# COVERINGS: VARIATIONS ON A RESULT OF ROGERS AND ON THE EPSILON-NET THEOREM OF HAUSSLER AND WELZL 

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#### Abstract

We consider four problems. Rogers proved that for any convex body $K$, we can cover $\mathbb{R}^{d}$ by translates of $K$ of density very roughly $d \ln d$. First, we extend this result by showing that, if we are given a family of positive homothets of $K$ of infinite total volume, then we can find appropriate translation vectors for each given homothet to cover $\mathbb{R}^{d}$ with the same (or, in certain cases, smaller) density.

Second, we extend Rogers' result to multiple coverings of space by translates of a convex body: we give a non-trivial upper bound on the density of the most economical covering where each point is covered by at least a certain number of translates.

Third, we show that for any sufficiently large $n$, the sphere $\mathbb{S}^{2}$ can be covered by $n$ strips of width $20 n / \ln n$, where no point is covered too many times.

Finally, we give another proof of the previous result based on a combinatorial observation: an extension of the Epsilon-net Theorem of Haussler and Welzl. We show that for a hypergraph of bounded Vapnik-Chervonenkis dimension, in which each edge is of a certain measure, there is a not-too large transversal set which does not intersect any edge too many times.


## 1. Introduction

For a convex body $K$ we denote its translative covering density (the minimum density of the covering of $\mathbb{R}^{d}$ by translates of $K$ ) by $\vartheta(K)$. We recall Rogers' estimate Rog57:

$$
\begin{equation*}
\vartheta(K) \leq d \ln d+d \ln \ln d+5 d . \tag{1}
\end{equation*}
$$

Our first result is an extension of (11). For a family $\mathcal{F}$ of sets in $\mathbb{R}^{d}$, we say that $\mathcal{F}$ permits a translative covering of a subset $A$ of $\mathbb{R}^{d}$ with density $\vartheta$, if we can select a translation vector $x_{F} \in \mathbb{R}^{d}$ for each member $F$ of $\mathcal{F}$ such that $A \subseteq \bigcup_{F \in \mathcal{F}} x_{F}+F$, and the density of this covering is $\vartheta$.

Theorem 1.1. Let $K$ be a convex body in $\mathbb{R}^{d}$, and let $\mathcal{F}=\left\{\lambda_{1} K, \lambda_{2} K, \ldots\right\}\left(0<\lambda_{i}\right)$ be a family of its homothets with

$$
\sum_{i=1}^{\infty} \lambda_{i}^{d}=\infty
$$

Let $\Lambda:=\left\{\lambda_{1}, \lambda_{2}, \ldots\right\}$.
(a) If $\Lambda$ is bounded, and has a limit point other than zero, then $\mathcal{F}$ permits a covering of space of density $\vartheta(K)$.
(b) If $\Lambda$ is bounded, and has no limit point other than zero, then $\mathcal{F}$ permits a covering of space of density one.
(c) If $\Lambda$ is unbounded, then $\mathcal{F}$ permits a covering of space with maximum multiplicity $4 d$ (that is, where no point is covered by more than $4 d$ sets).

[^0]In case (c), we will prove maximum multiplicity $2 d$ in a special case which includes all smooth bodies, see Theorem 2.5. The proofs are in Section 2.

In the proof of Theorem 1.1, we will use a result on covering $K$ by homothets of $K$.
Theorem 1.2. Let $K \subseteq \mathbb{R}^{d}$ be a convex body of volume one, and let $\mathcal{F}$ be a family of positive homothets of $K$ with total volume at least

$$
\begin{cases}\left(d^{3} \cdot \ln d \cdot \vartheta(K)+e\right) 2^{d}, & \text { if } K=-K, \\ d^{3} \cdot \ln d \cdot \vartheta(K) \cdot\binom{2 d}{d}+e \cdot 4^{d}, & \text { in general. }\end{cases}
$$

Then $\mathcal{F}$ permits a translative covering of $K$.
This result is a strengthening of a result of [Nas10], which, in turn is a strengthening of a result of Januszewski Jan03. We prove it in subsection 2.1. We learned that a stronger bound was recently obtained by Livshyts and Tikhomirov LT16.

Our second topic is multiple coverings of space. We denote the infimum of the densities of $k$-fold coverings of $\mathbb{R}^{d}$ by translates of $K$ by $\vartheta^{(k)}(K)$. Apart from the estimate that follows from (1) using the obvious fact $\vartheta^{(k)}(K) \leq k \vartheta(K)$, no general estimate has been known. For the Euclidean ball $\mathbf{B}_{2}^{d}$ in $\mathbb{R}^{d}$, G. Fejes Tóth FT76, FT79 gave the non-trivial lower bound $\vartheta^{(k)}\left(\mathbf{B}_{2}^{d}\right)>c_{d} k$ for some $c_{d}>1$, see more in the survey [FT04]. We prove
Theorem 1.3. Let $K \subseteq \mathbb{R}^{d}$ be a convex body and $k \leq d(\ln d+\ln \ln d)$. Then

$$
\vartheta^{(k)}(K) \leq 6 e d(3 \ln d+\ln \ln d+15) .
$$

This shows that G. Fejes Tóth's bound (up to a constant factor) is sharp if $k=d \ln d$.
To prove Theorem 1.3, we present in Section 3 a more general statement, Theorem 3.3, which extends AAS13, Theorem 1.6] and [Nas14, Theorem 1.2].

Our third topic is covering the sphere $\mathbb{S}^{2}:=\left\{x \in \mathbb{R}^{3}:|x|=1\right\}$ by strips. For a given point $x \in \mathbb{S}^{2}$, and $0 \leq w \leq 1$, we call $\left\{v \in \mathbb{S}^{2}:|\langle v, x\rangle| \leq w\right\}$ the strip centered at $x$, of Euclidean half-width $w$.

Theorem 1.4. For any sufficiently large integer $N$, there is a covering of $\mathbb{S}^{2}$ by $N$ strips of Euclidean half-width $\frac{10 \ln N}{N}$, with no point covered more than $c \ln N$ times, where $c$ is a universal constant.

Our study of this question was motivated by a problem at the 2015 Miklós Schweitzer competition posed by András Bezdek, Ferenc Fodor, Viktor Vígh and Tamás Zarnócz on covering the two-dimensional sphere by strips of a given width such that no point is covered too many times.

We note the following dual version of Theorem 1.4, and leave it to the reader to convince themselves that the two versions are equivalent: For any sufficiently large integer $N$, we can select $N$ points of $\mathbb{S}^{2}$ such that each strip of Euclidean half-width $\frac{10 \ln N}{N}$ contains at least one and at most $c \ln N$ points, where $c$ is a universal constant.

In Section 4, we present a direct, probabilistic proof of Theorem 1.4 .
Our third topic, presented in Section 5, is studying variants of the Epsilon-net theorem of Haussler and Welzl (HW87].

A set $X$ with a family $\mathcal{H} \subseteq 2^{X}$ of some of its subsets is called a hypergraph, its VapnikChervonenkis dimesion (VC-dimension, for short) is defined in Section 5 .

