r + s_{r+1} - 1$ .

**Theorem 1.6.** For any  $k \geq 2$  and integers  $s_1 \geq s_2 \geq \cdots \geq s_r \geq s_{r+1} \geq 1$  there exists  $n_0 = n_0(k, s_1, \ldots, s_{r+1})$  such that if  $n \geq n_0$  and  $\mathcal{F} \subseteq \binom{[n]}{k}$  is a family with  $\ell_{r+1}(\mathcal{F}) \geq s$  and  $K(n,k)[\mathcal{F}]$  is  $K_{s_1,s_2,\ldots,s_{r+1}}$ -free, then we have  $|\mathcal{F}| \leq \binom{n}{k} - \binom{n-r+1}{k} + \binom{n-r}{k-1} - \binom{n-s_{r+1}k-r}{k-1} + s_r + s_{r+1} - 1$ . Moreover, equality holds if and only if  $\mathcal{F}$  is isomorphic to some  $\mathcal{F}_{s_1,s_2,\ldots,s_{r+1}}$ .

Note that Frankl and Kupavskii [7] proved the special case  $s_1 = s_2 = \cdots = s_{r+1} = 1$  with the asymptotically best possible threshold  $n_0 = (2k + o_r(1))(r+1)k$ .

Actually, Theorem 1.6 is a special case of a more general result that shows that it is enough to solve the stability problem for bipartite graphs. For any graph F with  $\chi(F) \geq 3$  let us define  $\mathcal{B}_F$  to be the class of those bipartite graphs B such that there exists a subset U of vertices of F with F[U] = B and  $\chi(F[V(F) \setminus U]) = \chi(F) - 2$ . Note that by definition, for any  $B \in \mathcal{B}_F$  we have  $\eta(B) \geq \eta(F)$ . We define  $\mathcal{B}_{F,\eta}$  to be the subset of those bipartite graphs  $B \in \mathcal{B}_F$  for which  $\eta(B) = \eta(F)$  holds. To state our result let us introduce some notation. For any graph F let  $ex_v^{(2)}(n,k,F)$  denote the maximum size of a family  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $\ell_{\chi(F)}(\mathcal{F}) \geq \eta(F)$  and  $K(n,k)[\mathcal{F}]$  is F-free. Observe that Theorem 1.5 is about  $ex_v^{(2)}(n,k,K_{s,t})$  and Theorem 1.6 determines  $ex_v^{(2)}(n,k,K_{s_1,s_2,\ldots,s_{r+1}})$ . We define  $ex_v^{(2)}(n,k,\mathcal{B}_{F,\eta})$  to be the maximum size of a family  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $\ell_2(\mathcal{F}) \geq \eta(F)$  such that  $K(n,k)[\mathcal{F}]$  is B-free for any  $B \in \mathcal{B}_{F,\eta}$ . Similarly, let  $\widehat{ex}_v^{(2)}(n,k,\mathcal{B}_{F,\eta})$  be the maximum size of a family  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $\ell_2(\mathcal{F}) = \eta(F)$  such that  $K(n,k)[\mathcal{F}]$  is B-free for any  $B \in \mathcal{B}_{F,\eta}$ . Obviously we have  $\widehat{ex}_v^{(2)}(n,k,\mathcal{B}_{F,\eta}) \leq ex_v^{(2)}(n,k,\mathcal{B}_{F,\eta})$  and we do not know any graph F for which the two quantities differ.

**Theorem 1.7.** For any graph with  $\chi(F) \geq 3$  there exists an  $n_0 = n_0(F)$  such that if n is larger than  $n_0$ , then we have

$$\widehat{ex}_{v}^{(2)}(n - \chi(F), k, \mathcal{B}_{F,\eta}) \leq ex_{v}^{(2)}(n, k, F) - \left(\binom{n}{k} - \binom{n - \chi(F) + 2}{k}\right) \leq ex_{v}^{(2)}(n - \chi(F), k, \mathcal{B}_{F,\eta}).$$

Let us remark first that in the case of  $F = K_{s_1, s_2, \dots, s_{r+1}}$  we have  $\mathcal{B}_F = \{K_{s_i, s_j} : 1 \leq i < j \leq r+1\}$  and  $\mathcal{B}_{F,\eta} = \{K_{s_i, s_{r+1}} : 1 \leq i \leq r\}$  and obviously for both families the minimum is taken for  $K_{s_r, s_{r+1}}$ , so Theorems 1.7 and 1.5 yield the bound of Theorem 1.6.

