# APPROXIMATING SET MULTI-COVERS

### MÁRTON NASZÓDI AND ALEXANDR POLYANSKII

ABSTRACT. Johnson and Lovász and Stein proved independently that any hypergraph satisfies  $\tau \leq (1 + \ln \Delta)\tau^*$ , where  $\tau$  is the transversal number,  $\tau^*$  is its fractional version, and  $\Delta$  denotes the maximum degree. We prove  $\tau_f \leq 3.153\tau^* \max\{\ln \Delta, f\}$  for the *f*-fold transversal number  $\tau_f$ . Similarly to Johnson, Lovász and Stein, we also show that this bound can be achieved non-probabilistically, using a greedy algorithm.

As a combinatorial application, we prove an estimate on how fast  $\tau_f/f$  converges to  $\tau^*$ . As a geometric application, we obtain an upper bound on the minimal density of an f-fold covering of the d-dimensional Euclidean space by translates of any convex body.

### 1. INTRODUCTION AND PRELIMINARIES

A hypergraph is a pair  $(X, \mathcal{F})$ , where X is a finite set and  $\mathcal{F} \subseteq 2^X$  is a family of some subsets of X. We call the elements of X vertices, and the members of  $\mathcal{F}$  edges of the hypergraph. When a vertex is contained in an edge, we may say that 'the vertex covers the edge', or that 'the edge covers the vertex'.

Let f be a positive integer. An f-fold transversal of  $(X, \mathcal{F})$  is a multiset A of X such that each member of  $\mathcal{F}$  contains at least f elements (with multiplicity). The f-fold transversal number  $\tau_f$  of  $(X, \mathcal{F})$  is the minimum cardinality (with multiplicity) of an f-fold transversal. A 1-transversal is called a transversal, and the 1-transversal number is called the transversal number, and is denoted by  $\tau = \tau_1$ .

A fractional transversal is a function  $w : X \longrightarrow [0,1]$  with  $\sum_{x:x \in F} w(x) \ge 1$  for all  $F \in \mathcal{F}$ . The fractional transversal number of  $(X, \mathcal{F})$  is

$$\tau^* = \tau^*(\mathcal{F}) := \inf \left\{ \sum_{x:x \in X} w(x) : w \text{ is a fractional transversal} \right\}.$$

Clearly,  $\tau^* \leq \tau$ . In the opposite direction, Johnson [Joh74], Lovász [Lov75] and Stein [Ste74] independently proved that

(1) 
$$\tau \le (1 + \ln \Delta)\tau^*,$$

where  $\Delta$  denotes the maximum degree of  $(X, \mathcal{F})$ , that is, the maximum number of edges a vertex is contained in. They showed that the greedy algorithm, that is, picking vertices of X one by one, in such a way that we always pick one that is contained in the largest number of uncovered edges, yields a transversal set whose cardinality does not exceed the right hand side in (1). For more background, see Füredi's survey [Für88].

Our main result is an extension of this theorem to f-fold transversals.

<sup>2010</sup> Mathematics Subject Classification. 05D15, 52C17.

Key words and phrases. transversal, covering, Rogers' bound, multiple transversal.

M. Naszódi thanks the following agencies for their support: the Swiss National Science Foundation grants no. 200020-162884 and 200021-175977; the János Bolyai Research Scholarship of the Hungarian Academy of Sciences; and the National Research, Development and Innovation Office, NKFIH Grants PD-104744 and K119670.

A. Polyanskii was partially supported by the Russian Foundation for Basic Research, grants 15-31-20403 (mol\_a\_ved), 15-01-99563 A, 15-01-03530 A.

**Theorem 1.1.** Let  $\lambda \in (0,1)$  and let f be a positive integer. Then, with the above notation,

(2) 
$$\tau_f \le \frac{1 - \lambda^f}{1 - \lambda} \tau^* (1 + \ln \Delta - (f - 1) \ln \lambda)$$

moreover, for rational  $\lambda$ , the greedy algorithm using appropriate weights, yields an f-fold transversal of cardinality not exceeding the right hand side of (2).