Theorem 1.5. Let $X$ be a set, $\mathcal{H} \subset 2^{X}$ a hypergraph on $X$ of VC-dimension at most $d \geq 2$, and $0<\varepsilon<1$. Let $\mu$ be a probability measure on $X$ with $\mu(H)=\varepsilon$ for each $H \in \mathcal{H}$.
a) If $\varepsilon \leq \frac{1}{d}$, then one can choose $\left\lfloor C \frac{d}{\varepsilon} \ln (1 / \varepsilon)\right\rfloor$ elements of $X$ (not necessarily distinct), such that each edge of $\mathcal{H}$ contains at least one and at most $C_{1} d \ln \left(\frac{1}{\varepsilon}\right)$ chosen points (with multiplicity), where $C$ and $C_{1}$ are universal constants.
b) One can choose $\left\lfloor C \frac{d}{\varepsilon} \ln \left(\frac{1}{\varepsilon}+1\right)\right\rfloor$ elements of $X$ (not necessarily distinct), such that each edge of $\mathcal{H}$ contains at least one and at most $C_{1} d \ln d \ln \left(\frac{1}{\varepsilon}+1\right)$ chosen points (with multiplicity), where $C$ and $C_{1}$ are universal constants.

The dual (and equivalent) version of Theorem 1.4 clearly follows from Theorem 1.5 , since (using the uniform probability measure on $\mathbb{S}^{2}$ ) the measure of any strip of Euclidean half-width $w$ is $w$, and the VC-dimension of strips on $\mathbb{S}^{2}$ is bounded.

We also prove a similar result with essentially the same technique.
Theorem 1.6. Let $X$ be a set, $\mathcal{H} \subset 2^{X}$ a hypergraph on $X$ of VC-dimension at most $d \geq 2$, and $N \geq 2$ an integer. Let $\mu$ be a probability measure on $X$ with $\mu(H)=1 / N$ for each $H \in \mathcal{H}$. Then one can find a multisubset of $N$ elements of $X$ (with multiplicity), such that each edge of $\mathcal{H}$ contains at most $C d \frac{\ln N}{\ln \ln N}$ chosen points, where $C$ is a universal constant.

It was pointed out to us by Nabil Mustafa that Theorems 1.5 and 1.6 can be obtained directly from results on epsilon approximations.

## 2. Covering space with given homothets - Proof of Theorem 1.1

Remark 2.1. Given a family $\mathcal{F}$ of compact sets in $\mathbb{R}^{d}$. We want to cover the space with translates of members of $\mathcal{F}$. The minimum covering density that we can reach does not change whether we require that we use every member of $\mathcal{F}$, or we may use only a subfamily. Indeed, once we have a desired covering using a sub-family, we can take a zero-density arrangement of the rest of the members of $\mathcal{F}$.

In case (a) of Theorem 1.1, there is a subfamily of $\mathcal{F}$ which consists of essentially translates of $K$. The proof of case (a) now easily follows from Remark 2.1.

We make some preparations for the proof of case (b).
Definition 2.2. A collection $\mathcal{V}$ of Lebesgue-measurable subsets of $\mathbb{R}^{d}$ is a regular family if there is a constant $C$ for which $\operatorname{diam}(V)^{d} \leq C \operatorname{vol}(V)$ holds for every $V \in \mathcal{V}$.

Definition 2.3. A collection $\mathcal{V}$ of subsets of $\mathbb{R}^{d}$ is a Vitali-covering of $E \subseteq \mathbb{R}^{d}$, if for every $x \in E$ and $\delta>0$, there is an element $U$ of $\mathcal{V}$ such that $x \in U$ and $0<\operatorname{diam}(U)<\delta$.

We recall Vitali's covering theorem [Vit08].
Theorem 2.4. Let $E \subset \mathbb{R}^{d}$ be a measurable set with finite Lebesgue-measure, and let $\mathcal{V}$ be a regular family of closed subsets of $\mathbb{R}^{d}$ that is a Vitali covering for $E$. Then there is a finite or countably infinite subcollection $\left\{U_{j}\right\} \subseteq \mathcal{V}$ of disjoint sets such that

$$
\operatorname{vol}\left(E \backslash \bigsqcup_{j} U_{j}\right)=0
$$

Proof of (b) of Theorem 1.1. We may assume that $\operatorname{vol}(K)=1$ and fix an $\varepsilon_{0}>0$.
For a subcollection $\mathcal{G}$ of $\mathcal{F}$ we denote by $\mathcal{G}_{\varepsilon}$ the subset of those elements of $\mathcal{G}$ in which the ratio of homothety does not exceed $\varepsilon$.

Now, for every $\varepsilon>0$, the total volume in $\mathcal{F}_{\varepsilon}$ is infinite. A bit more is true: for any subfamily $\mathcal{G}$ of $\mathcal{F}$ of infinite total volume, and for every $\varepsilon>0$, the total volume in $\mathcal{G}_{\varepsilon}$ is infinite.

We partition

$$
\mathcal{F}_{\varepsilon_{0}}=\left(\bigsqcup_{i} \mathcal{A}_{i}\right) \bigsqcup\left(\bigsqcup_{j} \mathcal{B}_{j}\right)
$$

into countably many sub-collections, so that the total volume in each $\mathcal{A}_{i}$ and in each $\mathcal{B}_{j}$ is infinite.

We will cover most of the cube $[0,1]^{d}$ by a subfamily of $\mathcal{A}_{1}$, in which the sum of the volumes is at most $\left(1+\varepsilon_{0} \operatorname{diam}(K)\right)^{d}$, and the rest of the cube by a subfamily of $\mathcal{B}_{1}$, in which the sum of the volumes is at most $\varepsilon_{0}$. If we can achieve this for any $\varepsilon_{0}>0$, the density bound for the whole space clearly follows.

Again, we partition

$$
\mathcal{A}_{1}=\bigsqcup_{j} \mathcal{C}_{j}
$$

into countably many subcollections, so that the total volume in each $\mathcal{C}_{j}$ is infinite.
Using Theorem 1.2 for every $j \in \mathbb{N}$, we can cover the cube $[0,1]^{d}$ by the translates of the elements $\left(\mathcal{C}_{j}\right)_{\frac{1}{j}}$. Since we use homothets of a fixed convex body, $K$, the union (over $j$ ) of these coverings is clearly a regular Vitali covering.

Therefore we can apply Theorem 2.4. There is a subcollection $\left\{U_{l}\right\}$ of disjoint sets for which

$$
\operatorname{vol}\left([0,1]^{d} \backslash \bigsqcup_{l} U_{l}\right)=\operatorname{vol}(E)=0
$$

Next, we will cover $E$ by a subcollection of $\mathcal{B}_{1}$, in which the sum of the volumes is at most $\varepsilon_{0}$. We partition $\mathcal{B}_{1}=\bigcup_{l} \mathcal{D}_{l}$ into countably many subcollection, each of infinite total volume.

Since $\operatorname{vol}(E)=0$, for every $\varepsilon^{\prime}>0$ there is collection $\mathcal{E}=\left\{K_{1}, K_{2}, \ldots,\right\}$ of homothets of $K$ so that $E \subseteq \bigcup_{l} K_{l}$ and $\sum_{l} \operatorname{vol}\left(K_{l}\right) \leq \varepsilon^{\prime}$.

Note that $\left(\mathcal{D}_{l}\right)_{\varepsilon}$ is of infinite total volume for any $\varepsilon>0$. Thus, using Theorem 1.2 for each $l$, we can cover $K_{l}$ by translates of members of a subfamily of $\mathcal{D}_{l}$ of total volume at most $C \operatorname{vol}\left(K_{l}\right)$ for some constant $C>0$. If $\varepsilon^{\prime}$ is small enough, we obtain a covering of $E$ of total volume $\varepsilon_{0}$, as promised.