In view of Theorem 1.7, we turn our attention to bipartite graphs, namely to the case of even cycles:  $F = C_{2s}$ . According to Theorem 1.1, the largest families  $\mathcal{F}$  such that  $K(n,k)[\mathcal{F}]$  is  $C_{2s}$ -free have  $\ell(\mathcal{F}) = s - 1$ , so once again we will be interested in families for which  $\ell(\mathcal{F}) \geq s$ . The case  $C_4 = K_{2,2}$  is solved by Theorem 1.5 (at least for large enough n). Here we define a construction that happens to be asymptotically extremal for any  $s \geq 3$ .

Construction 1.8. Let us define  $\mathcal{G}_6 \subseteq \binom{[n]}{k}$  as

$$\mathcal{G}_6 = \left\{ G \in \binom{[n]}{k} : 1 \in G, G \cap [2, 2k+1] \neq \emptyset \right\} \cup \{ [2, k+1], [k+2, 2k+1], [2k+2, 3k+1] \}.$$

So 
$$|\mathcal{G}_6| = \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + 3$$

So  $|\mathcal{G}_6| = \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + 3$ . For  $s \geq 4$  we define the family  $\mathcal{G}_{2s} \subseteq \binom{[n]}{k}$  in the following way: let  $K = [2, k+1], K' = (n-1)^{k-1}$ [k+2,2k] and let  $H_1,H_2,\ldots,H_{s-1}$  be k-sets containing K' and not containing 1. Then

$$\mathcal{G}_{2s} = \left\{ G \in {[n] \choose k} : 1 \in G, G \cap (K \cup K') \neq \emptyset \right\} \cup \{K, H_1, H_2, \dots, H_{s-1}\}.$$

So 
$$|\mathcal{G}_{2s}| = \binom{n-1}{k-1} - \binom{n-2k}{k-1} + s$$
.

Somewhat surprisingly, it turns out that the asymptotics of the size of the largest family is  $(2k+o(1))\binom{n-2}{k-2}$  for s=2 and s=3 if k is fixed and n tends to infinity, and it is  $(2k-1+o(1))\binom{n-2}{k-2}$ 

Observe that  $K(n,k)[\mathcal{G}_{2s}]$  is  $C_{2s}$ -free and  $\ell(\mathcal{G}_{2s})=s$ . Indeed, if  $K(n,k)[\mathcal{G}_{2s}]$  contained a copy of  $C_{2s}$ , then this copy should contain all s sets not containing 1 as the sets containing 1 form an independent set in K(n,k). In the case s=3,  $\mathcal{F}_6$  does not contain any set that is disjoint from both [2, k+1] and [k+2, 2k+1], so no  $C_6$  exists in  $K(n,k)[\mathcal{G}_6]$ . In the case  $s \geq 4$ , there is no set in  $\mathcal{G}_{2s}$  that is disjoint from both K and  $H_i$  for some  $i=1,2,\ldots,s-1$ , so no copy of  $C_{2s}$  can exist in  $\mathcal{G}_{2s}$ .

The next theorems state that if n is large enough, then Construction 1.8 is asymptotically optimal. Moreover, as the above proofs show that  $K(n,k)[\mathcal{G}_{2s}]$  does not even contain a path on 2s vertices, Construction 1.8 is asymptotically optimal for the problem of forbidding paths as well.

**Theorem 1.9.** For any  $k \geq 2$ , there exists  $n_0 = n_0(k)$  with the following property: if  $n \geq n_0$ and  $\mathcal{F} \subseteq \binom{[n]}{k}$  is a family with  $\ell(\mathcal{F}) \geq 3$  and  $K(n,k)[\mathcal{F}]$  is  $C_6$ -free, then we have  $|\mathcal{F}| < \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + 10^6 (\binom{n-1}{k-1} - \binom{n-2k-1}{k-1})^{3/4}$ .

**Theorem 1.10.** For any  $s \ge 4$  and  $k \ge 3$  there exists  $n_0 = n_0(k, s)$  such if  $n \ge n_0$  and  $\mathcal{F} \subseteq \binom{[n]}{k}$  is a family with  $\ell(\mathcal{F}) \ge s$  and  $K(n, k)[\mathcal{F}]$  is  $C_{2s}$ -free, then we have  $|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-2k}{k-1} + (k^2 + k)$  $1)\binom{n-3}{k-3}.$ 

Let us finish the introduction by a remark on the second order term in Theorem 1.10.