Substituting  $\lambda = 0.287643$  (which is a bit less than 1/e), we obtain

Corollary 1.2. With the above notation, we have

(3) 
$$\tau_f \le 3.153\tau^* \max\{\ln\Delta, f\}.$$

This result may be interpreted in two ways. First, it gives an algorithm that approximates the integer programming (IP) problem of finding  $\tau_f$ , with a better bound on the output of the algorithm than the obvious estimate  $\tau_f \leq f\tau \leq f\tau^*(1 + \ln \Delta)$ .

A similar result was obtained by Rajagopalan and Vazirani in [RV98] (an improvement of [Dob82]), where, the (multi)-set (multi)-cover problem is considered, that is, the goal is to cover vertices by sets. This is simply the combinatorial dual (and therefore, equivalent) formulation of our problem. In [RV98], each set can be chosen at most once. They present generalizations of the greedy algorithm of [Joh74], [Lov75] and [Ste74], and prove that it finds an approximation of the (multi)-set (multi)-cover problem within an  $\ln \Delta$  factor of the optimal solution of the corresponding linear programming (LP) problem. Moreover, they give parallelized versions of the algorithms.

The main difference between [RV98] and the present paper is that there, the optimal solution of an IP problem is compared to the optimal solution of the LP-relaxation of the same IP problem, whereas here, we compare  $\tau_f$  with  $\tau^*$ , where the latter is the optimal solution of a weaker LP problem: the problem with f = 1.

We note that, using the fact that  $f\tau^* \leq \tau_f$ , (3) also implies that the *performance ratio* (that is, the ratio of the value obtained by the algorithm to the optimal value, in the worst case) of our algorithm is constant when  $\ln \Delta \leq f$ . Compare this with [BDS04, Lemma 1 in Section 3.1], where it is shown that, even for large f, the standard greedy algorithm yields a performance ratio of  $\Omega(\ln m)$ , where m is the number of sets in the hypergraph. Further recent results on the performance ratio of another modified greedy algorithm for variants of the set cover problem can be found in [FK06]. See also Chapter 2 of the book [Vaz01] by Vazirani.

The second interpretation of our result is the following. It is easy to see that  $\frac{\tau_f}{f}$  converges to  $\tau^*$  as f tends to infinity. Now, (3) quantifies the speed of this convergence in some sense. In particular, it yields that for  $f = \ln \Delta$  we have  $\frac{\tau_f}{f} \leq 3.153\tau^*$ . We have better approximation for larger f.

**Corollary 1.3.** For every  $0 < \varepsilon \leq 1$ , if we set  $f := \left\lceil \frac{2(1+\ln \Delta)}{\varepsilon(1-\lambda)} \right\rceil$ , where  $0 < \lambda < 1$  is such that  $-\ln \lambda/(1-\lambda) \leq 1 + \varepsilon/2$ , then the f-fold transversal constructed in Theorem 1.1 yields a fractional transversal which gives

$$\tau^* \le \frac{\tau_f}{f} \le \tau^* (1 + \varepsilon).$$

We prove Theorem 1.1 and Corollary 1.3 in Section 2, where, at the end, we discuss the running time of our algorithm. 1.1. A geometric application. Next, we turn to a classical geometric covering problem. Rogers [Rog57] showed that for any convex body K in  $\mathbb{R}^d$ , there is a covering of  $\mathbb{R}^d$  with translates of K of density at most

$$(4) d\ln d + d\ln\ln d + 5d.$$

For the definition of density cf. [PA95]. G. Fejes Tóth [FT76, FT79] gave the non-trivial lower bound  $c_d f$  for the density of an f-fold covering of  $\mathbb{R}^d$  by Euclidean unit balls (with some  $c_d > 1$ ). For more information on multiple coverings in geometry, see the survey [FT04]. As an application of Theorem 1.1, we give a similar estimate for f-fold coverings.