Case (c) of Theorem 1.1 clearly follows from the following statement.
Theorem 2.5. Let $K$ be a convex body, and let $\mathcal{F}=\left\{\lambda_{1} K, \lambda_{2} K, \ldots\right\}$ be a family of its homothets so that the $\lambda_{i}$-s are not bounded. Then $\mathcal{F}$ permits a translative covering of $\mathbb{R}^{d}$ so that every point is covered at most $4 d$ times.

Moreover, if $K$ is smooth at the points of intersection of $K$ with supporting hyperplanes that are parallel to one of the $d$ coordinate hyperplanes, then $\mathcal{F}$ permits a translative covering of $\mathbb{R}^{d}$ so that every point is covered at most $2 d$ times.

Clearly, if an affine image of $K$ has the special property that 'coordinate-hyperplane touching points' are smooth, then the $2 d$ bound on the covering multiplicity also follows.

Proof of Theorem 2.5 in the second case. Fix $\varepsilon>0$. We may assume that $\mathcal{F}$ has an element $\mu_{0} K$ so that $Q_{0}=[-\varepsilon, \varepsilon]^{d} \subseteq \mu_{0} K \subseteq[-1,1]^{d}$. We present an algorithm to produce the desired covering. We will define inductively a sequence of cubes $Q_{i}(i \in \mathbb{N})$, which are centered at the origin and have side length at least $i$, a sequence of translation vectors $x_{1}^{1}, x_{1}^{2}, \ldots x_{1}^{2 d}, x_{2}^{1}, \ldots, x_{2}^{2 d}, x_{3}^{1}, \ldots$, and a sequence $\mu_{1}^{1} K, \mu_{1}^{2} K, \ldots \mu_{1}^{2 d} K, \mu_{2}^{1} K, \ldots, \mu_{2}^{2 d} K, \mu_{3}^{1} K, \ldots$ of elements of $\mathcal{F}$ so that the following hold with the convention $x_{0}^{j}=0$ and $\mu_{0}^{j}=\mu_{0}$ :
(1) $Q_{k} \subseteq \bigcup_{i=0}^{k} \bigcup_{j=1}^{2 d} x_{i}^{j}+\mu_{i}^{j} K$ for $k \in \mathbb{N}$


Figure 1. The squares with bold edges are $Q_{1}, Q_{2}$ and $Q_{3}$ (counting from inside out).
(2) $\left(\bigcup_{i=0}^{k} \bigcup_{j=1}^{2 d} x_{i}^{j}+\mu_{i}^{j} K\right)+B(0, \varepsilon) \subseteq Q_{k+1}$ for $k \in \mathbb{N}$
(3) $\left(x_{i}^{j}+\mu_{i}^{j} K\right) \cap\left(x_{i}^{j+d}+\mu_{i}^{j+d} K\right)=\emptyset$ for $1 \leq j \leq d$
(4) $\left(x_{i}^{j}+\mu_{i}^{j} K\right) \cap\left(x_{l}^{m}+\mu_{l}^{m} K\right)=\emptyset$ if $|i-l| \geq 2$.

Indeed, assume that we found the $x_{i}^{j}$-s, $\mu_{i}^{j}$-s and $Q_{i}$-s for $i \leq k$. Choose $Q_{k+1}$ so that

$$
\left(\bigcup_{i=0}^{k} \bigcup_{j=1}^{2 d} x_{i}^{j}+\mu_{i}^{j} K\right)+B(0, \varepsilon) \subseteq Q_{k+1} .
$$

Let $H_{i}$ denote the support hyperplane of the $i$-th facet of $Q_{k}$, and $H_{i,+}$ the half-space bounded by $H_{i}$ that does not contain $Q_{k}$. Since the set of $\lambda_{i}$-s is unbounded, by the smoothness of $K$ at the touching points with the coordinate hyperplanes, we can choose a so far unused element $\mu_{k+1}^{i} K$ of $\mathcal{F}$, and a translation vector $x_{k+1}^{i}$ such that

$$
Q_{k+1} \cap H_{i,+} \subseteq x_{k+1}^{i}+\mu_{k+1}^{i} K
$$

and

$$
\left(x_{k+1}^{i}+\mu_{k+1}^{i} K\right) \cap\left(x_{k-1}^{j}+\mu_{k-1}^{j} K\right)=\emptyset
$$

for all $j$.
Also we have that if $H_{i}(i \leq d)$ and $H_{i+d}$ support opposite sides of $Q_{k+1}$ then

$$
\left(x_{k+1}^{i}+\mu_{k+1}^{i} K\right) \cap\left(x_{k+1}^{i+d}+\mu_{k+1}^{i+d} K\right)=\emptyset,
$$

and

$$
Q_{k+1} \backslash Q_{k} \subseteq \bigcup_{i=1}^{2 d} x_{k+1}^{i}+\mu_{k+1}^{i} K
$$

Hence we can find the desired $Q_{i}$-s and translates.
Since $Q_{i}$ has side length at least $i$,

$$
\mathbb{R}^{d}=\bigcup_{i=1}^{\infty} \bigcup_{\substack{j=1 \\ 5}}^{2 d} x_{i}^{j}+\mu_{i}^{j} K
$$

Property (3) ensures that, the subfamily $\bigcup_{i=1}^{2 d} x_{k}^{i}+\mu_{k}^{i} K$ covers every point at most $d$ times, and property (4) yields that every point of $\mathbb{R}^{n}$ is covered by at most two subfamilies $\bigcup_{i=1}^{2 d} x_{k}^{i}+\mu_{k}^{i} K$, which finishes the proof.
Remark 2.6. At first, one may believe that, by some approximation argument, the condition of smoothness can be dropped in Theorem 2.5. Unfortunately, this is not the case, the standard argument does not work.
Let $K$ be a convex body in $\mathbb{R}^{d}$ and $\mathcal{F}=\left\{\lambda_{1} K, \lambda_{2} K, \ldots\right\}$ a family of its homothets, such that the coefficients $\lambda_{i}$-s are not bounded. Let $L$ be a convex body with smooth boundary such that $L \subseteq K \subseteq(1+\varepsilon) L$. Consider the family $\mathcal{F}^{\prime}=\left\{\lambda_{1} L, \lambda_{2} L, \ldots\right\}$, and follow the steps of the proof of the smooth case in Theorem 2.5 for $\mathcal{F}^{\prime}$.

We obtain a covering $x_{i}^{j}+\lambda_{i}^{j} L$ of $\mathbb{R}^{d}$, where every point is covered at most $2 d$ times. Then $x_{i}^{j}+(1+\varepsilon) \lambda_{i}^{j} L$ is also a covering. However, it may happen that $x_{i}^{j}+(1+\varepsilon) \lambda_{i}^{j} L$ covers every point infinitely many times: If $\lambda_{i}^{k}$ is sufficiently large, then $x_{i}^{j}+(1+\varepsilon) \lambda_{i}^{j} K$ may contain $B(0, i)$ for all $i$.

Proof of Theorem 2.5 in the general case. We leave the proof of the following Lemma to the reader as an exercise.

