# Stability results for vertex Turán problems in Kneser graphs 

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#### Abstract

The vertex set of the Kneser graph $K(n, k)$ is $V=\binom{[n]}{k}$ and two vertices are adjacent if the corresponding sets are disjoint. For any graph $F$, the largest size of a vertex set $U \subseteq V$ such that $K(n, k)[U]$ is $F$-free, was recently determined by Alishahi and Taherkhani, whenever $n$ is large enough compared to $k$ and $F$. In this paper, we determine the second largest size of a vertex set $W \subseteq V$ such that $K(n, k)[W]$ is $F$-free, in the case when $F$ is an even cycle or a complete multi-partite graph. In the latter case, we actually give a more general theorem depending on the chromatic number of $F$.


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## 1 Introduction

Turán-type problems are fundamental in extremal (hyper)graph theory. For a pair $H$ and $F$ of graphs, they ask for the maximum number of edges that a subgraph $G$ of the host graph $H$ can

[^0]have without containing the forbidden graph $F$. A variant of this problem is the so-called vertex Turán problem where given a host graph $H$ and a forbidden graph $F$, one is interested in the maximum size of a vertex set $U \subset V(H)$ such that the induced subgraph $H[U]$ is $F$-free.

This problem has been studied in the context of several host graphs. In this paper we follow the recent work of Alishahi and Taherkhani [1], who determined the exact answer to the vertex Turán problem when $H$ is the Kneser graph $K(n, k)$, which is defined on the vertex set $\binom{[n]}{k}=\{K \subseteq[n]=\{1,2, \ldots, n\}:|K|=k\}$ where two vertices $K, K^{\prime}$ are adjacent if and only if $K \cap K^{\prime}=\emptyset$.

Theorem 1.1 (Alishahi, Taherkhani [1]). For any graph F, let $\chi$ denote its chromatic number and let $\eta=\eta(F)$ denote the minimum possible size of a color class of $G$ over all possible proper $\chi$-colorings of $F$. Then for any $k$ there exists an integer $n_{0}=n_{0}(k, F)$ such that if $n \geq n_{0}$ and for a family $\mathcal{G} \subseteq\binom{[n]}{k}$ the induced subgraph $K(n, k)[\mathcal{G}]$ is $F$-free, then $|\mathcal{G}| \leq\binom{ n}{k}-\binom{n-\chi+1}{k}+\eta-1$. Moreover, if equality holds, then there exists a $(\chi-1)$-set $L$ such that $|\{G \in \mathcal{G}: G \cap L=\emptyset\}|=$ $\eta-1$.

Observe that the vertex Turán problem in the Kneser graph $K(n, k)$ generalizes several intersection problems in $\binom{[n]}{k}$ :

- If $F=K_{2}$, the graph consisting a single edge, then the vertex Turán problem asks for the maximum size of an independent set in $K(n, k)$ or equivalently the size of a largest intersecting family $\mathcal{F} \subseteq\binom{[n}{k}$ (i.e. $F \cap F^{\prime} \neq \emptyset$ for all $F, F^{\prime} \in \mathcal{F}$ ). The celebrated theorem of Erdős, Ko, and Rado states that this is $\binom{n-1}{k-1}$ if $2 k \leq n$ holds. Furthermore, for intersecting families $\mathcal{F} \subseteq\binom{[n]}{k}$ of size $\binom{n-1}{k-1}$ we have $\cap_{F \in \mathcal{F}} F \neq \emptyset$ provided $n \geq 2 k+1$.
- If $F=K_{s}$ for some $s \geq 3$, then the vertex Turán problem is equivalent to Erdős's famous matching conjecture: $K(n, k)[\mathcal{F}]$ is $K_{s}$-free if and only if $\mathcal{F}$ does not contain a matching of size $s$ ( $s$ pairwise disjoint sets). Erdős conjectured that the maximum size of such a family is $\max \left\{\binom{s k-1}{k},\binom{n}{k}-\binom{n-s+1}{k}\right\}$.
- Gerbner, Lemons, Palmer, Patkós, and Szécsi [8] considered $l$-almost intersecting families $\mathcal{F} \subseteq\binom{[n]}{k}$ such that for any $F \in \mathcal{F}$ there are at most $l$ sets in $\mathcal{F}$ that are disjoint from $F$. This is equivalent to $K(n, k)[\mathcal{F}]$ being $K_{1, l}$-free.
- Katona and Nagy [10] considered $(s, t)$-union intersecting families $\mathcal{F} \subseteq\binom{[n]}{k}$ such that for any $F_{1}, F_{2}, \ldots, F_{s}, F_{1}^{\prime}, F_{2}^{\prime}, \ldots, F_{t}^{\prime} \in \mathcal{F}$ we have $\left(\cup_{i=1}^{s} F_{i}\right) \cap\left(\cup_{j=1}^{t} F_{j}^{\prime}\right) \neq \emptyset$. This is equivalent to $K(n, k)[\mathcal{F}]$ being $K_{s, t}$-free.

Theorem 1.1 leads into several directions. One can try to determine the smallest value of the threshold $n_{0}(k, G)$. Alishahi and Taherkhani [1] improved the upper bound on $n_{0}$ for $l$-almost intersecting and $(s, t)$-union intersecting families. Erdős's matching conjecture is known to hold
if $n \geq(2 s+1) k-s$. This is due to Frankl [6] and he also showed [5] that the conjecture is true if $k=3$.

Another direction is to determine the "second largest" family with $K(n, k)[\mathcal{F}]$ being $G$-free. In the case of $F=K_{2}$ this means that we are looking for the largest intersecting family $\mathcal{F} \subseteq\binom{[n]}{k}$ with $\cap_{F \in \mathcal{F}} F=\emptyset$. This is the following famous result of Hilton and Milner.
Theorem 1.2 (Hilton, Milner [9]). If $\mathcal{F} \subseteq\binom{[n]}{k}$ is an intersecting family with $n \geq 2 k+1$ and $\cap_{F \in \mathcal{F}} F=\emptyset$, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}-\binom{n-k-1}{k-1}+1$.