**Remark.** If  $s-1 \leq k$ , then the family  $\mathcal{G}_{2s}$  can be extended to a family  $\mathcal{G}_{2s}^+ \cup \mathcal{G}_{2s}$  so that  $K(n,k)[\mathcal{G}_{2s}^+ \cup \mathcal{G}_{2s}]$  is still  $C_{2s}$ -free. Suppose the sets  $H_1, H_2, \ldots, H_{s-1}$  are all disjoint from K, say  $H_i = K' \cup \{2k+i\}$  for  $i = 1, 2, \dots, s-1$ . Then we can define

$$\mathcal{G}_{2s}^{+} = \left\{ G \in {[n] \choose k} : \{1, 2k+1, 2k+2, \dots, 2k+s-2\} \subseteq G \right\}$$

and observe that  $K(n,k)[\mathcal{G}_{2s}\cup\mathcal{G}_{2s}^+]$  is still  $C_{2s}$ -free. Indeed, a copy of  $C_{2s}$  would have to contain  $K, H_1, H_2, \ldots, H_{s-1}$  as other vertices form an independent set. Moreover, K and  $H_i$  have a common neighbour in  $\mathcal{G}_{2s} \cup \mathcal{G}_{2s}^+$  if and only if i = s - 1, so K cannot be contained in  $C_{2s}$ .

Clearly,  $|\mathcal{G}_{2s}^+ \setminus \mathcal{G}_{2s}| = \binom{n-k-s+1}{k-s+1}$ , so in particular if s=4, then the order of magnitude of the second order term in Theorem 1.10 is sharp (when n is large enough compared to k).

All our results resemble the original Hilton-Milner theorem in the following sense. In Theorem 1.5, Theorem 1.9, Theorem 1.10, almost all sets of the (asymptotically) extremal family share a common element x and meet some set S ( $x \notin S$ ) of fixed size. We wonder whether this phenomenon is true for all bipartite graphs.

**Question 1.11.** Is it true that for any bipartite graph B and integer  $k \geq 3$  there exists an integer s such that the following holds:

- for any family  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $\ell(\mathcal{F}) \geq \eta(B)$  if  $K(n,k)[\mathcal{F}]$  is B-free, then  $|\mathcal{F}| \leq \binom{n-1}{k-1} \binom{n-1-s}{k-1} + o(n^{k-2})$
- the family  $\{G \in {[n] \choose k} : 1 \in G, G \cap [2, s+1] \neq \emptyset\}$  is contained in a family  $\mathcal{G} \subseteq {[n] \choose k}$  with  $\ell(\mathcal{G}) \geq \eta(B)$  such that  $K(n, k)[\mathcal{G}]$  is B-free.

## 2 Proofs

Let us start this section by stating the original Turán number results on the maximum number of edges in  $K_{s,t}$ -free and  $C_{2s}$ -free graphs.

**Theorem 2.1** (Kővári, Sós, Turán [11]). For any pair  $1 \le s \le t$  of integers if a graph G on n vertices is  $K_{s,t}$ -free, then  $e(G) \le (1/2 + o(1))(t-1)^{1/s}n^{2-\frac{1}{s}}$  holds.

**Theorem 2.2** (Bondy, Simonovits [3]). If G is a graph on n vertices that does not contain a cycle of length 2s, then  $e(G) \le 100sn^{1+1/s}$  holds.

We will also need the following lemma by Balogh, Bollobás and Narayanan. (It was improved by a factor of 2 in [1], but for our purposes the original lemma will be sufficient.)

**Lemma 2.3** (Balogh, Bollobás, Narayanan [2]). For any family  $\mathcal{F} \subseteq \binom{[n]}{k}$  we have  $e(K(n,k)[\mathcal{F}]) \ge \frac{l(\mathcal{F})^2}{2\binom{2k}{k}}$ .

We start with the following simple lemma.

**Lemma 2.4.** Let  $s \leq t$  and let  $H_1, H_2, \ldots, H_s, H_{s+1}$  be sets in  $\binom{[n]}{k}$  and  $x \in [n] \setminus \bigcup_{i=1}^{s+1} H_i$ . Suppose that  $\mathcal{F} \subseteq \{F \in \binom{[n]}{k} : x \in F\}$  such that for  $\mathcal{F}' := \mathcal{F} \cup \{H_1, H_2, \ldots, H_{s+1}\}$  the induced subgraph  $K(n,k)[\mathcal{F}']$  is  $K_{s,t}$ -free. Then there exists  $n_0 = n_0(k,s,t)$  such that if  $n \geq n_0$  holds, then we have

$$|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+1)k}{2} \rfloor - 1}{k-1} + (s+1)(t-1).$$