Lemma 2.7. Let $K \subseteq \mathbb{R}^{d}$ be a convex body. Then there exist $j \leq 2 d$ points $\left\{x_{1}, x_{2}, \ldots x_{j}\right\}$ on the boundary of $K$, so that $K$ is smooth in $x_{i}(1 \leq i \leq j)$ and $\bigcap_{i} H_{i+}=L$ is a bounded convex set with non-empty interior, where $H_{i+}$ is the half-space that contains $K$, bounded by the tangent hyperplane $H_{i}$ at $x_{i}$.
Let $L$ be the polytope obtained in Lemma 2.7. We may assume that $L$ contains the origin. Let $\varepsilon>0$ be fixed. We may also assume that $\mathcal{F}$ has an element $\mu_{0} K$ so that $-L_{0}=-\varepsilon L=\subseteq \mu_{0} K \subseteq-L$.

Similarly to the proof of the smooth case, we can inductively define a sequence $-\alpha_{1} L,-\alpha_{2} L, \ldots$ of homothets of $-L$, a sequence of translation vectors $x_{1}^{1}, x_{1}^{2}, \ldots x_{1}^{2 d}, x_{2}^{1}, \ldots, x_{2}^{2 d}, x_{3}^{1}, \ldots$ and a sequence $\mu_{1}^{1} K, \mu_{1}^{2} K, \ldots \mu_{1}^{2 d} K, \mu_{2}^{1} K, \ldots, \mu_{2}^{2 d} K, \mu_{3}^{1} K, \ldots$ of members of $\mathcal{F}$, so that $\alpha_{i} \geq i$ and the following hold:
(1) $-L_{k} \subseteq \bigcup_{i=0}^{k} \bigcup_{j=1}^{2 d} x_{i}^{j}+\mu_{i}^{j} K$ for $k \in \mathbb{N}$
(2) $\left(\bigcup_{i=0}^{k} \bigcup_{j=1}^{2 d} x_{i}^{j}+\mu_{i}^{j} K\right)+B(0, \varepsilon) \subseteq-L_{k+1}$ for $k \in \mathbb{N}$
(3) $\left(x_{i}^{j}+\mu_{i}^{j} K\right) \cap\left(x_{l}^{m}+\mu_{l}^{m} K\right)=\emptyset$ if $|i-l| \geq 2$.

Now, the general case of Theorem 2.5 easily follows.

### 2.1. Covering $K$ by its homothets.

Theorem 2.8. For any $\varepsilon>0$, dimension $d$ and any convex body $K$ of volume one in $\mathbb{R}^{d}$ with $o \in \operatorname{int} K$, if a family $\mathcal{F}$ of positive homothets of $K$ has total volume at least

$$
d^{2}\left\lceil\frac{-\log \varepsilon}{\log 1+\varepsilon}\right\rceil \vartheta(K) \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)}+\left(1+\frac{\varepsilon}{2}\right)^{d} 2^{d} \frac{\operatorname{vol}(K)}{\operatorname{vol}(K \cap K)} .
$$

then $\mathcal{F}$ permits a translative covering of $K$.
Theorem 1.2 clearly follows from this result. Indeed, we choose $\varepsilon=\frac{1}{d}$, and recall two facts. First, that there is a point $x \in K$ such that $\operatorname{vol}(K \cap(2 x-K)) \geq \frac{1}{2^{d}} \operatorname{vol}(K)$. And second, that by RS57, $\operatorname{vol}(K-K) \leq\binom{ 2 d}{d} \operatorname{vol}(K)$.
Proof of Theorem 2.8. First, we restate [Nas10, Theorem 1.3] in a slightly more general form than the original, which is easily obtained from the proof therein. The proof there easily yields this form.

Theorem 2.9. Let $K$ and $L$ be convex bodies in $\mathbb{R}^{d}$ with $o \in \operatorname{int} K$, and $\mathcal{F}=$ $\left\{\lambda_{1} K, \lambda_{2} K, \ldots\right\}$ be a family of its homothets with $0<\lambda_{i} \leq \lambda_{1}<1$. Assume that

$$
\sum_{i=1}^{\infty} \lambda_{i}^{d} \geq 2^{d} \frac{\operatorname{vol}\left(L+\lambda_{1} \frac{K \cap(-K)}{2}\right)}{\operatorname{vol}(K \cap(-K))}
$$

Then $\mathcal{F}$ permits a translative covering of $L$.
We fix $\varepsilon>0$. Now, we are given $\mathcal{F}=\left\{\lambda_{1} K, \lambda_{2} K, \ldots\right\}$ with $0<\lambda_{i}<1$ for all $i$. First, we consider the case when there is a subfamily $\mathcal{F}^{\prime}=\left\{\mu_{1} K, \mu_{2} K, \ldots\right\}$ of $\mathcal{F}$ in which

$$
(1+\varepsilon)^{-1} \leq \frac{\mu_{i}^{d}}{\mu_{j}^{d}} \leq(1+\varepsilon)
$$

for all $i$ and $j$, and

$$
\sum_{i=1}^{\infty} \mu_{i}^{d} \geq \vartheta(K)(1+\varepsilon) \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)}
$$

In this case, $\mathcal{F}^{\prime}$ has at least $\frac{1}{\mu_{1}^{d}(1+\varepsilon)} \vartheta(K)(1+\varepsilon) \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)}=\frac{1}{\mu_{1}^{d}} \vartheta(K) \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)}$ members.
We may assume that $\mu_{1}$ is the smallest homothety ratio in $\mathcal{F}^{\prime}$. By the main result of RZ97, we can cover $K$ by at most $\frac{\operatorname{vol}\left(K-\mu_{1} K\right) \cdot \vartheta(K)}{\operatorname{vol}\left(\mu_{1} K\right)}$ translates of $\mu_{1} K$. The statement of the Theorem in this case easily follows.

Next, we assume that there is no such subfamily $\mathcal{F}^{\prime}$. Consider the intervals

$$
\left(\varepsilon^{d}, \varepsilon^{d}(1+\varepsilon)\right],\left(\varepsilon^{d}(1+\varepsilon), \varepsilon^{d}(1+\varepsilon)^{2}\right], \ldots,\left(\varepsilon^{d}(1+\varepsilon)^{c(\varepsilon)-1}, \varepsilon^{d}(1+\varepsilon)^{c(\varepsilon)}\right]
$$

where $c(\varepsilon)=d\left\lceil\frac{-\log \varepsilon}{\log 1+\varepsilon}\right\rceil$. Since $\varepsilon^{d}(1+\varepsilon)^{c(\varepsilon)} \geq 1$, we have

$$
\sum_{\lambda_{i} K \in \mathcal{F}, \lambda_{i}^{d}>\varepsilon^{d}} \lambda_{i}^{d}<c(\varepsilon) d(1+\varepsilon) \vartheta(K) \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)} .
$$

This implies that there exists a subfamily $\mathcal{F}^{\prime}=\left\{\mu_{1} K, \mu_{2} K, \ldots\right\}$, in which

$$
\mu_{i}^{d} \leq \varepsilon^{d}
$$

and

$$
\sum_{i=1}^{\infty} \mu_{i}^{d} \geq\left(1+\frac{\varepsilon}{2}\right)^{d} 2^{d} \frac{\operatorname{vol}(K)}{\operatorname{vol}(K \cap(-K))}
$$

Then, by Theorem [2.9, $\mathcal{F}^{\prime}$ permits a translative covering of $K$.

## 3. Multiple covering - Proof of Theorem 1.3

Definition 3.1. Let $\mathcal{F}$ be a family of subsets of a base set $X$, and $k \in \mathbb{Z}^{+}$. The $k$-fold covering number of $\mathcal{F}$, denoted by $\tau_{k}(\mathcal{F})$, is the minimum cardinality of a multi-subfmaily of $\mathcal{F}$ such that each point of $X$ is contained in at least $k$ (with multiplicity) members of the subfamily.