**Theorem 1.4.** Let  $K \subseteq \mathbb{R}^d$  be a convex body and  $f \ge 1$  an integer. Then there is an arrangement of translates of K with density at most

 $(1+o(1)) \cdot 3.153 \max\{d \ln d, f\},\$ 

where every point of  $\mathbb{R}^d$  is covered at least f times.

The key in proving Theorem 1.4 is a general statement, Theorem 3.1, presented in Section 3. Both theorems are proved the same way as corresponding results in [Nas14], where the case f = 1 is considered.

Earlier versions of Theorems 1.4 and 3.1 were proved in [FNN16]. There, in place of the main result of the present paper, a probabilistic argument is used which yields quantitatively weaker bounds. The quantitative gain here comes from the fact that in the probabilistic bound on  $\tau_f$  presented in [FNN16], one has the size of the edge set  $\mathcal{F}$ as opposed to the maximum degree  $\Delta$ , which is what we have in (3).

# 2. Proof of Theorem 1.1

2.1. The algorithm. First, we imagine that each member of  $\mathcal{F}$  has f bank notes, the denominations are  $\$1, \$\lambda, \ldots, \$\lambda^{f-1}$ , where  $\lambda < 1$  is fixed. We pick vertices one by one. At each step, we pick a vertex, and each edge that contains it pays the largest bank note that it has. So, each edge pays \$1 for the first vertex selected from it, then  $\$\lambda$  for the second, etc., and finally,  $\$\lambda^{f-1}$  for the f-th vertex that it contains. Later on, it does not pay for any additional selected vertex that it contains. Now, we follow the greedy algorithm: at each step, we pick the vertex that yields the largest payout at that step. We finish once each edge is covered at least f times, that is, when we collected all the money.

2.2. Notation. Given a positive integer f, we define the truncated exponential function denoted by  ${}^{k}\lambda$  as follows: for any  $\lambda > 0$ , and any  $0 \le k < f$ , let  ${}^{k}\lambda = \lambda^{k}$ , and let  ${}^{k}\lambda = 0$  for any  $k \ge f$ . Note that the value of f is implicitly present in any formula involving the truncated exponential function.

For each  $F \in \mathcal{F}$ , let k(F) denote the number of chosen vertices (with multiplicity) contained in F. We call the function  $k : \mathcal{F} \to \mathbb{Z}^{\geq 0}$  the *current state*, where  $\mathbb{Z}^{\geq 0}$  is the set of non-negative integers. At the start, k is identically zero.

Given a function  $k : \mathcal{F} \to \mathbb{Z}^{\geq 0}$ , we define the value of a vertex  $x \in X$  with respect to k as

$$v_k(x) := \sum_{F:x\in F\in\mathcal{F}} {}^{k(F)}\lambda.$$

The total remaining value of k is defined as

$$v(k) := \sum_{F:F\in\mathcal{F}} \sum_{i=k(F)}^{f} {}^{i}\lambda,$$

which is the total pay out that will be earned in the subsequent steps.

2.3. Fractional matchings. A fractional matching of the hypergraph  $(X, \mathcal{F})$  is a function  $w : \mathcal{F} \longrightarrow [0, 1]$  with  $\sum_{F:x \in F \in \mathcal{F}} w(F) \leq 1$  for all  $x \in X$ . The fractional matching number of  $(X, \mathcal{F})$  is

$$\nu^* = \nu^*(\mathcal{F}) := \sup\left\{\sum_{F:F\in\mathcal{F}} w(F) : w \text{ is a fractional matching}\right\}$$

By the duality of linear programming,  $\nu^* = \tau^*$ .

We will need the following simple observation.