Proof. The number of sets in  $\mathcal{F}$  that meet at most one  $H_j$  is at most (s+1)(t-1) as  $K(n,k)[\mathcal{F}']$  is  $K_{s,t}$ -free. Let us define  $T = \{y \in [n] : \exists i \neq j \ y \in H_i \cap H_j\}$ . Those sets in  $\mathcal{F}$  that meet at least two of the  $H_j$ 's must either a) intersect T or b) intersect at least two of the  $(H_j \setminus T)$ 's. Clearly,  $|T| \leq \lfloor \frac{(s+1)k}{2} \rfloor$ , so the number of sets in  $\mathcal{F}$  meeting T is at most  $\binom{n-1}{k-1} - \binom{n-1-|T|}{k-1} \leq \binom{n-1}{k-1} - \binom{n-1-|T|}{k-1} =: B$ .

Assume first  $|T| < \lfloor \frac{(s+1)k}{2} \rfloor$ , then  $B - (\binom{n-1}{k-1} - \binom{n-1-|T|}{k-1}) = \Omega(n^{k-2})$ . Observe that the number of sets in  $\mathcal{F}$  that are disjoint with T and meet at least two  $H_j \setminus T$  is at most  $\sum_{i,j} |H_i \setminus T| \cdot |H_j \setminus T| \binom{n-3}{k-3} \le \binom{s+1}{2} k^2 \binom{n-3}{k-3} = O(n^{k-3})$ . Therefore if n is large enough, then  $|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-\lfloor \frac{(s+1)k}{k-1} \rfloor - 1}{k-1} - \varepsilon n^{k-2}$  for some  $\varepsilon > 0$ .

Assume now  $T = \lfloor \frac{(s+1)k}{2} \rfloor$ . This implies that at most one of the  $H_j \setminus T$  is non-empty, so  $\mathcal{F}$  does not contain sets of type b). Thus we have  $|\mathcal{F}| \leq B + (s+1)(t-1)$ .

Now we are ready to prove our main result on families  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $K(n,k)[\mathcal{F}]$  being  $K_{s,t}$ -free.

Proof of Theorem 1.5. Let  $\mathcal{F} \subseteq {[n] \choose k}$  be a family such that  $K(n,k)[\mathcal{F}]$  is  $K_{s,t}$ -free and  $|\mathcal{F}| = {n-1 \choose k-1} - {n-sk-1 \choose k-1} + s + t - 1$ . We consider three cases according to the value of  $\ell(\mathcal{F})$ .

Case I:  $\ell(\mathcal{F}) = s$ .

Consider  $F_1, F_2, \ldots, F_s \in \mathcal{F}$  such that  $\mathcal{F}' = \mathcal{F} \setminus \{F_i : 1 \leq i \leq s\}$  is intersecting. Then, as

$$|\mathcal{F}'| = \binom{n-1}{k-1} - \binom{n-sk-1}{k-1} + t - 1 > \binom{n-1}{k-1} - \binom{n-k-1}{k-1},$$

Theorem 1.2 implies that the sets in  $\mathcal{F}'$  share a common element. Since  $K(n,k)[\mathcal{F}]$  is  $K_{s,t}$ -free  $\mathcal{F}'$  can contain at most t-1 sets disjoint from  $T:=\bigcup_{i=1}^s F_i$ . So the size of  $\mathcal{F}$  is at most

$$\binom{n-1}{k-1} - \binom{n-|T|-1}{k-1} + t - 1 + s \le \binom{n-1}{k-1} - \binom{n-sk-1}{k-1} + s + t - 1$$

with equality if and only if  $\mathcal{F}$  is isomorphic to some  $\mathcal{F}_{s,t}$ .

Case II: 
$$s + 1 \le \ell(\mathcal{F}) \le (\binom{n-1}{k-1} - \binom{n-sk-1}{k-1})^{1-\frac{1}{3s}}$$
.

Let  $\mathcal{F}'$  be a largest intersecting subfamily of  $\mathcal{F}$ . As the size of  $\mathcal{F}'$  is  $\binom{n-1}{k-1} - \binom{n-sk-1}{k-1} + s + t - 1 - l(\mathcal{F})$  which is larger than  $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$  if n is large enough, Theorem 1.2 implies that the sets in  $\mathcal{F}'$  share a common element. Let us apply Lemma 2.4 to  $\mathcal{F}'$  and s+1 sets  $F_1, F_2, \ldots, F_{s+1} \in \mathcal{F} \setminus \mathcal{F}'$  to obtain

$$|\mathcal{F}'| \le \binom{n-1}{k-1} - \binom{n - \frac{(s+1)k}{2} - 1}{k-1} + (s+1)(t-1).$$

Therefore, we have

$$|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n - \frac{(s+1)k}{2} - 1}{k-1} + (s+1)(t-1) + \left(\binom{n-1}{k-1} - \binom{n-sk-1}{k-1}\right)^{1 - \frac{1}{3s}},$$

which is smaller than  $\binom{n-1}{k-1} - \binom{n-sk-1}{k-1}$ , if n is large enough.