We recall that a fractional covering of $X$ by $\mathcal{F}$ is a mapping $w$ from $\mathcal{F}$ to $\mathbb{R}^{+}$with $\sum_{x \in F \in \mathcal{F}} w(F) \geq 1$ for all $x \in X$. The total weight of a fractional covering is denoted by $w(\mathcal{F}):=\sum_{F \in \mathcal{F}} w(F)$, and its infimum is the fractional covering number of $\mathcal{F}$ :

$$
\tau^{*}(X, \mathcal{F}):=\inf \left\{w(\mathcal{F}): w: \mathcal{F} \rightarrow \mathbb{R}^{+} \text {is a fractional-covering of } X\right\}
$$

For more on (fractional) coverings, cf. [Für88] in the abstract (combinatorial) setting and PA95 and Mat02 in the geometric setting.

We will use the following simple combinatorial statement.

Lemma 3.2. Let $\mathcal{F}$ be a family of subsets of a base set $X$ of fractional covering number $\tau^{*}:=\tau^{*}(\mathcal{F})$, and $k \in \mathbb{Z}^{+}$. Then

$$
\tau_{k} \leq\left\lceil\tau^{*}\left(k+\frac{3}{2} \ln |X|+\frac{3}{2} \sqrt{(4 k+\ln |X|) \ln |X|}\right)\right\rceil \leq\left\lceil 6 \tau^{*} \max \{\ln |X|, k\}\right\rceil
$$

The proof is a standard probabilistic argument.
Proof. Let $w$ be a fractional covering of $X$ by $\mathcal{F}$ of total weight $\tau^{*}:=\tau^{*}(\mathcal{F})$, and let $m=\left\lceil\tau^{*}\left(k+\frac{3}{2} \ln |X|+\frac{3}{2} \sqrt{(4 k+\ln |X|) \ln |X|}\right)\right]$.

We pick $m$ members of $\mathcal{F}$ randomly, independently with the same distribution: at each draw, each member $F$ of $\mathcal{F}$ is picked with probabilty $w(F) / w(\mathcal{F})$. For a fixed $x \in X$, the probability that $x$ is not covered at least $k$ times by the selected family is at most $\mathbb{P}(\xi<k)$, where $\xi=\xi_{1}+\ldots+\xi_{m}$, with independent random Bernouli (ie., $0 / 1$-valued) variables $\xi_{1}, \ldots, \xi_{m}$, each of expectation $1 / \tau^{*}$. By Chernoff's inequality, $\mathbb{P}(\xi<k) \leq \exp \left(-\frac{\left(m-k \tau^{*}\right)^{2}}{3 m \tau^{*}}\right)$. Thus, $\mathbb{P}$ (there is an $x \in X$ which is not covered) $\leq$ $|X| \exp \left(-\frac{\left(m-k \tau^{*}\right)^{2}}{3 m \tau^{*}}\right)$. The lemma now clearly follows.

For two sets $K$ and $L$ in $\mathbb{R}^{d}$, we define $N_{k}(L, K)$, the $k$-fold covering number of $K$ by $L$ as the minimum number of translates of $L$ that cover $K k$-fold. Note that $N_{k}(L, K)=$ $\tau_{k}(\mathcal{F})$, where $\mathcal{F}=\left\{(x+K) \cap L: x \in \mathbb{R}^{d}\right\}$. We also define the fractional covering number of $K$ by $L$ as $N^{*}(L, K)=\tau^{*}(\mathcal{F})$.

By AAS13, Theorem 1.7], we have .

$$
\begin{equation*}
\max \left\{\frac{\operatorname{vol}(L)}{\operatorname{vol}(K)}, 1\right\} \leq N^{*}(L, K) \leq \frac{\operatorname{vol}(L-K)}{\operatorname{vol}(K)} \tag{2}
\end{equation*}
$$

for any Borel measurable sets, $K$ and $L$ is $\mathbb{R}^{d}$.
The same proof as AAS13, Theorem 1.6] (or, Nas14, Theorem 1.2]) yields
Theorem 3.3. Let $K, L$ and $T$ be bounded Borel measurable sets in $\mathbb{R}^{d}$ and let $\Lambda \subset \mathbb{R}^{d}$ be a finite set with $L \subseteq \Lambda+T$. Then

$$
N_{k}(L, K) \leq\left\lceil 6 N^{*}(L-T, K \sim T) \max \{\ln |\Lambda|, k\}\right\rceil .
$$

If $\Lambda \subset K$, then we have

$$
N_{k}(L, K) \leq\left\lceil 6 N^{*}(L, K \sim T) \max \{\ln |\Lambda|, k\}\right\rceil
$$

Proof of Theorem 1.3. We may assume that

$$
B(0,1) \subseteq K \subseteq[-d, d]^{d}
$$

Let $C=\left[-\frac{a}{2}, \frac{a}{2}\right]^{d}$ be a cube of edge length $a$, where we will set $a$ later.
Let $\delta>0$ be fixed and let $\Lambda \subseteq \mathbb{R}^{d}$ be a finite set such that $\lambda+\frac{\delta}{2}(K \cap(-K))$ is a saturated (ie. maximal) packing of $\frac{\delta}{2}(K \cap(-K))$ in $C-\frac{\delta}{2}(K \cap(-K))$. Thus $C \subseteq$ $\Lambda+\delta K \subseteq \lambda+\delta(K \cap(-K)) \subseteq \Lambda+\delta K$. By considering volume, we have that

$$
|\Lambda| \leq \frac{\operatorname{vol}\left(C-\frac{\delta}{2}(K \cap(-K))\right)}{\operatorname{vol}\left(\frac{\delta}{2}(K \cap(-K))\right)} \leq \frac{\left(a+\frac{\delta d}{2}\right)^{d} 2^{d}}{\operatorname{vol}(B(0,1))\left(\frac{\delta}{2}\right)^{d}}
$$

Equation (2) yields that

$$
\left.\begin{array}{rl}
N^{*}(C-\delta(K \cap(-K)), K & \sim \delta(K \cap(-K)))
\end{array}\right) \leq \frac{(a+d)^{d}}{\operatorname{vol}((1-\delta) K)} \leq \frac{\operatorname{v}^{*}(C-\delta K,(1-\delta) K)}{(1-\delta)^{d} \operatorname{vol}(K)} .
$$

From Theorem 3.3 we have now

$$
\begin{align*}
N_{k}(C, K) \leq & {\left[6 \frac{(a+d)^{d}}{(1-\delta)^{d} \operatorname{vol}(K)} \ln \left(\frac{\left(a+\frac{\delta d}{2}\right)^{d} 2^{d}}{\operatorname{vol}(B(0,1))\left(\frac{\delta}{2}\right)^{d}}\right)\right] \leq } \\
& 6 \frac{(a+d)^{d}}{(1-\delta)^{d} \operatorname{vol}(K)} \ln \left(\frac{\left(a+\frac{\delta d}{2}\right)^{d} 2^{d}}{\operatorname{vol}(B(0,1))\left(\frac{\delta}{2}\right)^{d}}\right)+1 . \tag{4}
\end{align*}
$$