**Lemma 2.1.** Let z > 0, and  $\ell : \mathcal{F} \to \mathbb{Z}^{\geq 0}$  be such that  $v_{\ell}(x) \leq z$  for any  $x \in X$ . Then we have

$$\frac{v(\ell)}{z} \le (1+\lambda+\ldots+\lambda^{f-1})\nu^*(\mathcal{F}) = \frac{1-\lambda^f}{1-\lambda}\nu^*(\mathcal{F}).$$

Proof of Lemma 2.1. Let

$$w(F) := \frac{\sum_{i=\ell(F)}^{f} i\lambda}{z(1+\lambda+\ldots+\lambda^{f-1})}, \text{ for any } F \in \mathcal{F}.$$

First, we show that w is a fractional matching. Indeed, fix an  $x \in X$ .

$$\sum_{F:x\in F\in\mathcal{F}} w(F) = \frac{1}{z} \sum_{F:x\in F\in\mathcal{F}} \frac{\sum_{i=\ell(F)}^{J} i\lambda}{1+\lambda+\ldots+\lambda^{f-1}} \le \frac{1}{z} \sum_{F:x\in F\in\mathcal{F}} \ell(F)\lambda = \frac{v_\ell(x)}{z} \le 1.$$

Second, the total weight is

$$\sum_{F:F\in\mathcal{F}} w(F) = \frac{1}{z(1+\lambda+\ldots+\lambda^{f-1})} \sum_{F:F\in\mathcal{F}} \sum_{i=\ell(F)}^{f} \lambda = \frac{v(\ell)}{z(1+\lambda+\ldots+\lambda^{f-1})},$$

finishing the proof of the Lemma.

2.4. Finally, we count the steps of the algorithm. We may assume that  $\lambda = p/q \in (0,1)$  with  $p,q \in \mathbb{Z}^+$ . If  $\lambda$  is irrational, then the statement of Theorem 1.1 follows by continuity. Clearly,  $q^{f-1}$  is a common denominator for the pay outs at each step.

At the start,  $k_0(F) := k(F) = 0$  for all  $F \in \mathcal{F}$ . We group the steps according to the \$-amount (that is,  $v_k(x)$ ) that we get at each.

In the first  $t_1$  steps, each vertex x that we pick has value  $v_k(x) = \Delta =: z_1$ , where, we recall,  $\Delta$  is the maximum degree in the hypergraph. Let  $k_1 : \mathcal{F} \to \mathbb{Z}^{\geq 0}$  denote the current state after the first  $t_1$  steps.

Then, in the second group of steps, we make  $t_2$  steps, at each picking a vertex  $x \in V$  of value  $v_k(x) = \Delta - q^{1-f} =: z_2$ , where k changes at each step. Let  $k_2 : \mathcal{F} \to \mathbb{Z}^{\geq 0}$  denote the current state after the first  $t_1 + t_2$  steps.

In the *j*-th group of steps, we make  $t_j$  steps, at each picking a vertex  $x \in V$  of value  $v_k(x) = \Delta - (j-1)q^{1-f} =: z_j$ . Let  $k_j : \mathcal{F} \to \mathbb{Z}^{\geq 0}$  denote the current state after the first  $t_1 + \ldots + t_j$  steps.

Obviously,  $t_j \ge 0$ , moreover some  $t_j$  may be zero. For instance (the reader may check as an exercise), if f > 1, then  $t_2 = 0$ . For the last group, we have  $j = q^{f-1}\Delta - p^{f-1} + 1 =: N$ .