In the case of $F=K_{s, t}$ extremal families are not intersecting, so to describe the condition of being "second largest" precisely, we introduce the following parameter.
Definition 1.3. For a family $\mathcal{F}$ and integer $t \geq 2$ let $\ell_{t}(\mathcal{F})$ denote the minimum number $m$ such that one can remove $m$ sets from $\mathcal{F}$ with the resulting family not containing $t$ pairwise disjoint sets. We will write $\ell(\mathcal{F})$ instead of $\ell_{2}(\mathcal{F})$. Note that this is the minimum number of sets one needs to remove from $\mathcal{F}$ in order to obtain an intersecting family.

Observe that if $s \leq t$, then for any family $\mathcal{F}$ with $\ell(\mathcal{F}) \leq s-1$ the induced subgraph $K(n, k)[\mathcal{F}]$ is $K_{s, t}$-free. In [1], the following asymptotic stability result was proved.
Theorem 1.4 (Alishahi, Taherkhani [1). For any integers $s \leq t$ and $k$, and positive real number $\beta$, there exists an $n_{0}=n_{0}(k, s, t, \beta)$ such that the following holds for $n \geq n_{0}$. If for $\mathcal{F} \subseteq\binom{[n]}{k}$ with $\ell(\mathcal{F}) \geq s$, the induced subgraph $K(n, k)[\mathcal{F}]$ is $K_{s, t}-$ free, then $|\mathcal{F}| \leq(s+\beta)\left(\binom{n-1}{k-1}-\binom{n-k-1}{k-1}\right)$ holds.

Note that the above bound is asymptotically optimal as shown by any family $\mathcal{F}_{s, t}=\{F \in$ $\left.\binom{[n]}{k}: 1 \in F, F \cap S \neq \emptyset\right\} \cup\left\{H_{1}, H_{2}, \ldots, H_{s}\right\} \cup\left\{F_{1}^{\prime}, F_{2}^{\prime}, \ldots, F_{t-1}^{\prime}\right\}$, where $S=[2, s k+1], H_{i}=$ $[(i-1) k+2, i k+1]$ for all $i=1,2, \ldots, s$ and $F_{1}^{\prime}, F_{2}^{\prime}, \ldots, F_{t-1}^{\prime}$ are distinct sets containing 1 and disjoint with $S$.

We improve Theorem 1.4 to obtain the following precise stability result for families $\mathcal{F}$ for which $K(n, k)[\mathcal{F}]$ is $K_{s, t} t^{\text {free. }}$

Theorem 1.5. For any $2 \leq s \leq t$ and $k$ there exists $n_{0}=n_{0}(s, t, k)$ such that the following holds for $n \geq n_{0}$. If $\mathcal{F} \subseteq\binom{[n]}{k}$ is a family with $\ell(\mathcal{F}) \geq s$ and $K(n, k)[\mathcal{F}]$ is $K_{s, t}-$ free, then we have $|\mathcal{F}| \leq\binom{ n-1}{k-1}-\binom{n-s k-1}{k-1}+s+t-1$. Moreover, equality holds if and only if $\mathcal{F}$ is isomorphic to some $\mathcal{F}_{s, t}$.

Using Theorem [1.5, we obtain a general stability result for the case when $F$ is a complete multi-partite graph. We consider the family $\mathcal{F}_{s_{1}, s_{2}, \ldots, s_{r+1}}$ that consists of $s_{r+1}$ pairwise disjoint $k$-subsets $F_{1}, F_{2}, \ldots, F_{s_{r+1}}$ of $[n]$ that do not meet $[r]$ and those $k$-subsets of $[n]$ that either (i) intersect $[r-1]$ or (ii) contain $r$ and meet $\cup_{j=1}^{s_{r+1}} F_{j}$ and (iii) $s_{r}-1$ other $k$-sets containing $r$. Clearly, if $s_{1} \geq s_{2} \geq \cdots \geq s_{r} \geq s_{r+1}$ holds, then $K(n, k)\left[\mathcal{F}_{s_{1}, s_{2}, \ldots, s_{r+1}}\right]$ is $K_{s_{1}, s_{2}, \ldots, s_{r+1}}$-free and its size is $\binom{n}{k}-\binom{n-r+1}{k}+\binom{n-r}{k-1}-\binom{n-s_{r+1} k-r}{k-1}+s_{r}+s_{r+1}-1$.

Theorem 1.6. For any $k \geq 2$ and integers $s_{1} \geq s_{2} \geq \cdots \geq s_{r} \geq s_{r+1} \geq 1$ there exists $n_{0}=n_{0}\left(k, s_{1}, \ldots, s_{r+1}\right)$ such that if $n \geq n_{0}$ and $\mathcal{F} \subseteq\binom{[n]}{k}$ is a family with $\ell_{r+1}(\mathcal{F}) \geq s$ and $K(n, k)[\mathcal{F}]$ is $K_{s_{1}, s_{2}, \ldots, s_{r+1}}-$ free, then we have $|\mathcal{F}| \leq\binom{ n}{k}-\binom{n-r+1}{k}+\binom{n-r}{k-1}-\binom{n-s_{r+1} k-r}{k-1}+s_{r}+s_{r+1}-1$. Moreover, equality holds if and only if $\mathcal{F}$ is isomorphic to some $\mathcal{F}_{s_{1}, s_{2}, \ldots, s_{r+1}}$.

Note that Frankl and Kupavskii [7] proved the special case $s_{1}=s_{2}=\cdots=s_{r+1}=1$ with the asymptotically best possible threshold $n_{0}=\left(2 k+o_{r}(1)\right)(r+1) k$.