On the other hand

$$
\begin{equation*}
\vartheta^{(k)}(K) \leq N_{k}(C, K) \frac{\operatorname{vol}(K)}{\operatorname{vol}(C)} \leq 6 \frac{(a+d)^{d}}{(1-\delta)^{d}} \ln \left(\frac{\left(a+\frac{\delta d}{2}\right)^{d} 2^{d}}{\operatorname{vol}(B(0,1))\left(\frac{\delta}{2}\right)^{d}}\right)(\operatorname{vol}(C))^{-1}+\frac{\operatorname{vol}(K)}{\operatorname{vol}(C)} \tag{5}
\end{equation*}
$$

Choose now $\delta=\frac{1}{2 d \ln d}, a=d^{2}$, and estimate $\operatorname{vol}(B(0,1))$ by the volume of the cube of side length $\frac{1}{2 \sqrt{d}}$, which is contained in $B(0,1)$.

$$
\begin{gather*}
\vartheta^{(k)} \leq 6\left(\frac{d^{2}+d}{d^{2}}\right)^{d}\left(1-\frac{1}{2 d \ln d}\right)^{-1}\left(d \ln \left(4 d^{3} \ln d+d+2+2 d^{\frac{1}{2}}\right)\right)+1 \leq \\
6 d\left(1+\frac{1}{d}\right)^{d} \exp \left(\frac{1}{\ln d}\right) \ln \left(8 d^{3} \ln d\right) \leq  \tag{6}\\
6 d\left(1+\frac{2}{\ln d}\right)(3 \ln d+\ln \ln d+\ln 8) \leq 6 e d(3 \ln d+\ln \ln d+15)
\end{gather*}
$$

yields the desired bound.

## 4. Covering the sphere by strips - A Direct Proof of Theorem 1.4

In this section, we present a direct, probabilistic proof of Theorem 1.4. We use the uniform probability measure on the sphere $\mathbb{S}^{2}$, and recall that the measure of any strip of Euclidean half-width $w$ is $w$.

Let $r=\frac{\ln N}{N}$. By a standard saturated packing argument, we may fix a set of points $v_{1}, \ldots, v_{N^{2}}$ on $\mathbb{S}^{2}$ such that the caps around $v_{i}$ of radius $r$ cover the sphere.

Let $X_{i}(i=1, \ldots, N)$ be independent random variables distributed uniformly on $\mathbb{S}^{2}$. We prove that with positive probability, the points $X_{i}$ will satisfy the conditions of the theorem.

First consider the following probability:
$Q_{1}=\mathbb{P}\left(\exists v \in \mathbb{S}^{2}:\left|\left\langle v, X_{j}\right\rangle\right| \geq \frac{10 \ln N}{N}\right.$, for all $\left.j=1, \ldots, N\right)$.
Let
$P_{i}=\mathbb{P}\left(\exists v \in \mathbb{S}^{2}:\left|v-v_{i}\right| \leq r,\left|\left\langle v, X_{j}\right\rangle\right| \geq \frac{10 \ln N}{N}\right.$, for all $\left.j=1, \ldots, N\right)$, where $i=$ $1, \ldots, N^{2}$.

The union of the events corresponding to $P_{i}$ covers the event corresponding to $Q_{1}$, because the caps around $v_{i}$ with radius $r$ cover the sphere. On the other hand, clearly, $P_{i}$ does not depend on $i$. We obtain that $N^{2} P_{1} \geq Q_{1}$.

Assume that $\left|v-v_{i}\right| \leq r$.

$$
\left|\left\langle v, X_{j}\right\rangle\right|-\left|\left\langle v-v_{i}, X_{j}\right\rangle\right| \leq\left|\left\langle v_{i}, X_{j}\right\rangle\right|,
$$

thus,

$$
\left|\left\langle v, X_{j}\right\rangle\right|-r \leq\left|\left\langle v_{i}, X_{j}\right\rangle\right| .
$$

Hence, we can estimate from above $P_{1}$ as

$$
P_{1} \leq \mathbb{P}\left(\left|\left\langle v_{1}, X_{j}\right\rangle\right| \geq 9 r, j=1, \ldots, N\right)=(1-9 r)^{N}=\left(1-9 \frac{\ln N}{N}\right)^{N}
$$

thus,

$$
P_{1} \leq e^{-9 \ln n}=N^{-9}
$$

which yields $Q_{1} \leq \frac{1}{N^{7}}<\frac{1}{2}$.
Next, for any unit vector $v$, we denote the number of points in the strip $\left|\left\langle v, X_{j}\right\rangle\right| \leq 10 r$ from the set $X_{1}, \ldots, X_{N}$ by $k_{v}$, and denote the number of points in the strip $\left|\left\langle v, X_{j}\right\rangle\right| \leq 11 r$ from the set $X_{1}, \ldots, X_{N}$ by $h_{v}$. We will bound from above the probability $Q_{2}=\mathbb{P}(\exists v \in$ $\left.\mathbb{S}^{2} \mid k_{v} \geq c \ln N\right)$, where we will fix $c$ later.

Let $R_{i}=\mathbb{P}\left(\exists v \in \mathbb{S}^{2}| | v-v_{i} \mid \leq r, k_{v} \geq c \ln N\right)$ where $i=1, \ldots, N^{2}$.
Clearly, $R_{i}$ does not depend on $i$, and $N^{2} R_{1} \geq Q_{2}$. So, we will estimate $R_{1}$ from above.
Assume that $\left|v-v_{1}\right| \leq r$.

$$
\left|\left\langle v, X_{j}\right\rangle\right|+\left|\left\langle v-v_{1}, X_{j}\right\rangle\right| \geq\left|\left\langle v_{1}, X_{j}\right\rangle\right|,
$$

thus,

$$
\left|\left\langle v, X_{j}\right\rangle\right|+r \geq\left|\left\langle v_{1}, X_{j}\right\rangle\right|
$$

We denote the floor of $c \ln N$ by $t$, and let $z=11 \ln N$.

$$
R_{1} \leq P\left(h_{v_{i}} \geq c \ln N\right) \leq\left(\frac{z}{N}\right)^{t}\binom{N}{t}
$$

By Stirling's formula we easily get that there exists a universal constant $D$ such that $\binom{N}{t} \leq \frac{D N^{N}}{t^{t}(N-t)^{(N-t)}}$. Thus,

$$
R_{1} \leq \frac{D z^{t} \cdot N^{N}}{N^{t} \cdot t^{t}(N-t)^{(N-t)}} \leq \frac{D(e z)^{t}}{t^{t}}
$$

If $c \geq 100$ then $t \geq e^{2} z$, so we have $R_{1} \leq \frac{D}{e^{t}} \leq \frac{1}{N^{3}}$, if $N$ is large enough. It follows that

$$
\begin{equation*}
Q_{2} \leq N^{2} R_{1} \leq \frac{1}{2} \tag{7}
\end{equation*}
$$

Overall, we obtained that $Q_{1}+Q_{2}<1$, which means that if $N$ is large enough, then with positive probability the points $X_{i}$ will satisfy the the conditions of the theorem, with the constant $c=100$. This completes the proof of Theorem 1.4.

Following the proof of Theorem 1.4, with a very little modification in the calculation, one can easily get the following theorem:

Theorem 4.1. There are $N$ points $x_{i}(i=1, \ldots, N)$ on $\mathbb{S}^{2}$, such that for any unit vector $v$, there are at most $c \frac{\ln N}{\ln \ln N}$ chosen points in the strip $|\langle v, x\rangle| \leq \frac{1}{N}$, where $c$ is a universal constant.