Case III:  $\binom{n-1}{k-1} - \binom{n-sk-1}{k-1}^{1-\frac{1}{3s}} \le \ell(\mathcal{F}).$ 

Then by Lemma 2.3, we have

$$e(K(n,k)[\mathcal{F}]) \ge \frac{\left(\binom{n-1}{k-1} - \binom{n-sk-1}{k-1}\right)^{2-\frac{2}{3s}}}{2\binom{2k}{k}}.$$

For large enough n, this is larger than  $(1/2 + o(1))(t-1)^{\frac{1}{s}}|\mathcal{F}|^{2-\frac{1}{s}}$ , so  $K(n,k)[\mathcal{F}]$  contains  $K_{s,t}$  by Theorem 2.1.

Proof of Theorem 1.6. Let  $\mathcal{F} \subseteq \binom{[n]}{k}$  be a family of size  $\binom{n}{k} - \binom{n-r+1}{k} + \binom{n-r}{k-1} - \binom{n-s_{r+1}k-r}{k-1} + s_r + s_{r+1} - 1$  with  $\ell_{r+1}(\mathcal{F}) \geq s_{r+1}$  such that  $K(n,k)[\mathcal{F}]$  is  $K_{s_1,s_2,\dots,s_{r+1}}$ -free. The proof proceeds by a case analysis according to the number of large degree vertices. We say that  $x \in [n]$  has large degree if  $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}$  has size at least  $d = \binom{n-1}{k-1} - \binom{n-Qk-1}{k-1} + Q$  where  $Q := \sum_{i=1}^{r+1} s_i$ . Let D denote the set of large degree vertices. We will use the following claim in which  $G_1 \oplus G_2$  denotes the join of  $G_1$  and  $G_2$ , i.e. the graph consisting of disjoint copies of  $G_1$  and  $G_2$  with all possible edges between the  $G_1$  and  $G_2$ .

Claim 2.5. Suppose  $\mathcal{F}$  contains a subfamily  $\mathcal{G} \subseteq \binom{[n] \setminus D}{k}$  with  $|\mathcal{G}| \leq Q - \sum_{i=1}^{|D|} s_i$  and  $K(n,k)[\mathcal{G}]$  is isomorphic to G, then  $K(n,k)[\mathcal{F}]$  contains  $K_{s_1,s_2,...,s_{|D|}} \oplus G$ .

Proof of Claim. Note that d is Q plus the number of k-subsets of [n] containing a fixed element x of [n] and meeting a set S of size Qk. As  $K_{s_1,s_2,\ldots,s_{|D|}} \oplus G$  contains at most Qk vertices, we can pick the sets corresponding to  $K_{s_1,s_2,\ldots,s_{|D|}}$  greedily. Indeed, for each high degree vertex, we can choose  $s_i$  sets containing it which avoid the set spanned by the already chosen sets and the (at most Q) sets corresponding to G.

Case I:  $|D| \ge r$ .

Let  $D' \subset D$  be of size r and let  $F_1, F_2, \ldots, F_{s_{r+1}}$  be sets in  $\mathcal{F}$  not meeting D'. (There exists such sets as otherwise  $\ell_{r+1}(\mathcal{F}) < s_{r+1}$  would hold.) Applying Claim 2.5 with  $\mathcal{G} = \{F_1, F_2, \ldots, F_{s_{r+1}}\}$  we obtain that  $K(n, k)[\mathcal{F}]$  is not  $K_{s_1, s_2, \ldots, s_{r+1}}$ -free.

Case II: |D| = r - 1.

Then  $\mathcal{F}' = \mathcal{F} \setminus \bigcup_{x \in D} \mathcal{F}_x \subseteq {[n] \setminus D \choose k}$  has size at least  ${n-r \choose k-1} - {n-r-s_{r+1}k \choose k-1} + s_r + s_{r+1} - 1$  with equality if and only if  $\bigcup_{x \in D} \mathcal{F}_x$  contains all k-sets meeting D. Either  $K(n,k)[\mathcal{F}']$  contains  $K_{s_r,s_{r+1}}$  and thus, by Claim 2.5,  $\mathcal{F}$  contains  $K_{s_1,s_2,\ldots,s_{r+1}}$ . Otherwise note that  $\ell_{r+1}(\mathcal{F}) \geq s_{r+1}$  implies  $\ell_2(\mathcal{F}') = \ell(\mathcal{F}') \geq s_{r+1}$ , so Theorem 1.5 implies that if n is large enough, then  $\mathcal{F}'$  is some  $\mathcal{F}_{s_r,s_{r+1}}$  and thus,  $\mathcal{F}$  is some  $\mathcal{F}_{s_1,s_2,\ldots,s_{r+1}}$ .