Actually, Theorem 1.6 is a special case of a more general result that shows that it is enough to solve the stability problem for bipartite graphs. For any graph $F$ with $\chi(F) \geq 3$ let us define $\mathcal{B}_{F}$ to be the class of those bipartite graphs $B$ such that there exists a subset $U$ of vertices of $F$ with $F[U]=B$ and $\chi(F[V(F) \backslash U])=\chi(F)-2$. Note that by definition, for any $B \in \mathcal{B}_{F}$ we have $\eta(B) \geq \eta(F)$. We define $\mathcal{B}_{F, \eta}$ to be the subset of those bipartite graphs $B \in \mathcal{B}_{F}$ for which $\eta(B)=\eta(F)$ holds. To state our result let us introduce some notation. For any graph $F$ let $e x_{v}^{(2)}(n, k, F)$ denote the maximum size of a family $\mathcal{F} \subseteq\binom{[n]}{k}$ with $\ell_{\chi(F)}(\mathcal{F}) \geq \eta(F)$ and $K(n, k)[\mathcal{F}]$ is $F$-free. Observe that Theorem 1.5 is about $e x_{v}^{(2)}\left(n, k, K_{s, t}\right)$ and Theorem 1.6 determines $e x_{v}^{(2)}\left(n, k, K_{s_{1}, s_{2}, \ldots, s_{r+1}}\right)$. We define $e x_{v}^{(2)}\left(n, k, \mathcal{B}_{F, \eta}\right)$ to be the maximum size of a family $\mathcal{F} \subseteq\binom{[n]}{k}$ with $\ell_{2}(\mathcal{F}) \geq \eta(F)$ such that $K(n, k)[\mathcal{F}]$ is $B$-free for any $B \in \mathcal{B}_{F, \eta}$. Similarly, let $\widehat{e x}_{v}^{(2)}\left(n, k, \mathcal{B}_{F, \eta}\right)$ be the maximum size of a family $\mathcal{F} \subseteq\binom{[n]}{k}$ with $\ell_{2}(\mathcal{F})=\eta(F)$ such that $K(n, k)[\mathcal{F}]$ is $B$-free for any $B \in \mathcal{B}_{F, \eta}$. Obviously we have $\widehat{e x}_{v}^{(2)}\left(n, k, \mathcal{B}_{F, \eta}\right) \leq e x_{v}^{(2)}\left(n, k, \mathcal{B}_{F, \eta}\right)$ and we do not know any graph $F$ for which the two quantities differ.

Theorem 1.7. For any graph with $\chi(F) \geq 3$ there exists an $n_{0}=n_{0}(F)$ such that if $n$ is larger than $n_{0}$, then we have
$\widehat{e x}_{v}^{(2)}\left(n-\chi(F), k, \mathcal{B}_{F, \eta}\right) \leq e x_{v}^{(2)}(n, k, F)-\left(\binom{n}{k}-\binom{n-\chi(F)+2}{k}\right) \leq e x_{v}^{(2)}\left(n-\chi(F), k, \mathcal{B}_{F, \eta}\right)$.
Let us remark first that in the case of $F=K_{s_{1}, s_{2}, \ldots, s_{r+1}}$ we have $\mathcal{B}_{F}=\left\{K_{s_{i}, s_{j}}: 1 \leq i<j \leq\right.$ $r+1\}$ and $\mathcal{B}_{F, \eta}=\left\{K_{s_{i}, s_{r+1}}: 1 \leq i \leq r\right\}$ and obviously for both families the minimum is taken for $K_{s_{r}, s_{r+1}}$, so Theorems 1.7 and 1.5 yield the bound of Theorem 1.6,

In view of Theorem 1.7, we turn our attention to bipartite graphs, namely to the case of even cycles: $F=C_{2 s}$. According to Theorem [1.1, the largest families $\mathcal{F}$ such that $K(n, k)[\mathcal{F}]$ is $C_{2 s}$-free have $\ell(\mathcal{F})=s-1$, so once again we will be interested in families for which $\ell(\mathcal{F}) \geq s$. The case $C_{4}=K_{2,2}$ is solved by Theorem 1.5 (at least for large enough $n$ ). Here we define a construction that happens to be asymptotically extremal for any $s \geq 3$.
Construction 1.8. Let us define $\mathcal{G}_{6} \subseteq\binom{[n]}{k}$ as

$$
\mathcal{G}_{6}=\left\{G \in\binom{[n]}{k}: 1 \in G, G \cap[2,2 k+1] \neq \emptyset\right\} \cup\{[2, k+1],[k+2,2 k+1],[2 k+2,3 k+1]\} .
$$

So $\left|\mathcal{G}_{6}\right|=\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}+3$.
For $s \geq 4$ we define the family $\mathcal{G}_{2 s} \subseteq\binom{[n]}{k}$ in the following way: let $K=[2, k+1], K^{\prime}=$ [ $k+2,2 k$ ] and let $H_{1}, H_{2}, \ldots, H_{s-1}$ be $k$-sets containing $K^{\prime}$ and not containing 1. Then

$$
\mathcal{G}_{2 s}=\left\{G \in\binom{[n]}{k}: 1 \in G, G \cap\left(K \cup K^{\prime}\right) \neq \emptyset\right\} \cup\left\{K, H_{1}, H_{2}, \ldots, H_{s-1}\right\} .
$$

So $\left|\mathcal{G}_{2 s}\right|=\binom{n-1}{k-1}-\binom{n-2 k}{k-1}+s$.
Somewhat surprisingly, it turns out that the asymptotics of the size of the largest family is $(2 k+o(1))\binom{n-2}{k-2}$ for $s=2$ and $s=3$ if $k$ is fixed and $n$ tends to infinity, and it is $(2 k-1+o(1))\binom{n-2}{k-2}$ for $s \geq 4$.

Observe that $K(n, k)\left[\mathcal{G}_{2 s}\right]$ is $C_{2 s}$-free and $\ell\left(\mathcal{G}_{2 s}\right)=s$. Indeed, if $K(n, k)\left[\mathcal{G}_{2 s}\right]$ contained a copy of $C_{2 s}$, then this copy should contain all $s$ sets not containing 1 as the sets containing 1 form an independent set in $K(n, k)$. In the case $s=3, \mathcal{F}_{6}$ does not contain any set that is disjoint from both $[2, k+1]$ and $[k+2,2 k+1]$, so no $C_{6}$ exists in $K(n, k)\left[\mathcal{G}_{6}\right]$. In the case $s \geq 4$, there is no set in $\mathcal{G}_{2 s}$ that is disjoint from both $K$ and $H_{i}$ for some $i=1,2, \ldots, s-1$, so no copy of $C_{2 s}$ can exist in $\mathcal{G}_{2 s}$.

The next theorems state that if $n$ is large enough, then Construction 1.8 is asymptotically optimal. Moreover, as the above proofs show that $K(n, k)\left[\mathcal{G}_{2 s}\right]$ does not even contain a path on $2 s$ vertices, Construction 1.8 is asymptotically optimal for the problem of forbidding paths as well.