We pose the following open questions:
Conjecture 4.2. There is a function $f$ on the positive integers tending to infinity such that, for any $N$ points on $\mathbb{S}^{2}$, there is a unit vector $v$, for which the strip $|\langle v, x\rangle| \leq \frac{1}{N}$ contains at least $f(N)$ of the given points.
Conjecture 4.3. There is a function $g$ on the positive integers tending to infinity such that for any $N$ points in the unit disk on the plane, there is a strip of width $\frac{1}{N}$ containing at least $g(N)$ of the given points.

Conjecture 4.4. There is a function $h$ on the positive integers tending to infinity such that for any $N$ points on $\mathbb{S}^{2}$ and any width $w>0$, there is a unit vector $v$, for which there are at least $h(N)$ given points in the strip $|\langle v, x\rangle| \leq w$, or there is no chosen point in the strip $\mid\langle v, x>| \leq w$.
Note that Conjecture 4.2 would imply Conjecture 4.4 with $h=f$. This is because if $w \geq \frac{1}{N}$ then the definition of $f$ guarantees that, and if $w \leq \frac{1}{N}$, then computing the sum of the areas of the dual strips associated to the points $x_{i}, i=1, \ldots, N$, they cannot cover the sphere, so in that case there exists a unit vector $v$ such that there is no chosen point in the strip $|\langle v, x\rangle| \leq w$.

## 5. Some analogues of the Epsilon-net Theorem

In this section, we prove Theorems 1.5 and 1.6. Both proofs closely follow the doublesampling technique of Haussler and Welzl from (HW87).

We recall some basics notions from the theory of hypergraphs, for details, we refer to Mat02.

Definition 5.1. The shatter function of a hypergraph $\mathcal{H}$ on the set $X$ is $\pi_{\mathcal{H}}(m)=$ $\max _{A \subset X,|A|=m}|\mathcal{H}|_{A} \mid$. The Vapnik-Chervonenkis dimension (VC-dimension, in short) of $\mathcal{H}$ is the maximal $m$ for which $\pi(m)=2^{m}$ (if there is no maximum, th VC-dimension is infinite).

We recall the Sauer-Shelah Lemma Sau72, She72.
Lemma 5.2. Let $\mathcal{H}$ be a hypergraph of VC-dimension $d$. Then for any non-negative integer $m, \pi(m) \leq\binom{ m}{0}+\binom{m}{1}+\ldots+\binom{m}{d} \leq 2 m^{d}$.

In the proof of Theorem 1.5 and Theorem 1.6 we assume that the measure of every singleton is 0 , but this is not a restriction, because we can replace every singleton with measure greater than 0 with an interval having the same measure.
Proof of Theorem 1.5. We will assume, that the measure of every singleton of $X$ is 0 , to ensure that in a random sample the probability of having some element more than once is zero. The general case will follow in the following way. If $A:=\left\{p_{1}, p_{2}, \ldots\right\}$ is the set of elements of $X$ that, as singletons, are of positive measure, then we replace each element, say $p_{i}$, of $A$ by a 'labeled' interval $p_{i} \times[0,1]$. This way, we obtain the set $\hat{X}$ from $X$. We define the measure of $\hat{\mu}$ on $\hat{X}$ in such a way that the measure $\mu\left(p_{i}\right)$ is uniformly distributed on the labeled interval $p_{i} \times[0,1]$. The set family $\hat{\mathcal{F}}$ on $\hat{X}$ is essentially $\mathcal{F}$, where if $F \in \mathcal{F}$ contains $p_{i}$, then the corresponding $\hat{F} \in \hat{\mathcal{F}}$ contains the entire labeled interval $p_{i} \times[0,1]$.
Next, with this assumption of having no positive-measure singletons, let $X_{1}, \ldots, X_{2 N}$ be independent random variables according to $\mu$ taking values in $X$, where $N:=$ $\left\lfloor C_{\varepsilon}^{d} \ln (1 / \varepsilon)\right\rfloor$. Set $Q_{0}:=\left\{X_{1}, \ldots, X_{N}\right\}, Q_{1}:=\left\{X_{N+1}, \ldots, X_{2 N}\right\}$ and $Q:=Q_{0} \cup Q_{1}$.

The Epsilon-net theorem yields that the probability that $Q_{0}$ is a transversal to $\mathcal{H}$ (that is, that each edge $H \in \mathcal{H}$ intersects $Q_{0}$ ) is greater than $\frac{1}{2}$, if $C$ is sufficiently large.

For a given $H \in \mathcal{H}$, let $E_{0}^{H}$ be the event that $\left|Q_{0} \cap H\right|>C_{1} d \ln \left(\frac{1}{\varepsilon}\right)$, where $C_{1}>0$ is to be chosen later. Let $E_{1}^{H}$ be the event that $E_{0}^{H}$ holds and $\left|Q_{1} \cap H\right| \leq 2 \varepsilon N$.

Let $E_{0}$ be the union of the events $E_{0}^{H}$ for all $H \in \mathcal{H}$, that is, $E_{0}$ is the event that, for some $H \in \mathcal{H}$, we have $\left|Q_{0} \cap H\right|>C_{1} d \ln \left(\frac{1}{\varepsilon}\right)$. Let $E_{1}$ be the union of the events $E_{1}^{H}$ for all $H \in \mathcal{H}$.

We claim that $\mathbb{P}\left(E_{0}\right) \leq 2 \mathbb{P}\left(E_{1}\right)$.

Indeed,

$$
\frac{\mathbb{P}\left(E_{1}\right)}{\mathbb{P}\left(E_{0}\right)}=\mathbb{P}\left(E_{1} \mid E_{0}\right) \geq \min _{H \in \mathcal{H}} \mathbb{P}\left(E_{1}^{H} \mid E_{0}^{H}\right)>1 / 2
$$

by Markov's inequality.
Thus, it is sufficient to show that $\mathbb{P}\left(E_{1}\right)<\frac{1}{4}$ to obtain the theorem.
Next, we sample in a different way. We permute the indices of the variables $X_{1}, \ldots, X_{2 N}$ with a random permutation (taking each permutation with equal probability), and denote the resulting variables as $Y_{1}, \ldots, Y_{2 N}$. They are again independent. We estimate the probability of the event $E_{1}$ for these variables.

We fix a $2 N$-element subset $R$ of $X$, and let $L:=H \cap R$. We estimate the probability of the event $E_{1}^{H}$ under the condition that $Q=R$. By Lemma 5.2, there are at most $2(2 N)^{d}$ possibilities for $L$, so we have

$$
\begin{equation*}
\mathbb{P}\left(E_{1} \mid Q=R\right) \leq 2(2 N)^{d} \max _{H \in \mathcal{H}} \mathbb{P}\left(E_{1}^{H} \mid Q=R\right) \tag{8}
\end{equation*}
$$