Notice that  $v_{k_j}(x) \leq z_{j+1}$  for any  $x \in V$ . Therefore, by Lemma 2.1, we have

(5) 
$$\frac{v(k_j)}{z_{j+1}} \le \frac{1-\lambda^f}{1-\lambda} \nu^*(\mathcal{F}).$$

Clearly,

(6) 
$$v(k_j) = \sum_{i=j+1}^{N} t_i z_i, \text{ for } 0 \le j \le N-1.$$

In total, we choose  $t_1 + t_2 + \cdots + t_N$  vertices (that is the cardinality of A with multiplicity), and they form an f-fold transversal of  $(X, \mathcal{F})$ . Thus, by (6) and (5), we obtain

. . .

$$\tau_{f} \leq t_{1} + t_{2} + \dots + t_{N} = \\ = \left(\frac{v(k_{0})}{z_{1}} + \sum_{j=1}^{N-1} v(k_{j}) \left(\frac{1}{z_{j+1}} - \frac{1}{z_{j}}\right)\right) = \frac{v(k_{0})}{z_{1}} + \sum_{j=1}^{N-1} \frac{v(k_{j})q^{1-f}}{z_{j+1}z_{j}} \leq \\ \leq \frac{1 - \lambda^{f}}{1 - \lambda} \nu^{*}(\mathcal{F}) \left(1 + \sum_{j=1}^{N-1} \frac{q^{1-f}}{z_{j}}\right) = \frac{1 - \lambda^{f}}{1 - \lambda} \tau^{*}(\mathcal{F}) \left(1 + \sum_{k=p^{f-1}+1}^{q^{f-1}\Delta} \frac{1}{k}\right) \leq \\ \leq \frac{1 - \lambda^{f}}{1 - \lambda} \tau^{*}(\mathcal{F})(1 + \ln \Delta - (f-1)\ln \lambda), \end{aligned}$$

which completes the proof of Theorem 1.1.

2.5. Proof of Corollary 1.3. An f-fold transversal  $A \subset X$  (A is a multiset) easily yields a fractional transversal: one sets the weight  $w(x) = \frac{|\{x:x \in A\}|}{f}$  (cardinality counted with multiplicity) for any vertex  $x \in X$ . The total weight that we get from our construction in Theorem 1.1 is then

$$\tau^*(\mathcal{F}) \le \sum_{x:x \in V} w(x) \le \tau^*(\mathcal{F}) \frac{1 + \ln \Delta - (f-1) \ln \lambda}{f(1-\lambda)} \le \tau^*(\mathcal{F})(1+\varepsilon).$$

2.6. Running time. Let n denote the number of vertices and m be the number of edges of the hypergraph. The adjacency matrix and f are the inputs of the algorithm. As preprocessing, for each vertex, we create a list of edges that contain it (at most  $\Delta$ ), which takes nm operations. We keep track of the current state in an array k of length m.

At each step, the following operations are performed. Computing the value of a vertex takes the addition of at most  $\Delta$  numbers. Thus, finding the vertex of maximal value is  $n\Delta$  operations. Picking that vertex means decreasing at most  $\Delta$  entries of the array k by one. We make at most  $\frac{1-\lambda^f}{1-\lambda}\tau^*(\mathcal{F})(1+\ln\Delta-(f-1)\ln\lambda)$  steps. With the  $\lambda = 0.287643$  substitution, in total, the number of operations is at most

$$nm + O(\tau^* \max\{\ln \Delta, f\} \cdot \Delta n) \le O(\max\{\ln \Delta, f\} \cdot \Delta nm).$$

# 3. Multiple covering of space – Proof of Theorem 1.4

We denote by  $K \sim T := \{x \in \mathbb{R}^d : T + x \subseteq K\}$  the *Minkowski difference* of two sets K and T in  $\mathbb{R}^d$ . For  $K, L \subset \mathbb{R}^d$ , and  $f \ge 1$  integer, we denote the *f*-fold translative covering number of L by K, that is, the minimum number of translates of K such that each point of L is contained in at least f, by  $N_f(L, K)$ . We denote the fractional covering number of L by K by  $N^*(L, K) := \tau^*(\mathcal{F})$ , where  $\mathcal{F} := \{x - K : x \in L\}$  is a hypergraph with base set  $\mathbb{R}^d$ , see details in [Nas14], or [AAS15].