Theorem 1.9. For any $k \geq 2$, there exists $n_{0}=n_{0}(k)$ with the following property: if $n \geq n_{0}$ and $\mathcal{F} \subseteq\binom{[n]}{k}$ is a family with $\ell(\mathcal{F}) \geq 3$ and $K(n, k)[\mathcal{F}]$ is $C_{6}$-free, then we have $|\mathcal{F}|<\binom{n-1}{k-1}-$ $\binom{n-2 k-1}{k-1}+10^{6}\left(\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}\right)^{3 / 4}$.
Theorem 1.10. For any $s \geq 4$ and $k \geq 3$ there exists $n_{0}=n_{0}(k, s)$ such if $n \geq n_{0}$ and $\mathcal{F} \subseteq\binom{[n]}{k}$ is a family with $\ell(\mathcal{F}) \geq s$ and $K(n, k)[\mathcal{F}]$ is $C_{2 s}$-free, then we have $|\mathcal{F}| \leq\binom{ n-1}{k-1}-\binom{n-2 k}{k-1}+\left(k^{2}+\right.$ 1) $\binom{n-3}{k-3}$.

Let us finish the introduction by a remark on the second order term in Theorem 1.10.
Remark. If $s-1 \leq k$, then the family $\mathcal{G}_{2 s}$ can be extended to a family $\mathcal{G}_{2 s}^{+} \cup \mathcal{G}_{2 s}$ so that $K(n, k)\left[\mathcal{G}_{2 s}^{+} \cup \mathcal{G}_{2 s}\right]$ is still $C_{2 s}$-free. Suppose the sets $H_{1}, H_{2}, \ldots, H_{s-1}$ are all disjoint from $K$, say $H_{i}=K^{\prime} \cup\{2 k+i\}$ for $i=1,2 \ldots, s-1$. Then we can define

$$
\mathcal{G}_{2 s}^{+}=\left\{G \in\binom{[n]}{k}:\{1,2 k+1,2 k+2, \ldots, 2 k+s-2\} \subseteq G\right\}
$$

and observe that $K(n, k)\left[\mathcal{G}_{2 s} \cup \mathcal{G}_{2 s}^{+}\right]$is still $C_{2 s}$-free. Indeed, a copy of $C_{2 s}$ would have to contain $K, H_{1}, H_{2}, \ldots, H_{s-1}$ as other vertices form an independent set. Moreover, $K$ and $H_{i}$ have a common neighbour in $\mathcal{G}_{2 s} \cup \mathcal{G}_{2 s}^{+}$if and only if $i=s-1$, so $K$ cannot be contained in $C_{2 s}$.

Clearly, $\left|\mathcal{G}_{2 s}^{+} \backslash \mathcal{G}_{2 s}\right|=\binom{n-k-s+1}{k-s+1}$, so in particular if $s=4$, then the order of magnitude of the second order term in Theorem 1.10 is sharp (when $n$ is large enough compared to $k$ ).

All our results resemble the original Hilton-Milner theorem in the following sense. In Theorem 1.5, Theorem 1.9, Theorem 1.10, almost all sets of the (asymptotically) extremal family share a common element $x$ and meet some set $S(x \notin S)$ of fixed size. We wonder whether this phenomenon is true for all bipartite graphs.

Question 1.11. Is it true that for any bipartite graph $B$ and integer $k \geq 3$ there exists an integer $s$ such that the following holds:

- for any family $\mathcal{F} \subseteq\binom{[n]}{k}$ with $\ell(\mathcal{F}) \geq \eta(B)$ if $K(n, k)[\mathcal{F}]$ is $B$-free, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$ -$\binom{n-1-s}{k-1}+o\left(n^{k-2}\right)$
- the family $\left\{G \in\binom{[n]}{k}: 1 \in G, G \cap[2, s+1] \neq \emptyset\right\}$ is contained in a family $\mathcal{G} \subseteq\binom{[n]}{k}$ with $\ell(\mathcal{G}) \geq \eta(B)$ such that $K(n, k)[\mathcal{G}]$ is $B$-free.


## 2 Proofs

Let us start this section by stating the original Turán number results on the maximum number of edges in $K_{s, t}$-free and $C_{2 s}$-free graphs.

Theorem 2.1 (Kővári, Sós, Turán [11]). For any pair $1 \leq s \leq t$ of integers if a graph $G$ on $n$ vertices is $K_{s, t}-$ free, then $e(G) \leq(1 / 2+o(1))(t-1)^{1 / s} n^{2-\frac{1}{s}}$ holds.

Theorem 2.2 (Bondy, Simonovits [3). If $G$ is a graph on $n$ vertices that does not contain a cycle of length 2 s , then $\mathrm{e}(G) \leq 100 \mathrm{sn} \mathrm{n}^{1+1 / s}$ holds.

We will also need the following lemma by Balogh, Bollobás and Narayanan. (It was improved by a factor of 2 in [1], but for our purposes the original lemma will be sufficient.)

Lemma 2.3 (Balogh, Bollobás, Narayanan [2]). For any family $\mathcal{F} \subseteq\binom{[n]}{k}$ we have e $(K(n, k)[\mathcal{F}]) \geq$ $\frac{l(\mathcal{F})^{2}}{2\binom{2 k}{k}}$.

