The Visible Perimeter of an Arrangement of Disks*

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Abstract

Given a collection of n opaque unit disks in the plane, we want to find a stacking order for them that maximizes their *visible perimeter*, the total length of all pieces of their boundaries visible from above. We prove that if the centers of the disks form a *dense* point set, *i.e.*, the ratio of their maximum to their minimum distance is $O(n^{1/2})$, then there is a stacking order for which the visible perimeter is $\Omega(n^{2/3})$. We also show that this bound cannot be improved in the case of a sufficiently small $n^{1/2} \times n^{1/2}$ uniform grid. On the other hand, if the set of centers is dense and the maximum distance between them is small, then the visible perimeter is $O(n^{3/4})$ with respect to any stacking order. This latter bound cannot be improved either.

Finally, we address the case where no more than c disks can have a point in common.

These results partially answer some questions of Cabello, Haverkort, van Kreveld, and Speckmann.

Keywords: Visible perimeter, disk, unit disk, dense set.

1 Introduction

In cartography and data visualization, one often has to place similar copies of a symbol, typically an opaque disk, on a map or a figure at given locations [De99], [Gr90]. The size of the symbol is sometimes proportional to the quantitative data associated with the location. On a cluttered map, it is difficult to identify the symbols. Therefore, it has been investigated in several studies how to minimize the amount of overlap [GrC78], [SlM03].

In the present note, we follow the approach of Cabello, Haverkort, van Kreveld, and Speckmann [CaH10]. We assume that the symbols used are opaque circular disks of the same size. Given a collection $\mathcal{D}$ of n distinct *unit* disks in the (x, y)-plane, a *stacking order* is a one-to-one assignment $f : \mathcal{D} \to \{1, 2, \ldots, n\}$. We consider the integer $f(D)$ to be the z-coordinate of the disk $D \in \mathcal{D}$. The *map* corresponding to this stacking order is the 2-dimensional view of this arrangement from the point at negative infinity of the z-axis (for notational convenience,

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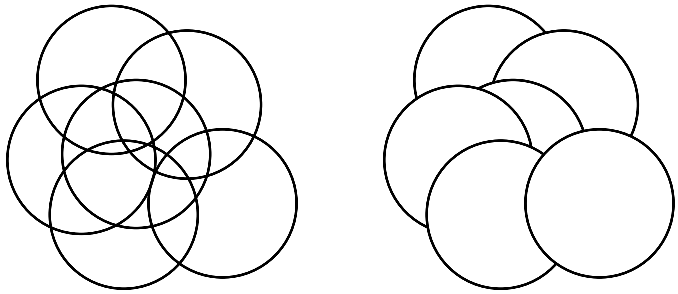

Figure 1: Left: A collection of unit disks in the plane. Right: A stacking order for them.

we look at the arrangement from below rather than from above.) In particular, for the lowest disk D, we have $f(D) = 1$, and this disk, including its full perimeter, is visible from below. The total length of the boundary pieces of the disks visible from below is the *visible perimeter* of $\mathcal{D}$ with respect to the stacking order f, denoted by visible$(\mathcal{D}, f)$. We are interested in finding a stacking order for which the visible perimeter of $\mathcal{D}$ is as large as possible. See Figure 1.

There are other situations in which this setting is relevant. Sometimes the vertices of a graph are not represented as points but as circles of a given radius. It may happen that some vertices overlap in the visualization (especially if they have further constraints on their geometric position), and then it becomes important to choose a convenient stacking order that maximizes the visible perimeter.

Given an integer n, we define

$$v(n) = \inf_{|\mathcal{D}|=n} \max_{f} \text{visible}(\mathcal{D}, f), \tag{1}$$

where the maximum is taken over all stacking orders f. We would like to describe the asymptotic behavior of $v(n)$, as n tends to infinity.