**Theorem 3.1.** Let K, L and T be bounded Borel measurable sets in  $\mathbb{R}^d$  and let  $\Lambda \subset \mathbb{R}^d$ be a finite set with  $L \subseteq \Lambda + T$ . Then (7)

$$N_f(L,K) \le \left\lceil 3.153N^*(L-T,K \sim T) \max\left\{ \ln\left(\max_{x \in L-K} |(x+(K \sim T)) \cap \Lambda|\right), f \right\} \right\rceil.$$

If  $\Lambda \subset L$ , then we have

(8) 
$$N_f(L,K) \le \left\lceil 3.153N^*(L,K \sim T) \max\left\{ \ln\left(\max_{x \in L-K} \left| (x + (K \sim T)) \cap \Lambda \right| \right), f \right\} \right\rceil.$$

Theorem 3.1 is the f-fold analogue of [Nas14, Theorem 1.2], where the case f = 1 is considered. For completeness, we give an outline the proof.

Proof of Theorem 3.1. To prove (7), consider the hypergraph with base set  $\mathbb{R}^d$  and hyperedges of the form  $u - (K \sim T)$ , where  $u \in \Lambda$ . An *f*-fold transversal of this hypergraph clearly yields an *f*-fold covering of *L* by translates of *K*. A substitution into (3) yields the desired bound. We omit the proof of (8), which is very similar.

Using this result, one may prove Theorem 1.4 following [Nas14, proof of Theorem 2.1], which is the proof of Rogers' density bound (4). We give an outline of this proof.

Proof of Theorem 1.4. Let C denote the cube  $C = [-a, a]^d$ , where a > 0 is large. Our goal is to cover C by translates of K economically. We only consider the case when K = -K, as treating the general case would add only minor technicalities.

Let  $\delta > 0$  be fixed (to be chosen later) and let  $\Lambda \subset \mathbb{R}^d$  be a finite set such that  $\Lambda + \frac{\delta}{2}K$  is a saturated (i.e. maximal) packing of  $\frac{\delta}{2}K$  in  $C - \frac{\delta}{2}K$ . By the maximality of the packing, we have that  $\Lambda + \delta K \supseteq C$ . By considering volume, for any  $x \in \mathbb{R}^d$  we have

(9) 
$$|\Lambda \cap (x + (1 - \delta)K)| \le \frac{\operatorname{vol}\left((1 - \delta)K + \frac{\delta}{2}K\right)}{\operatorname{vol}\left(\frac{\delta}{2}K\right)} \le \left(\frac{2}{\delta}\right)^d.$$

Let  $\varepsilon > 0$  be fixed. Clearly, if a is sufficiently large, then

(10) 
$$N^*(C - \delta K, (1 - \delta)K) \le (1 + \varepsilon) \frac{\operatorname{vol}(C)}{(1 - \delta)^d \operatorname{vol}(K)}.$$

By (7), (9) and (10) we have

$$N_f(C,K) \le \left\lceil 3.153 \frac{1+\varepsilon}{(1-\delta)^d} \frac{\operatorname{vol}\left(C\right)}{\operatorname{vol}\left(K\right)} \max\left\{ d\ln\left(\frac{2}{\delta}\right), f \right\} \right\rceil.$$

Thus, we obtain an f-fold covering of C. We repeat this covering periodically for all translates of C in a tiling of  $\mathbb{R}^d$  by translates of C, which yields an f-fold covering of  $\mathbb{R}^d$ . The density of this covering is at most

$$N_f(C, K) \operatorname{vol}(K) / \operatorname{vol}(C) \le \left\lceil 3.153 \frac{1 + \varepsilon}{(1 - \delta)^d} \max\left\{ d \ln\left(\frac{2}{\delta}\right), f \right\} \right\rceil.$$

We choose  $\delta = \frac{2}{d \ln d}$ , and a standard computation yields the desired result.