We start with the following simple lemma.
Lemma 2.4. Let $s \leq t$ and let $H_{1}, H_{2}, \ldots, H_{s}, H_{s+1}$ be sets in $\binom{[n]}{k}$ and $x \in[n] \backslash \cup_{i=1}^{s+1} H_{i}$. Suppose that $\mathcal{F} \subseteq\left\{F \in\binom{[n]}{k}: x \in F\right\}$ such that for $\mathcal{F}^{\prime}:=\mathcal{F} \cup\left\{H_{1}, H_{2}, \ldots, H_{s+1}\right\}$ the induced subgraph $K(n, k)\left[\mathcal{F}^{\prime}\right]$ is $K_{s, t}-$ free. Then there exists $n_{0}=n_{0}(k, s, t)$ such that if $n \geq n_{0}$ holds, then we have

$$
|\mathcal{F}| \leq\binom{ n-1}{k-1}-\binom{n-\left\lfloor\frac{(s+1) k}{2}\right\rfloor-1}{k-1}+(s+1)(t-1)
$$

Proof. The number of sets in $\mathcal{F}$ that meet at most one $H_{j}$ is at most $(s+1)(t-1)$ as $K(n, k)\left[\mathcal{F}^{\prime}\right]$ is $K_{s, t}$-free. Let us define $T=\left\{y \in[n]: \exists i \neq j \quad y \in H_{i} \cap H_{j}\right\}$. Those sets in $\mathcal{F}$ that meet at least two of the $H_{j}$ 's must either a) intersect $T$ or b) intersect at least two of the ( $H_{j} \backslash T$ )'s. Clearly, $|T| \leq\left\lfloor\frac{(s+1) k}{2}\right\rfloor$, so the number of sets in $\mathcal{F}$ meeting $T$ is at most $\binom{n-1}{k-1}-\binom{n-1-|T|}{k-1} \leq$ $\binom{n-1}{k-1}-\binom{n-\left\lfloor\frac{(s+1) k}{2}\right\rfloor-1}{k-1}=: B$.

Assume first $|T|<\left\lfloor\frac{(s+1) k}{2}\right\rfloor$, then $\left.B-\binom{n-1}{k-1}-\binom{n-1-|T|}{k-1}\right)=\Omega\left(n^{k-2}\right)$. Observe that the number of sets in $\mathcal{F}$ that are disjoint with $T$ and meet at least two $H_{j} \backslash T$ is at most $\sum_{i, j} \mid H_{i} \backslash$ $\left.T|\cdot| H_{j} \backslash T \left\lvert\, \begin{array}{c}n-3 \\ k-3\end{array}\right.\right) \leq\binom{ s+1}{2} k^{2}\binom{n-3}{k-3}=O\left(n^{k-3}\right)$. Therefore if $n$ is large enough, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}-$ $\left({ }^{n-\left\lfloor\frac{(s+1) k}{2}\right\rfloor-1}\right)-\varepsilon n^{k-2}$ for some $\varepsilon>0$.

Assume now $T=\left\lfloor\frac{(s+1) k}{2}\right\rfloor$. This implies that at most one of the $H_{j} \backslash T$ is non-empty, so $\mathcal{F}$ does not contain sets of type b ). Thus we have $|\mathcal{F}| \leq B+(s+1)(t-1)$.

Now we are ready to prove our main result on families $\mathcal{F} \subseteq\binom{[n]}{k}$ with $K(n, k)[\mathcal{F}]$ being $K_{s, t}$-free.

Proof of Theorem 1.5. Let $\mathcal{F} \subseteq\binom{[n]}{k}$ be a family such that $K(n, k)[\mathcal{F}]$ is $K_{s, t}$ free and $|\mathcal{F}|=$ $\binom{n-1}{k-1}-\binom{n-s k-1}{k-1}+s+t-1$. We consider three cases according to the value of $\ell(\mathcal{F})$.

Case I: $\ell(\mathcal{F})=s$.
Consider $F_{1}, F_{2}, \ldots, F_{s} \in \mathcal{F}$ such that $\mathcal{F}^{\prime}=\mathcal{F} \backslash\left\{F_{i}: 1 \leq i \leq s\right\}$ is intersecting. Then, as

$$
\left|\mathcal{F}^{\prime}\right|=\binom{n-1}{k-1}-\binom{n-s k-1}{k-1}+t-1>\binom{n-1}{k-1}-\binom{n-k-1}{k-1}
$$

Theorem 1.2 implies that the sets in $\mathcal{F}^{\prime}$ share a common element. Since $K(n, k)[\mathcal{F}]$ is $K_{s, t}$-free $\mathcal{F}^{\prime}$ can contain at most $t-1$ sets disjoint from $T:=\cup_{i=1}^{s} F_{i}$. So the size of $\mathcal{F}$ is at most

$$
\binom{n-1}{k-1}-\binom{n-|T|-1}{k-1}+t-1+s \leq\binom{ n-1}{k-1}-\binom{n-s k-1}{k-1}+s+t-1
$$

with equality if and only if $\mathcal{F}$ is isomorphic to some $\mathcal{F}_{s, t}$.

## CASE II: $\left.s+1 \leq \ell(\mathcal{F}) \leq\binom{ n-1}{k-1}-\binom{n-s k-1}{k-1}\right)^{1-\frac{1}{3 s}}$.

Let $\mathcal{F}^{\prime}$ be a largest intersecting subfamily of $\mathcal{F}$. As the size of $\mathcal{F}^{\prime}$ is $\binom{n-1}{k-1}-\binom{n-s k-1}{k-1}+s+$ $t-1-l(\mathcal{F})$ which is larger than $\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1$ if $n$ is large enough, Theorem 1.2 implies that the sets in $\mathcal{F}^{\prime}$ share a common element. Let us apply Lemma 2.4 to $\mathcal{F}^{\prime}$ and $s+1$ sets $F_{1}, F_{2}, \ldots, F_{s+1} \in \mathcal{F} \backslash \mathcal{F}^{\prime}$ to obtain

$$
\left|\mathcal{F}^{\prime}\right| \leq\binom{ n-1}{k-1}-\binom{n-\frac{(s+1) k}{2}-1}{k-1}+(s+1)(t-1)
$$

Therefore, we have

$$
|\mathcal{F}| \leq\binom{ n-1}{k-1}-\binom{n-\frac{(s+1) k}{2}-1}{k-1}+(s+1)(t-1)+\left(\binom{n-1}{k-1}-\binom{n-s k-1}{k-1}\right)^{1-\frac{1}{3 s}}
$$

which is smaller than $\binom{n-1}{k-1}-\binom{n-s k-1}{k-1}$, if $n$ is large enough.
CASE III: $\left.\binom{n-1}{k-1}-\binom{n-s k-1}{k-1}\right)^{1-\frac{1}{3 s}} \leq \ell(\mathcal{F})$.
Then by Lemma 2.3, we have

$$
e(K(n, k)[\mathcal{F}]) \geq \frac{\left(\binom{n-1}{k-1}-\binom{n-s k-1}{k-1}\right)^{2-\frac{2}{3 s}}}{2\binom{2 k}{k}}
$$

For large enough $n$, this is larger than $(1 / 2+o(1))(t-1)^{\frac{1}{s}}|\mathcal{F}|^{2-\frac{1}{s}}$, so $K(n, k)[\mathcal{F}]$ contains $K_{s, t}$ by Theorem 2.1.