We fix $H \in \mathcal{H}$. If $t:=|L|<C_{1} d \ln \left(\frac{1}{\varepsilon}\right)$, then $E_{1}^{H}$ does not hold. We consider the case when $t \geq C_{1} d \ln \left(\frac{1}{\varepsilon}\right)$.
In order to bound $\mathbb{P}\left(E_{1}^{H} \mid Q=R\right)$, we first note the following simple combinatorial fact. Let $V$ be a subset of $L$ with $0 \leq m:=|V| \leq t$. Then,

$$
\mathbb{P}\left(Q_{1} \cap H=V \mid Q=R\right)=\binom{2 N-t}{N-m} /\binom{2 N}{N} \leq\binom{ 2 N-t}{N-\lfloor t / 2\rfloor} /\binom{2 N}{N} \frac{D N^{3}}{2^{t}}
$$

where $D$ is a universal constant. Since, for any $0 \leq m \leq t$, we have $\binom{t}{m}$ ways to choose an $m$-element subset of $L$. Thus,

$$
\mathbb{P}\left(E_{1}^{H} \mid Q=R\right) \leq D \cdot\left(\binom{t}{0}+\binom{t}{1}+\ldots+\binom{t}{\lfloor 2 \varepsilon N\rfloor}\right) 2^{-t} N^{3} \leq 3 D \varepsilon N\binom{t}{\lfloor 2 \varepsilon N\rfloor} 2^{-t} N^{3},
$$

using $t>4 \varepsilon N$ if $C_{1}$ is large enough compared to $C$. By Stirling's approximation, with the notations $l:=D N^{3}, k:=\lfloor 2 \varepsilon N\rfloor$ and $r:=t / k$, if $C_{1}$ is large enough compared to $C$, then $r>10$, and the right hand side is less than

$$
\frac{3 l \varepsilon N}{2^{t}} \cdot \frac{t^{t}}{k^{k}(t-k)^{t-k}} \leq \frac{2 l k r^{k}}{2^{r k}} \cdot\left(\frac{r}{r-1}\right)^{(r-1) k}<\left(\frac{3}{4}\right)^{(r-1) k} \cdot l<\left(\frac{3}{4}\right)^{\frac{C_{1} d \ln \left(\frac{1}{\varepsilon}\right)}{2}} \cdot l
$$

if $C_{1}$ (and hence, $r$ ) is sufficiently large. So, by (8), it is sufficient to show, that if $C_{1}$ is large enough, then

$$
2(2 N)^{d}\left(\frac{3}{4}\right)^{\frac{C_{1} d \ln \left(\frac{1}{\varepsilon}\right)}{2}} \cdot D N^{3}<1 / 4 .
$$

Clearly, it follows from

$$
D\left(4 C \frac{d}{\varepsilon} \ln (1 / \varepsilon)\right)^{d+3}\left(\frac{3}{4}\right)^{\frac{C_{1} d \ln \left(\frac{1}{\varepsilon}\right)}{2}}<1 / 4
$$

The latter holds by the restriction $\varepsilon \leq 1 / d$, if $C_{1}$ is large enough, because:

$$
\sqrt{D}(4 C d)^{d+3}\left(\frac{3}{4}\right)^{\frac{C_{1} d \ln \left(\frac{1}{\varepsilon}\right)}{4}}<\frac{1}{4}
$$

and

$$
\sqrt{D}\left(\frac{\ln (1 / \varepsilon)}{\varepsilon}\right)^{d+3}\left(\frac{3}{4}\right)^{\frac{C_{1} d \ln \left(\frac{1}{\varepsilon}\right)}{4}}<1
$$

Thus, by (8), we have that $\mathbb{P}\left(E_{1} \mid Q=R\right)<1 / 4$. Since $R$ was arbitrary, we obtain $\mathbb{P}\left(E_{1}\right)<1 / 4$ completing the proof of the first part of Theorem 1.5. The second part follows by the same calculation and the inequality

$$
D\left(4 C \frac{d}{\varepsilon} \ln \left(\frac{1}{\varepsilon}+1\right)\right)^{d+3}\left(\frac{3}{4}\right)^{\frac{d \ln d \ln \left(\frac{1}{\varepsilon}+1\right)}{2}}<1 / 4 .
$$

Proof of Theorem 1.6. Similarly to the proof of Theorem 1.5, we will assume, that the measure of every singleton of $X$ is 0 .

Let $X_{1}, \ldots, X_{N(\ln N+1)}$ be independent random variables according to $\mu$ taking values in $X$. Set $Q_{0}:=\left\{X_{1}, \ldots, X_{N}\right\}, Q_{1}:=\left\{X_{N+1}, \ldots, X_{N(\ln N+1)}\right\}$ and $Q:=Q_{0} \cup Q_{1}$. Denote by $r=C d \frac{\ln N}{\ln \ln N}$, where $C$ is to be chosen later.

For a given $H \in \mathcal{H}$, let $E_{0}^{H}$ be the event that $\left|Q_{0} \cap H\right|>r$. Let $E_{1}^{H}$ be the event that $E_{0}^{H}$ holds and $\left|Q_{1} \cap H\right|=\emptyset$.

Let $E_{0}$ be the union of the events $E_{0}^{H}$ for all $H \in \mathcal{H}$, that is, $E_{0}$ is the event that, for some $H \in \mathcal{H}$, we have $\left|Q_{0} \cap H\right|>r$. Let $E_{1}$ be the union of the events $E_{1}^{H}$ for all $H \in \mathcal{H}$.

We claim that $\mathbb{P}\left(E_{0}\right) \leq N^{2} \mathbb{P}\left(E_{1}\right)$.
Indeed,

$$
\frac{\mathbb{P}\left(E_{1}\right)}{\mathbb{P}\left(E_{0}\right)}=\mathbb{P}\left(E_{1} \mid E_{0}\right) \geq \min _{H \in \mathcal{H}} \mathbb{P}\left(E_{1}^{H} \mid E_{0}^{H}\right)>\left(\frac{N-1}{N}\right)^{N \ln N} \geq \frac{1}{N^{2}},
$$

where, in the last inequality we used the fact that $(1-x / 2) \geq e^{-x}$ for $0<x<1.59$.
Thus, it is sufficient to show that $\mathbb{P}\left(E_{1}\right)<\frac{1}{N^{2}}$ to obtain the theorem.
Next, we sample in a different way. We permute the indices of the variables $X_{1}, \ldots, X_{N(\ln N+1)}$ with a random permutation (taking each permutation with equal probability), and let the resulting variables be $Y_{1}, \ldots, Y_{N(\ln N+1)}$. They are again independent random variables. We will estimate the probability of the event $E_{1}$ for these variables.

We fix a subset $R$ of $X$ of $N(\ln N+1)$ elements, and let $L:=H \cap R$. We estimate the probability of the event $E_{1}^{H}$ under the condition that $Q=R$. By Lemma 5.2, there are at most $2(N \ln N)^{d}$ possibilities for $L=H \cap R$, so we have

$$
\begin{equation*}
\mathbb{P}\left(E_{1} \mid Q=R\right) \leq 2(N \ln N)^{d} \max _{H \in \mathcal{H}} \mathbb{P}\left(E_{1}^{H} \mid Q=R\right) \tag{9}
\end{equation*}
$$

We fix $H \in \mathcal{H}$. If $t:=|L| \leq r$, then $E_{1}^{H}$ does not hold.
If $t>r$, then

$$
\mathbb{P}\left(E_{1}^{H} \mid Q=R\right) \leq \frac{\binom{N}{t}}{\binom{N(\ln N+1)}{t}} \leq\left(\frac{N}{N \ln N}\right)^{t} \leq\left(\frac{1}{\ln N}\right)^{r}=N^{-C d} .
$$

If $C$ is large enough, then by $(9), \mathbb{P}\left(E_{1} \mid Q=R\right)<1 / N^{2}$. Since this bound does not depend on the choice of $R$, we have $\mathbb{P}\left(E_{1}\right)<1 / N^{2}$ finishing the proof of Theorem 1.6.

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