Case III:  $|D| \le r - 2$ .

In this case  $\mathcal{F}' = \mathcal{F} \setminus \bigcup_{x \in D} \mathcal{F}_x \subseteq \binom{[n] \setminus D}{k}$  has size at least  $\binom{n-r+1}{k-1} + \binom{n-r}{k-1} - \binom{n-r-s_{r+1}k}{k-1} + s_r + s_{r+1} - 1$ . The order of magnitude of this is  $n^{k-1}$ , thus it is larger than Qkd if n is large enough. We claim that  $K(n,k)[\mathcal{F}']$  contains  $K_Q$  and therefore a copy of  $K_{s_1,s_2,\dots,s_{r+1}}$ . Indeed, for any  $F \in \mathcal{F}'$  there are at most kd sets in  $\mathcal{F}'$  that intersect F, thus we can pick Q pairwise disjoint sets greedily.

Proof of Theorem 1.7. First we show the construction for the lower bound. For a graph F with  $\chi(F) \geq 3$ , let  $\mathcal{G}_F \subseteq \binom{[n-\chi(F)+2]}{k}$  be a family of size  $\widehat{ex}_v^{(2)}(n-\chi(F)+2,k,\mathcal{B}_{F,\eta})$  such that  $K(n-\chi(F)+2,k)[\mathcal{G}_F]$  is B-free for any  $B \in \mathcal{B}_{F,\eta}$  and  $\ell(\mathcal{G}_F) = \eta(F)$ . Let us define  $\mathcal{F}_F \subseteq \binom{[n]}{k}$  as

$$\mathcal{F}_F = \mathcal{G}_F \cup \left\{ K \in {[n] \choose k} : K \cap [n - \chi(F) + 3, n] \neq \emptyset \right\}.$$

Clearly, we have

$$|\mathcal{F}_F| = \binom{n}{k} - \binom{n - \chi(F) + 2}{k} + \widehat{ex}_v^{(2)}(n - \chi(F) + 2, k, \mathcal{B}_{F,\eta})$$

and we claim that  $K(n,k)[\mathcal{F}_F]$  is F-free. Indeed, if  $K(n,k)[\mathcal{F}_F]$  contains F, then  $K(n,k)[\mathcal{G}_F]$  contains some  $B \in \mathcal{B}_F$ , as  $\{K \in {[n] \choose k} : K \cap [n-\chi(F)+3,n] \neq \emptyset\}$  is the union  $\chi(F)-2$  intersecting families. This is impossible for  $B \in \mathcal{B}_{F,\eta}$  by definition of  $\mathcal{G}_F$ , and it is also impossible for  $B \in \mathcal{B}_F \setminus \mathcal{B}_{F,\eta}$  as  $\ell(\mathcal{G}_F) = \eta(F) < \eta(B)$ .

The proof of the upper bound is basically identical to that of the upper bound in Theorem 1.6, so we just outline it. Let  $\mathcal{F} \subseteq \binom{[n]}{k}$  with  $\ell_{\chi(F)}(\mathcal{F}) \geq \eta(F)$  and  $|\mathcal{F}| \geq \binom{n}{k} - \binom{n-\chi(F)+2}{k}$  be such that  $K(n,k)[\mathcal{F}]$  is F-free. Let us define  $d = \binom{n-1}{k} - \binom{n-|v(F)|k-1}{k} + |V(F)|$  and let  $D \subseteq V(F)$  be the set of vertices with degree at least d in  $\mathcal{F}$ .

Case I:  $|D| \ge \chi(F) - 1$ .

Then one can pick sets of  $\mathcal{F}$  greedily to form a copy of F in  $K(n,k)[\mathcal{F}]$ , a contradiction.

Case II:  $|D| = \chi(F) - 2$ .