Proof of Theorem 1.6. Let $\mathcal{F} \subseteq\binom{[n]}{k}$ be a family of size $\binom{n}{k}-\binom{n-r+1}{k}+\binom{n-r}{k-1}-\binom{n-s_{r+1} k-r}{k-1}+s_{r}+$ $s_{r+1}-1$ with $\ell_{r+1}(\mathcal{F}) \geq s_{r+1}$ such that $K(n, k)[\mathcal{F}]$ is $K_{s_{1}, s_{2}, \ldots, s_{r+1}}$ free. The proof proceeds by a case analysis according to the number of large degree vertices. We say that $x \in[n]$ has large degree if $\mathcal{F}_{x}=\{F \in \mathcal{F}: x \in F\}$ has size at least $d=\binom{n-1}{k-1}-\binom{n-Q k-1}{k-1}+Q$ where $Q:=\sum_{i=1}^{r+1} s_{i}$. Let $D$ denote the set of large degree vertices. We will use the following claim in which $G_{1} \oplus G_{2}$ denotes the join of $G_{1}$ and $G_{2}$, i.e. the graph consisting of disjoint copies of $G_{1}$ and $G_{2}$ with all possible edges between the $G_{1}$ and $G_{2}$.
Claim 2.5. Suppose $\mathcal{F}$ contains a subfamily $\mathcal{G} \subseteq\binom{[n] \backslash D}{k}$ with $|\mathcal{G}| \leq Q-\sum_{i=1}^{|D|} s_{i}$ and $K(n, k)[\mathcal{G}]$ is isomorphic to $G$, then $K(n, k)[\mathcal{F}]$ contains $K_{s_{1}, s_{2}, \ldots, s_{|D|}} \oplus G$.
Proof of Claim. Note that $d$ is $Q$ plus the number of $k$-subsets of $[n]$ containing a fixed element $x$ of $[n]$ and meeting a set $S$ of size $Q k$. As $K_{s_{1}, s_{2}, \ldots, s_{|D|}} \oplus G$ contains at most $Q k$ vertices, we can pick the sets corresponding to $K_{s_{1}, s_{2}, \ldots, s_{|D|}}$ greedily. Indeed, for each high degree vertex, we can choose $s_{i}$ sets containing it which avoid the set spanned by the already chosen sets and the (at most $Q$ ) sets corresponding to $G$.

CASE I: $|D| \geq r$.
Let $D^{\prime} \subset D$ be of size $r$ and let $F_{1}, F_{2}, \ldots, F_{s_{r+1}}$ be sets in $\mathcal{F}$ not meeting $D^{\prime}$. (There exists such sets as otherwise $\ell_{r+1}(\mathcal{F})<s_{r+1}$ would hold.) Applying Claim 2.5 with $\mathcal{G}=$ $\left\{F_{1}, F_{2}, \ldots, F_{s_{r+1}}\right\}$ we obtain that $K(n, k)[\mathcal{F}]$ is not $K_{s_{1}, s_{2}, \ldots, s_{r+1}}$-free.

CASE II: $|D|=r-1$.

Then $\mathcal{F}^{\prime}=\mathcal{F} \backslash \cup_{x \in D} \mathcal{F}_{x} \subseteq\binom{[n] \backslash D}{k}$ has size at least $\binom{n-r}{k-1}-\binom{n-r-s_{r+1} k}{k-1}+s_{r}+s_{r+1}-1$ with equality if and only if $\cup_{x \in D} \mathcal{F}_{x}$ contains all $k$-sets meeting $D$. Either $K(n, k)\left[\mathcal{F}^{\prime}\right]$ contains $K_{s_{r}, s_{r+1}}$ and thus, by Claim 2.5, $\mathcal{F}$ contains $K_{s_{1}, s_{2}, \ldots, s_{r+1}}$. Otherwise note that $\ell_{r+1}(\mathcal{F}) \geq s_{r+1}$ implies $\ell_{2}\left(\mathcal{F}^{\prime}\right)=\ell\left(\mathcal{F}^{\prime}\right) \geq s_{r+1}$, so Theorem 1.5 implies that if $n$ is large enough, then $\mathcal{F}^{\prime}$ is some $\mathcal{F}_{s_{r}, s_{r+1}}$ and thus, $\mathcal{F}$ is some $\mathcal{F}_{s_{1}, s_{2}, \ldots, s_{r+1}}$.

Case III: $|D| \leq r-2$.
In this case $\mathcal{F}^{\prime}=\mathcal{F} \backslash \cup_{x \in D} \mathcal{F}_{x} \subseteq\binom{[n] \backslash D}{k}$ has size at least $\binom{n-r+1}{k-1}+\binom{n-r}{k-1}-\binom{n-r-s_{r+1} k}{k-1}+s_{r}+$ $s_{r+1}-1$. The order of magnitude of this is $n^{k-1}$, thus it is larger than $Q k d$ if $n$ is large enough. We claim that $K(n, k)\left[\mathcal{F}^{\prime}\right]$ contains $K_{Q}$ and therefore a copy of $K_{s_{1}, s_{2}, \ldots, s_{r+1}}$. Indeed, for any $F \in \mathcal{F}^{\prime}$ there are at most $k d$ sets in $\mathcal{F}^{\prime}$ that intersect $F$, thus we can pick $Q$ pairwise disjoint sets greedily.