Cabello *et al.* have already noted that $v(n) = \Omega(n^{1/2})$; in other words, every set $\mathcal{D}$ of n disks of unit radii admits a stacking order with respect to which its visible perimeter is $\Omega(n^{1/2})$. Indeed, by a well-known result or Erdős and Szekeres [ErSz35], we can select a sequence of $\lceil n^{1/2} \rceil$ disks $D_i \in \mathcal{D}$ $(1 \le i \le \lceil n^{1/2} \rceil)$ such that their centers form a monotone sequence. More precisely, letting x_i and y_i denote the coordinates of the center of D_i, we have $x_1 \le x_2 \le x_3 \le \ldots$ and either $y_1 \le y_2 \le y_3 \le \ldots$ or $y_1 \ge y_2 \ge y_3 \ge \ldots$. Then, in any stacking order f such that $f(D_i) = i$ for every i, $1 \le i \le \lceil n^{1/2} \rceil$, a full quarter of the perimeter of each D_i $(1 \le i \le \lceil n^{1/2} \rceil)$ is visible from below. Therefore, the visible perimeter of $\mathcal{D}$ with respect to f satisfies

$$\text{visible}(\mathcal{D}, f) \ge \frac{\pi}{2} \lceil n^{1/2} \rceil.$$

At the problem session of *EuroCG'11* (Morschach, Switzerland), Cabello, Haverkort, van Kreveld, and Speckmann asked whether $v(n) = \Omega(n)$; in other words, does there exist a positive constant c such that every set of n unit disks in the plane admits a stacking order, with respect to which its visible perimeter is at least cn? We answer this question in the negative; *cf.* Theorems 2 and 5 below.

Given a set of points P in the plane, let $\mathcal{D}(P)$ denote the collection of disks of radius 1 centered at the elements of P. For any positive real ε, let εP stand for a similar copy of P, scaled by a factor of ε. For a stacking order f of $\mathcal{D}(P)$ we will study the quantity visible($\mathcal{D}(\varepsilon P), f$). (Note the slight abuse of notation: We denote the stacking order of $\mathcal{D}(P)$ and the corresponding stacking order of $\mathcal{D}(\varepsilon P)$ by the same symbol f. The two orders are also identified in Lemmas 1 and 7 and in Theorems 2, 3, and 5.) It is not hard to verify that, as ε gets smaller, the function visible($\mathcal{D}(\varepsilon P), f$) decreases. To see this, it is enough to observe, as was also done by Cabello *et al.* (unpublished), that as we contract the set of centers, the part of the boundary of each unit disk visible from below shrinks. As we will see in Lemma 7, the limit in the following lemma has a simple alternative geometric interpretation.

Lemma 1. *For every point set P in the plane and for every stacking order f of the collection of disks $\mathcal{D}(P)$, we have*

$$\text{visible}(\mathcal{D}(P), f) \geq \lim_{\varepsilon \to 0} \text{visible}(\mathcal{D}(\varepsilon P), f).$$

As in [AlKP89], [Va92], and [Va96], we consider C-*dense* n-element point sets P, *i.e.*, point sets in which the ratio of the maximum distance between two points to the minimum distance satisfies

$$\frac{\max(|pq| : p, q \in P)}{\min(|pq| : p, q \in P, p \neq q)} \leq Cn^{1/2}.$$

(The above ratio is sometimes called the *spread* of P [Er03]; thus, we consider point sets with spread at most $Cn^{1/2}$.)

Theorem 2. *For any C-dense n-element point set P in the plane and for any stacking order f, we have*

$$\lim_{\varepsilon \to 0} \text{visible}(\mathcal{D}(\varepsilon P), f) \leq C' n^{3/4},$$

where C' is a constant depending only on C.

The order of magnitude of the upper bound in Theorem 2 cannot be improved:

Theorem 3. *For every positive integer n, there exists a 4-dense n-element point set P_n in the plane and a stacking order f such that*

$$\lim_{\varepsilon \to 0} \text{visible}(\mathcal{D}(\varepsilon P_n), f) \geq n^{3/4}.$$

In the general case, where P is an arbitrary n-element point set in the plane, we have been unable to improve on the easy lower bound

$$\max_f \text{visible}(\mathcal{D}(P), f) = \Omega(n^{1/2}),$$

sketched above. However, under special assumptions on P, we can do better.

Theorem 4. *Every C-dense n-element point set P in the plane admits a stacking order f with*

$$\text{visible}(\mathcal{D}(P), f) \geq C'' n^{2/3},$$

where $C'' > 0$ depends only on C.

In particular, Theorem 4 provides an $\Omega(n^{2/3})$ lower bound for the visible perimeter of a collection of n unit disks centered at the points of an $n^{1/2} \times n^{1/2}$ uniform grid, under a suitable stacking order. If the side length of the grid is very small, this is better than the line-by-line "lexicographic" stacking order, for which the visible perimeter is only $\Theta(n^{1/2} \log n)$. It turns out that in this case there is no stacking order for which the order of the magnitude of the visible perimeter would exceed $n^{2/3}$.