Then  $\mathcal{F}' = \{K \in \mathcal{F} : K \cap D \neq \emptyset\}$  has size at most  $\binom{n}{k} - \binom{n - \chi(F) + 2}{k}$ . Also  $K(n, k)[\mathcal{F} \setminus \mathcal{F}']$  cannot contain any  $B \in \mathcal{B}_{F,\eta}$ , as otherwise  $K(n, k)[\mathcal{F}]$  would contain F. Observe that  $\ell_{\chi(F)}(\mathcal{F}) \geq \eta(F)$  implies  $\ell(\mathcal{F} \setminus \mathcal{F}') \geq \eta(F)$ , so we have  $|\mathcal{F} \setminus \mathcal{F}'| \leq ex_v^{(2)}(n, k, \mathcal{B}_{F,\eta})$ .

Case III:  $|D| \leq \chi(F) - 3$ .

Then  $\mathcal{F}' = \{K \in \mathcal{F} : K \cap D \neq \emptyset\}$  has size at most  $\binom{n}{k} - \binom{n-\chi(F)+3}{k}$ . Therefore  $\mathcal{F} \setminus \mathcal{F}'$  is of size at least  $\binom{n-\chi(F)+2}{k-1}$ . If n is large enough compared to k, then one can pick greedily a copy of  $K_{|V(F)|}$  in  $K(n,k)[\mathcal{F} \setminus \mathcal{F}']$ .

Now we turn our attention to proving theorems on families that induce cycle-free subgraphs in the Kneser graph.

Proof of Theorem 1.9. Let  $\mathcal{F} \subseteq \binom{[n]}{k}$  be a family of subsets such that  $K(n,k)[\mathcal{F}]$  is  $C_6$ -free,  $\ell(\mathcal{F}) \geq 3$  and  $|\mathcal{F}| = \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + 10^6 \binom{n-1}{k-1} - \binom{n-2k-1}{k-1})^{3/4}$ .

Case I: 
$$\ell(\mathcal{F}) \le \frac{10^6}{2} (\binom{n-1}{k-1} - \binom{n-2k-1}{k-1})^{3/4}$$
.

Let  $H_1, H_2, \ldots, H_{\ell(\mathcal{F})}$  be sets in  $\mathcal{F}$  such that  $\mathcal{F}' := \mathcal{F} \setminus \{H_1, H_2, \ldots, H_{\ell(\mathcal{F})}\}$  is intersecting. Then as

$$|\mathcal{F}'| \geq \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + \frac{10^6}{2} \left( \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} \right)^{3/4} > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1,$$

Theorem 1.2 implies that the sets in  $\mathcal{F}'$  share a common element x. The  $H_i$ 's do not contain this x, since  $\mathcal{F}'$  is a largest intersecting family in  $\mathcal{F}$ . As  $|H_i \cup H_j| \leq 2k$  and

$$|\mathcal{F}'| \ge \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + \frac{10^6}{2} \left( \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} \right)^{3/4},$$

for any  $i \neq j$  there exist 3 sets  $F_{i,j,1}, F_{i,j,2}, F_{i,j,3} \in \mathcal{F}'$  that are disjoint from  $H_i \cup H_j$ . So we can find a copy of  $C_6$  in  $\mathcal{F}$ , in which the sets  $H_1, H_2, H_3$  represent three independent vertices and the other three sets can be chosen from  $\{F_{i,j,k}: 1 \leq i < j \leq 3, 1 \leq k \leq 3\}$  greedily.

Case II: 
$$\ell(\mathcal{F}) \ge \frac{10^6}{2} (\binom{n-1}{k-1} - \binom{n-2k-1}{k-1})^{3/4}$$
.

By Lemma 2.3  $K(n,k)[\mathcal{F}]$  contains at least  $\frac{10^{12}}{8\binom{2k}{k}}(\binom{n-1}{k-1}-\binom{n-2k-1}{k-1})^{3/2}$  edges and when n is large enough, this is bigger than  $300|\mathcal{F}|^{4/3}$ , so by Theorem 2.2 it contains a copy of  $C_6$ , as desired.

Proof of Theorem 1.10. Let  $\mathcal{F} \subseteq {n \brack k}$  be a family of subsets such that  $K(n,k)[\mathcal{F}]$  is  $C_{2s}$ -free,  $\ell(\mathcal{F}) \geq 3$  and  $|\mathcal{F}| = {n-1 \choose k-1} - {n-2k \choose k-1} + (k^2+1){n-3 \choose k-3}$ .

Case I: 
$$\ell(\mathcal{F}) \le 20s2^k {\binom{n-1}{k-1} - \binom{n-2k}{k-1}}^{\frac{s+1}{2s}}$$
.