Proof of Theorem 1.7. First we show the construction for the lower bound. For a graph $F$ with $\chi(F) \geq 3$, let $\mathcal{G}_{F} \subseteq\binom{[n-\chi(F)+2]}{k}$ be a family of size $\widehat{e x}_{v}^{(2)}\left(n-\chi(F)+2, k, \mathcal{B}_{F, \eta}\right)$ such that $K(n-\chi(F)+2, k)\left[\mathcal{G}_{F}\right]$ is $B$-free for any $B \in \mathcal{B}_{F, \eta}$ and $\ell\left(\mathcal{G}_{F}\right)=\eta(F)$. Let us define $\mathcal{F}_{F} \subseteq\binom{[n]}{k}$ as

$$
\mathcal{F}_{F}=\mathcal{G}_{F} \cup\left\{K \in\binom{[n]}{k}: K \cap[n-\chi(F)+3, n] \neq \emptyset\right\} .
$$

Clearly, we have

$$
\left|\mathcal{F}_{F}\right|=\binom{n}{k}-\binom{n-\chi(F)+2}{k}+\widehat{e x}_{v}^{(2)}\left(n-\chi(F)+2, k, \mathcal{B}_{F, \eta}\right)
$$

and we claim that $K(n, k)\left[\mathcal{F}_{F}\right]$ is $F$-free. Indeed, if $K(n, k)\left[\mathcal{F}_{F}\right]$ contains $F$, then $K(n, k)\left[\mathcal{G}_{F}\right]$ contains some $B \in \mathcal{B}_{F}$, as $\left\{K \in\binom{[n]}{k}: K \cap[n-\chi(F)+3, n] \neq \emptyset\right\}$ is the union $\chi(F)-2$ intersecting families. This is impossible for $B \in \mathcal{B}_{F, \eta}$ by definition of $\mathcal{G}_{F}$, and it is also impossible for $B \in \mathcal{B}_{F} \backslash \mathcal{B}_{F, \eta}$ as $\ell\left(\mathcal{G}_{F}\right)=\eta(F)<\eta(B)$.

The proof of the upper bound is basically identical to that of the upper bound in Theorem 1.6, so we just outline it. Let $\mathcal{F} \subseteq\binom{[n]}{k}$ with $\ell_{\chi(F)}(\mathcal{F}) \geq \eta(F)$ and $|\mathcal{F}| \geq\binom{ n}{k}-\binom{n-\chi(F)+2}{k}$ be such that $K(n, k)[\mathcal{F}]$ is $F$-free. Let us define $d=\binom{n-1}{k}-\binom{n-|v(F)| k-1}{k}+|V(F)|$ and let $D \subseteq V(F)$ be the set of vertices with degree at least $d$ in $\mathcal{F}$.

Case I: $|D| \geq \chi(F)-1$.
Then one can pick sets of $\mathcal{F}$ greedily to form a copy of $F$ in $K(n, k)[\mathcal{F}]$, a contradiction.
CASE II: $|D|=\chi(F)-2$.

Then $\mathcal{F}^{\prime}=\{K \in \mathcal{F}: K \cap D \neq \emptyset\}$ has size at most $\binom{n}{k}-\binom{n-\chi(F)+2}{k}$. Also $K(n, k)\left[\mathcal{F} \backslash \mathcal{F}^{\prime}\right]$ cannot contain any $B \in \mathcal{B}_{F, \eta}$, as otherwise $K(n, k)[\mathcal{F}]$ would contain $F$. Observe that $\ell_{\chi(F)}(\mathcal{F}) \geq \eta(F)$ implies $\ell\left(\mathcal{F} \backslash \mathcal{F}^{\prime}\right) \geq \eta(F)$, so we have $\left|\mathcal{F} \backslash \mathcal{F}^{\prime}\right| \leq e x_{v}^{(2)}\left(n, k, \mathcal{B}_{F, \eta}\right)$.

Case III: $|D| \leq \chi(F)-3$.
Then $\mathcal{F}^{\prime}=\{K \in \mathcal{F}: K \cap D \neq \emptyset\}$ has size at most $\binom{n}{k}-\binom{n-\chi(F)+3}{k}$. Therefore $\mathcal{F} \backslash \mathcal{F}^{\prime}$ is of size at least $\binom{n-\chi(F)+2}{k-1}$. If $n$ is large enough compared to $k$, then one can pick greedily a copy of $K_{|V(F)|}$ in $K(n, k)\left[\mathcal{F} \backslash \mathcal{F}^{\prime}\right]$.

Now we turn our attention to proving theorems on families that induce cycle-free subgraphs in the Kneser graph.
Proof of Theorem 1.9. Let $\mathcal{F} \subseteq\binom{[n]}{k}$ be a family of subsets such that $K(n, k)[\mathcal{F}]$ is $C_{6}$-free, $\ell(\mathcal{F}) \geq 3$ and $|\mathcal{F}|=\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}+10^{6}\left(\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}\right)^{3 / 4}$.

Case I: $\ell(\mathcal{F}) \leq \frac{10^{6}}{2}\left(\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}\right)^{3 / 4}$.
Let $H_{1}, H_{2}, \ldots, H_{\ell(\mathcal{F})}$ be sets in $\mathcal{F}$ such that $\mathcal{F}^{\prime}:=\mathcal{F} \backslash\left\{H_{1}, H_{2}, \ldots, H_{\ell(\mathcal{F})}\right\}$ is intersecting. Then as

$$
\left|\mathcal{F}^{\prime}\right| \geq\binom{ n-1}{k-1}-\binom{n-2 k-1}{k-1}+\frac{10^{6}}{2}\left(\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}\right)^{3 / 4}>\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1
$$

Theorem 1.2 implies that the sets in $\mathcal{F}^{\prime}$ share a common element $x$. The $H_{i}$ 's do not contain this $x$, since $\mathcal{F}^{\prime}$ is a largest intersecting family in $\mathcal{F}$. As $\left|H_{i} \cup H_{j}\right| \leq 2 k$ and

$$
\left|\mathcal{F}^{\prime}\right| \geq\binom{ n-1}{k-1}-\binom{n-2 k-1}{k-1}+\frac{10^{6}}{2}\left(\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}\right)^{3 / 4}
$$

for any $i \neq j$ there exist 3 sets $F_{i, j, 1}, F_{i, j, 2}, F_{i, j, 3} \in \mathcal{F}^{\prime}$ that are disjoint from $H_{i} \cup H_{j}$. So we can find a copy of $C_{6}$ in $\mathcal{F}$, in which the sets $H_{1}, H_{2}, H_{3}$ represent three independent vertices and the other three sets can be chosen from $\left\{F_{i, j, k}: 1 \leq i<j \leq 3,1 \leq k \leq 3\right\}$ greedily.