Theorem 5. *Let n be a perfect square and let G_n denote an $n^{1/2}$ by $n^{1/2}$ uniform grid in the plane. For any stacking order f, we have*

$$\lim_{\varepsilon \to 0} \text{visible}(\mathcal{D}(\varepsilon G_n), f) = O(n^{2/3}).$$

Consequently, we have $v(n) = O(n^{2/3})$.

Lemma 1 implies that the worst collections of disks are those whose centers are very close to each other, so all disks have a point in common. This is, of course, not a realistic assumption in the labeling problem in cartography that has motivated our investigations. In practical applications, only a bounded number of unit disks share a point. For such a case, we have the following result:

Theorem 6. *Let $\mathcal{D}$ be a collection of n unit disks in which at most c disks have a point in common. Then there exists a stacking order f for which*

$$\text{visible}(\mathcal{D}, f) = \Omega(v(c)n/c),$$

where $v(c)$ is given in (1). This bound is worst-case asymptotically tight.

In Section 2, we establish Theorems 2 and 3. The proof of Theorem 4 is presented in Section 3. In Section 4, we consider the square grid and present a much simpler proof of this special case of Theorem 4 based on Jarnik's theorem [Ja25]; we then prove Theorem 5, which states that the bound of Theorem 4 is tight in this case. In Section 5, we prove Theorem 6. The last section contains concluding remarks and open problems.

2 Dense Sets with Largest Visible Perimeter

In this section, we prove Theorems 2 and 3.

First, we express the limit of visible perimeters in a simpler form. Given a set of points P in the plane, let $\text{conv}\, P$ stand for its convex hull. Let $D(p)$ denote the unit disk centered at p and let $\mathcal{D}(P)$ stand for the set $\{D(p) : p \in P\}$.

Fix an orthogonal system of coordinates in the plane. For any point $p = (x, y)$ and for any $\varepsilon > 0$, let εp denote the point with coordinates $(\varepsilon x, \varepsilon y)$.

Lemma 7. *Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of points in the plane, let $\varepsilon > 0$, and let f be the stacking order of $\mathcal{D}(\varepsilon P)$ given by $f(D(\varepsilon p_i)) = i$ for $i = 1, 2, \ldots, n$.*

We have

$$\lim_{\varepsilon \to 0} \text{visible}(\mathcal{D}(\varepsilon P), f) = \sum_{i=1}^{n} \tau_i,$$

where $\tau_1 = 2\pi$, and for all other indices, $\tau_i = 0$ if p_i belongs to $\text{conv}\{p_1, p_2, \ldots, p_{i-1}\}$, and τ_i is equal to the external angle of the convex polygon $\text{conv}\{p_1, p_2, \ldots, p_i\}$ at vertex p_i, otherwise.

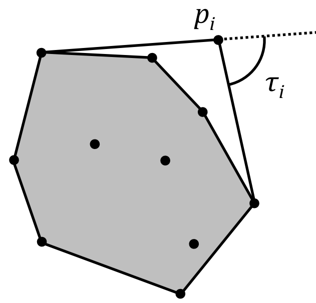

Figure 2: If p_i lies outside the convex hull of the preceding points, then τ_i is defined as the external angle of the polygon $\operatorname{conv}\{p_1, \ldots, p_i\}$ at vertex p_i.

See Figure 2.

Proof of Lemma 7. We prove that the contribution of $\mathcal{D}(\varepsilon p_i)$ to the visible perimeter tends to τ_i as $\varepsilon \to 0$ for each $1 \leq i \leq n$.

Since $D(\varepsilon p_1)$ is the lowest disk in $\mathcal{D}(\varepsilon P)$, its whole boundary is visible from below. Therefore, its contribution is 2π. Let $i > 1$. If p_i belongs to the interior of $\operatorname{conv}\{p_1, p_2, \ldots, p_{i-1}\}$, then there is a threshold $\varepsilon_0 > 0$ such that

$$D(\varepsilon p_i) \subset \bigcup_{j=1}^{i-1} D(\varepsilon p_j),$$

for every $\varepsilon < \varepsilon_0$. In this case, no portion of the boundary of $D(\varepsilon p_i)$ is visible from below, provided that ε is sufficiently small. If p_i lies on the boundary of $\operatorname{conv}\{p_1, p_2, \ldots, p_i\}$, then it is in between some points p_j and p_k with $1 \leq j < k < i$ and although $\mathcal{D}(\varepsilon p_i)$ will not be entirely covered by earlier disks for any $\varepsilon > 0$, the part of its boundary outside $\mathcal{D}(\varepsilon p_j) \cup \mathcal{D}(\varepsilon p_k)$ tends to zero as $\varepsilon \to 0$.