Acknowledgement. We thank the referees, whose comments helped greatly to improve the presentation.

#### References

- [AAS15] S. Artstein-Avidan and B. A. Slomka, On weighted covering numbers and the Levi-Hadwiger conjecture, Israel Journal of Mathematics 209 (2015), no. 1, 125–155.
- [BDS04] P. Berman, B. DasGupta, and E. Sontag, Randomized Approximation Algorithms for Set Multicover Problems with Applications to Reverse Engineering of Protein and Gene Networks, Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques: 7th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems, APPROX 2004, and 8th International Workshop on Randomization and Computation, RANDOM 2004, Cambridge, MA, USA, August 22-24, 2004. Proceedings, 2004, pp. 39– 50.
- [Dob82] G. Dobson, Worst-Case Analysis of Greedy Heuristics for Integer Programming with Nonnegative Data, Mathematics of Operations Research 7 (1982), no. 4, 515–531.
- [FK06] T. Fujito and H. Kurahashi, A Better-Than-Greedy Algorithm for k-Set Multicover, Approximation and Online Algorithms: Third International Workshop, WAOA 2005, Palma de Mallorca, Spain, October 6-7, 2005, Revised Papers, 2006, pp. 176–189.
- [FNN16] N. Frankl, J. Nagy, and M. Naszódi, Coverings: variations on a result of Rogers and on the Epsilon-net theorem of Haussler and Welzl, arXiv:1607.02888 [math.MG] (2016).
  - [FT04] G. Fejes Tóth, Handbook of discrete and computational geometry (2nd edition), 2004, pp. 25–53.
  - [FT76] G. Fejes Tóth, Multiple packing and covering of the plane with circles, Acta Math. Acad. Sci. Hungar. 27 (1976), no. 1-2, 135–140. MR0417930
  - [FT79] G. Fejes Tóth, Multiple packing and covering of spheres, Acta Math. Acad. Sci. Hungar. 34 (1979), no. 1-2, 165–176. MR546731
- [Für88] Z. Füredi, Matchings and covers in hypergraphs, Graphs Combin. 4 (1988), no. 2, 115–206.
- [Joh74] D. S. Johnson, Approximation algorithms for combinatorial problems, Journal of Computer and System Sciences 9 (1974), no. 3, 256 –278.
- [Lov75] L. Lovász, On the ratio of optimal integral and fractional covers, Discrete Math. 13 (1975), no. 4, 383–390.
- [Nas14] M. Naszódi, On some covering problems in geometry, arXiv:1404.1691 [math] (April 2014). To appear in Proc. AMS.
- [PA95] J. Pach and P. K. Agarwal, Combinatorial geometry., New York, NY: John Wiley & Sons, 1995 (English).
- [Rog57] C. A. Rogers, A note on coverings, Mathematika 4 (1957), 1–6.
- [RV98] S. Rajagopalan and V. V. Vazirani, Primal-dual rnc approximation algorithms for set cover and covering integer programs, SIAM Journal on Computing 28 (1998), no. 2, 525–540, available at http://dx.doi.org/10.1137/S0097539793260763.
- [Ste74] S. K Stein, Two combinatorial covering theorems, Journal of Combinatorial Theory, Series A 16 (1974), no. 3, 391–397.
- [Vaz01] V. V. Vazirani, Approximation algorithms, Springer-Verlag New York, Inc., New York, NY, USA, 2001.

(M. Naszódi) ELTE, DEPT. OF GEOMETRY, LORAND EÖTVÖS UNIVERSITY, PÁZMÁNY PÉTER SÉTÁNY 1/C BUDAPEST, HUNGARY 1117

(A. Polyanskii) Moscow Institute of Physics and Technology, Technion, Institute for Information Transmission Problems RAS.

E-mail address: marton.naszodi@math.elte.hu and alexander.polyanskii@yandex.ru