CASE II: $\ell(\mathcal{F}) \geq \frac{10^{6}}{2}\left(\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}\right)^{3 / 4}$.
By Lemma $2.3 K(n, k)[\mathcal{F}]$ contains at least $\left.\frac{10^{12}}{8\binom{(2 k}{k}}\binom{n-1}{k-1}-\binom{n-2 k-1}{k-1}\right)^{3 / 2}$ edges and when $n$ is large enough, this is bigger than $300|\mathcal{F}|^{4 / 3}$, so by Theorem 2.2 it contains a copy of $C_{6}$, as desired.

Proof of Theorem 1.10. Let $\mathcal{F} \subseteq\binom{[n]}{k}$ be a family of subsets such that $K(n, k)[\mathcal{F}]$ is $C_{2 s}$-free, $\ell(\mathcal{F}) \geq 3$ and $|\mathcal{F}|=\binom{n-1}{k-1}-\binom{n-2 k}{k-1}+\left(k^{2}+1\right)\binom{n-3}{k-3}$.

CASE I: $\ell(\mathcal{F}) \leq 20 s 2^{k}\left(\binom{n-1}{k-1}-\binom{n-2 k}{k-1}\right)^{\frac{s+1}{2 s}}$.
Let $H_{1}, H_{2}, \ldots, H_{\ell(\mathcal{F})}$ be sets in $\mathcal{F}$ such that $\mathcal{F}^{\prime}:=\mathcal{F} \backslash\left\{H_{1}, H_{2}, \ldots, H_{\ell(\mathcal{F})}\right\}$ is intersecting. Then as $\left|\mathcal{F}^{\prime}\right| \geq\binom{ n-1}{k-1}-\binom{n-2 k}{k-1}+\left(k^{2}+1\right)\binom{n-3}{k-3}-20 s 2^{k}\left(\binom{n-1}{k-1}-\binom{n-2 k}{k-1}\right)^{\frac{s+1}{2 s}}>\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1$, Theorem 1.2 implies that the sets in $\mathcal{F}^{\prime}$ share a common element $x$. The $H_{i}$ 's do not contain this $x$, since $\mathcal{F}^{\prime}$ is a maximal intersecting family in $\mathcal{F}$. Let us define the following auxiliary graph $\Gamma$ with vertex set $\left\{H_{1}, H_{2}, \ldots, H_{s}\right\}$ : two sets $H_{i}, H_{j}$ are adjacent if and only if there exist $s$ sets in $\mathcal{F}^{\prime}$ that are disjoint from $H_{i} \cup H_{j}$. Observe that if $\Gamma$ contains a Hamiltonian cycle, then $\mathcal{F}$ contains a copy of $C_{2 s}$. Indeed, if $H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(s)}$ is a Hamiltonian cycle, then for any pair $H_{\sigma(i)}, H_{\sigma(i+1)}$ (with $s+1=1$ ) we can greedily pick different sets $F_{i} \in \mathcal{F}^{\prime}$ with $F_{i} \cap\left(H_{\sigma(i)} \cup H_{\sigma(i+1)}\right)=\emptyset$ to get $H_{\sigma(1)}, F_{1}, H_{\sigma(2)}, F_{2}, \ldots, H_{\left.\sigma_{( }\right)}, F_{s}$ a copy of $C_{2 s}$ in $K(n, k)[\mathcal{F}]$. Therefore the next claim and Dirac's theorem [4] finishes the proof of Case I.

Claim 2.6. The minimum degree of $\Gamma$ is at least $s-2$.
Proof of Claim. First note that if $H_{i}$ and $H_{j}$ are not joined in $\Gamma$, then they must be disjoint. Indeed, otherwise $\left|H_{i} \cup H_{j}\right| \leq 2 k-1$ and as $\left|\mathcal{F}^{\prime}\right| \geq\binom{ n-1}{k-1}-\binom{n-2 k}{k-1}+s$, there are at least $s$ sets in $\mathcal{F}^{\prime}$ avoiding $H_{i} \cup H_{j}$. Now assume for contradiction that $H_{1}$ is not connected to $H_{2}$ and $H_{3}$, so in particular $H_{1} \cap\left(H_{2} \cup H_{3}\right)=\emptyset$. Observe the following

- there are at most $s-1$ sets in $\mathcal{F}^{\prime}$ that avoid $H_{1} \cup H_{2}$ and another $s-1$ sets avoiding $H_{1} \cup H_{3}$,
- as $\left|H_{1} \cup\left(H_{2} \cap H_{3}\right)\right| \leq 2 k-1$, there are at most $\binom{n-1}{k-1}-\binom{n-2 k}{k-1}$ sets in $q c F^{\prime}$ that meet $H_{1} \cup\left(H_{2} \cap H_{3}\right)$.
So there are at least $\left(k^{2}+1\right)\binom{n-3}{k-3}-20 s 2^{k}\left(\binom{n-1}{k-1}-\binom{n-2 k}{k-1}\right)^{\frac{s+1}{2 s}}$ sets of $\mathcal{F}^{\prime}$ containing at least one element $h_{2} \in H_{2} \backslash H_{3}$ and one element $h_{3} \in H_{3} \backslash H_{2}$. Since the number of such pairs is at most $k^{2}$, there exists a pair $h_{2}, h_{3}$ such that the number of sets in $\mathcal{F}^{\prime}$ containing both $h_{2}, h_{3}$ is more than $\binom{n-3}{k-3}$. But this is clearly impossible as the total number of $k$-sets containing $x, h_{2}, h_{3}$ is $\binom{n-3}{k-3}$.

Case II: $\ell(\mathcal{F}) \geq 20 s 2^{k}\left(\binom{n-1}{k-1}-\binom{n-2 k}{k-1}\right)^{\frac{s+1}{2 s}}$.
By Lemma $2.3 K(n, k)[\mathcal{F}]$ contains at least $\frac{400 s^{2} 2^{2 k}}{2\binom{k}{k}}\left(\binom{n-1}{k-1}-\binom{n-2 k}{k-1}\right)^{\frac{s+1}{s}}>100 s|\mathcal{F}|^{1+1 / s}$ edges, and thus by Theorem 2.2 it contains a copy of $C_{2 s}$.

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