Finally, if p_i lies outside $\operatorname{conv}\{p_1, \ldots, p_{i-1}\}$, then it is a vertex of $\operatorname{conv}\{p_1, \ldots, p_i\}$. Consider the external unit normal vectors to the two sides of $\operatorname{conv}\{p_1, \ldots p_i\}$ that meet at εp_i (or in case the convex hull is a single segment, the two unit normal vectors for this segment). Drawing these vectors from εp_i, the arc on the boundary of $\mathcal{D}(p_i)$ between them is of length τ_i and it is not covered by $\bigcup_{j=1}^{i-1} D(\varepsilon p_j)$. Thus, it is visible from below, and, as $\varepsilon \to 0$, the total contribution of the remaining part of the boundary of $D(\varepsilon p_i)$ to the visible perimeter tends to 0, concluding the proof. □

Theorem 2. *For any C-dense n-element point set P in the plane and for any stacking order f, we have*

$$\lim_{\varepsilon \to 0} \operatorname{visible}(\mathcal{D}(\varepsilon P), f) \leq C' n^{3/4},$$

where C' is a constant depending only on C.

Proof. Consider a C-dense point set P in the plane and let f be a stacking order for $\mathcal{D}(P)$. Using Lemma 7, it is enough to prove $\sum_{i=1}^{n} \tau_i \leq C' n^{3/4}$ for the angles τ_i defined in the lemma. As $\tau_i = 0$ whenever p_i is contained in $\operatorname{conv}\{p_1, \ldots, p_{i-1}\}$, we can assume this is never the case.

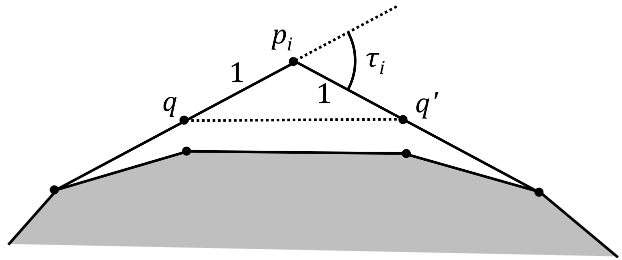

Figure 3: The triangle $p_i q q'$ lies entirely outside the convex hull of $p_1, \ldots, p_{i-1}$.

Since the quantity $\sum \tau_i$ is independent of scale, we can assume without loss of generality that the minimum distance between points is 1; thus, the maximum distance (diameter) is at most $Cn^{1/2}$. We write $P = \{p_1, p_2, \ldots, p_n\}$ with $f(D(p_i)) = i$.

For every i ($1 \leq i \leq n$), let per(i) denote the perimeter of conv$\{p_1, p_2, \ldots, p_i\}$. We define the perimeter of a segment to be twice its length and the perimeter of a point to be 0. Let $2 \leq i \leq n$, consider the two sides of the polygon conv$\{p_1, p_2, \ldots, p_i\}$ meeting at p_i, and denote by q and q' the points on these sides at unit distance from p_i. Since no point of P is closer to p_i than 1, the triangle $p_i q q'$ does not contain any element of $\{p_1, p_2, \ldots, p_{i-1}\}$. (See Figure 3.) Hence, conv$\{p_1, p_2, \ldots, p_{i-1}\}$ is contained in the convex region obtained from conv$\{p_1, p_2, \ldots, p_i\}$ by cutting off the triangle $p_i q q'$. (In the degenerate case when conv$\{p_1, \ldots, p_i\}$ is a segment, we have $q = q'$, and the empty "triangle" becomes just a unit segment.) This observation implies that the perimeter of conv$\{p_1, p_2, \ldots, p_{i-1}\}$ satisfies

$$\text{per}(i-1) \leq \text{per}(i) - |p_i q| - |p_i q'| + |qq'| = \text{per}(i) - 2 + 2\cos\frac{\tau_i}{2} \leq \text{per}(i) - \frac{\tau_i^2}{5}.$$

Here we used that the external angle of the triangle $p_i q q'$ at vertex p_i is τ_i.