Let  $H_1, H_2, \ldots, H_{\ell(\mathcal{F})}$  be sets in  $\mathcal{F}$  such that  $\mathcal{F}' := \mathcal{F} \setminus \{H_1, H_2, \ldots, H_{\ell(\mathcal{F})}\}$  is intersecting. Then as  $|\mathcal{F}'| \geq \binom{n-1}{k-1} - \binom{n-2k}{k-1} + (k^2+1)\binom{n-3}{k-3} - 20s2^k(\binom{n-1}{k-1} - \binom{n-2k}{k-1})^{\frac{s+1}{2s}} > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ , Theorem 1.2 implies that the sets in  $\mathcal{F}'$  share a common element x. The  $H_i$ 's do not contain this x, since  $\mathcal{F}'$  is a maximal intersecting family in  $\mathcal{F}$ . Let us define the following auxiliary graph  $\Gamma$  with vertex set  $\{H_1, H_2, \ldots, H_s\}$ : two sets  $H_i, H_j$  are adjacent if and only if there exist s sets in  $\mathcal{F}'$  that are disjoint from  $H_i \cup H_j$ . Observe that if  $\Gamma$  contains a Hamiltonian cycle, then  $\mathcal{F}$  contains a copy of  $C_{2s}$ . Indeed, if  $H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(s)}$  is a Hamiltonian cycle, then for any pair  $H_{\sigma(i)}, H_{\sigma(i+1)}$  (with s+1=1) we can greedily pick different sets  $F_i \in \mathcal{F}'$  with  $F_i \cap (H_{\sigma(i)} \cup H_{\sigma(i+1)}) = \emptyset$  to get  $H_{\sigma(1)}, F_1, H_{\sigma(2)}, F_2, \ldots, H_{\sigma(s)}, F_s$  a copy of  $C_{2s}$  in  $K(n,k)[\mathcal{F}]$ . Therefore the next claim and Dirac's theorem [4] finishes the proof of Case I.

#### Claim 2.6. The minimum degree of $\Gamma$ is at least s-2.

Proof of Claim. First note that if  $H_i$  and  $H_j$  are not joined in  $\Gamma$ , then they must be disjoint. Indeed, otherwise  $|H_i \cup H_j| \leq 2k-1$  and as  $|\mathcal{F}'| \geq \binom{n-1}{k-1} - \binom{n-2k}{k-1} + s$ , there are at least s sets in  $\mathcal{F}'$  avoiding  $H_i \cup H_j$ . Now assume for contradiction that  $H_1$  is not connected to  $H_2$  and  $H_3$ , so in particular  $H_1 \cap (H_2 \cup H_3) = \emptyset$ . Observe the following

- there are at most s-1 sets in  $\mathcal{F}'$  that avoid  $H_1 \cup H_2$  and another s-1 sets avoiding  $H_1 \cup H_3$ ,
- as  $|H_1 \cup (H_2 \cap H_3)| \leq 2k-1$ , there are at most  $\binom{n-1}{k-1} \binom{n-2k}{k-1}$  sets in qcF' that meet  $H_1 \cup (H_2 \cap H_3)$ .

So there are at least  $(k^2+1)\binom{n-3}{k-3}-20s2^k(\binom{n-1}{k-1}-\binom{n-2k}{k-1})^{\frac{s+1}{2s}}$  sets of  $\mathcal{F}'$  containing at least one element  $h_2 \in H_2 \setminus H_3$  and one element  $h_3 \in H_3 \setminus H_2$ . Since the number of such pairs is at most  $k^2$ , there exists a pair  $h_2, h_3$  such that the number of sets in  $\mathcal{F}'$  containing both  $h_2, h_3$  is more than  $\binom{n-3}{k-3}$ . But this is clearly impossible as the total number of k-sets containing  $x, h_2, h_3$  is  $\binom{n-3}{k-3}$ .

Case II: 
$$\ell(\mathcal{F}) \ge 20s2^k {\binom{n-1}{k-1} - \binom{n-2k}{k-1}}^{\frac{s+1}{2s}}$$
.

By Lemma 2.3  $K(n,k)[\mathcal{F}]$  contains at least  $\frac{400s^22^{2k}}{2\binom{2k}{k}}(\binom{n-1}{k-1}-\binom{n-2k}{k-1})^{\frac{s+1}{s}} > 100s|\mathcal{F}|^{1+1/s}$  edges, and thus by Theorem 2.2 it contains a copy of  $C_{2s}$ .

**Funding**: Research supported by the ÚNKP-17-3 New National Excellence Program of the Ministry of Human Capacities, by National Research, Development and Innovation Office - NKFIH under the grants SNN 116095 and K 116769, by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and the Taiwanese-Hungarian Mobility Program of the Hungarian Academy of Sciences.

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