Thus, we have

$$\text{per}(i) - \text{per}(i-1) \geq \frac{\tau_i^2}{5},$$

for all $i > 1$. Adding up these inequalities, we obtain

$$\text{per}(n) \geq \sum_{i=2}^{n} \frac{\tau_i^2}{5}.$$

Since per(n) is at most π times the diameter of P, that is, $\text{per}(n) \leq \pi C n^{1/2}$, we have

$$\sum_{i=2}^{n} \tau_i^2 \leq 5\pi C n^{1/2}.$$

Applying the relationship between the arithmetic and quadratic means, we can conclude that

$$\sum_{i=2}^{n} \tau_i \leq (n-1)^{1/2} \left(\sum_{i=2}^{n} \tau_i^2 \right)^{1/2} < (5\pi C)^{1/2} n^{3/4}.$$

Taking into account that $\tau_1 = 2\pi$, the theorem follows by Lemma 7. $\square$

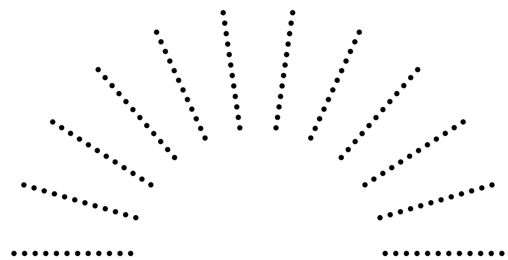

Figure 4: A dense point set that has a good stacking order.

Theorem 3. *For every positive integer n, there exists a 4-dense n-element point set P_n in the plane and a stacking order f such that*

$$\lim_{\varepsilon \to 0} \text{visible}(\mathcal{D}(\varepsilon P_n), f) \geq n^{3/4}.$$

Proof. Suppose for simplicity that $n = k^2$ for some integer $k \geq 3$. Our point set P_n consists of the points having polar coordinates $(r, \theta) = (i, j\pi/(k-1))$ for $i \in \{k, k+1, \ldots, 2k-1\}$ and $j \in \{0, 1, \ldots, k-1\}$. See Figure 4. The smallest distance between two points in P_n is 1, and the largest distance is less than $4k$; thus, P_n is 4-dense, as required.

Our stacking order f takes the points by increasing r, and for each r by increasing θ.

We apply Lemma 7 and calculate the sum of the external angles determined by f. Denote by C_i the circle of radius i centered at the origin. Consider a point $p \in P_n$ on C_i. Let ℓ be the ray leaving p towards the right tangent to C_i, and let ℓ' be the ray leaving p towards the left tangent to C_{i-1}. Let q be the point of tangency between ℓ' and C_{i-1}. Then all the points of P_n that precede p in the order f lie below ℓ and ℓ'. Thus, the external angle τ contributed by p is at least the supplement α of the angle between ℓ and ℓ'. We have $\alpha = \measuredangle p0q \geq \sin\alpha = \sqrt{2i-1}/i \geq n^{-1/4}$. The theorem follows. □

3 All Dense Sets Have Good Stacking Orders

We now turn to Theorem 4.

Theorem 4. *Every C-dense n-element point set P in the plane admits a stacking order f with*

$$\text{visible}(\mathcal{D}(P), f) \geq C'' n^{2/3},$$

where $C'' > 0$ depends only on C.

Throughout this section, let P be a C-dense n-point set in the plane. We will define a stacking order f for $\mathcal{D}(P)$ for which the external angles τ_i defined in Lemma 7 satisfy $\sum_{i=1}^{n} \tau_i \geq C'' n^{2/3}$, for some constant $C'' > 0$ depending only on C. Then the theorem follows from Lemma 7.

Assume without loss of generality that the minimum distance in P is 1. Then, since P is C-dense, there exists a disk of radius $Cn^{1/2}$ that contains all of P. Let D be such a disk, and let K be a circle of radius $2Cn^{1/2}$ concentric with D.

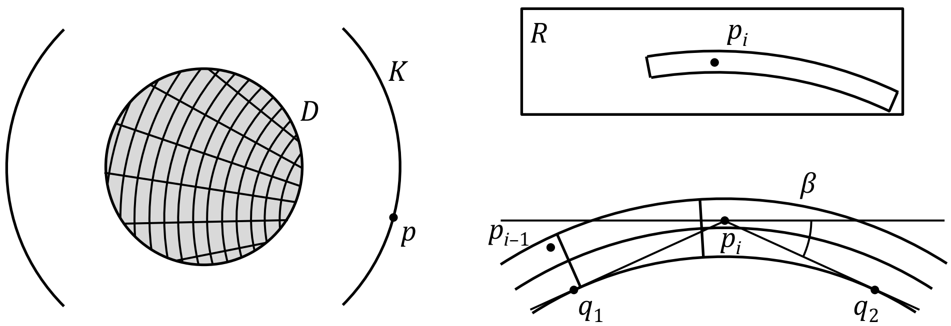

Figure 5: Left: Partition of D into annular sectors centered at a point $p \in K$. Top right: The sector containing p_i is contained in the rectangle R centered at p_i. Bottom right: Point p_i contributes external angle at least β.

Given a point $p \in K$, we define a family $F = F(p)$ of annular sectors that disjointly cover the plane, as follows: For each positive integer i, let $K_i = K_i(p)$ be a circle centered at p with radius $in^{-1/6}$; then divide each annulus between two consecutive circles into sectors of angular length $\alpha = C^* n^{-1/3}$ for a large enough constant C^* (as will be specified below). See Figure 5 (left).

Note that each annular sector that intersects D has area $\Theta(1)$ (since the radius of such a sector is $\Theta(n^{1/2})$). The number of annular sectors that intersect D is $\Theta(n^{1/2} n^{1/6} n^{1/3}) = \Theta(n)$. Call a sector *occupied* if it contains at least one point of P.

Lemma 8. *There exists a point $p \in K$ for which $\Omega(n)$ sectors of $F(p)$ are occupied.*

Proof. Choose p uniformly at random on K and construct the sectors using p and dividing the annuli into the correct-length sectors in an arbitrary way. For each point $p_i \in P$, define the random variable $n(p_i)$ to be the number of points of P contained in the sector of $F(p)$ that contains p_i. We claim that the expected value $E[n(p_i)]$ of $n(p_i)$ satisfies

$$E[n(p_i)] \le k$$

for some constant k.

Indeed, let $R = R_{p_i}(p)$ be a rectangle centered at p_i, with dimensions $(k' n^{1/6}) \times (k' n^{-1/6})$, and with short sides parallel to the line pp_i, for an appropriate constant k'. If k' is large enough (but constant with respect to n), then R completely contains the sector of $F(p)$ that contains p_i. See Figure 5 (top right). Thus, it suffices to bound the expected number of points of P in R. Note that, as p rotates around K, R rotates around its center together with p.

Partition the plane into annuli centered at p_i by tracing circles around p_i of radii $1, 2, 4, 8, \ldots$. The annulus with inner radius r and outer radius $2r$ contains at most $k_2 r^2$ points of P, for some constant k_2. Each such point has probability at most $k_3 n^{-1/6} r^{-1}$ of falling in R (over the choice of p), for another constant k_3; therefore, the expected contribution of this annulus to $n(p_i)$ is at most $k_2 k_3 r n^{-1/6}$. Summing up for all annuli with inner radius $r \le k' n^{1/6}$, we obtain that $E[n(p_i)] \le k$ for some constant k, as claimed.

Now, call point p_i *isolated* if $n(p_i) \le 2k$. By Markov's inequality, each point p_i has probability at least $1/2$ of being isolated. Therefore, the expected number of isolated points is at least $n/2$. There must exist a p that achieves this expectation, and for it we obtain at least $n/(4k)$ occupied sectors, proving the lemma. □

Proof of Theorem 4. Fix a point p for which $F(p)$ has $\Omega(n)$ occupied sectors. Color the sectors with four colors, using colors 1 and 2 alternatingly on the odd-numbered annuli and colors 3 and 4 alternatingly on the even-numbered annuli.

There must be a color for which $\Omega(n)$ sectors are occupied. Consider only the occupied sectors with this color. Let these sectors be $S_1, S_2, \ldots, S_m$, listed by increasing distance from p, and for each fixed distance, in clockwise order around p. Select one point $p_i \in P \cap S_i$ from each of these sectors. Let the stacking order f start with these points, that is, $f(\mathcal{D}(p_i)) = i$ for $i = 1, \ldots, m$. The order of the remaining points in P is arbitrary.

We claim that each selected point p_i contributes an external angle of $\tau_i = \Omega(n^{-1/3})$, which implies that $\sum \tau_i = \Omega(n^{2/3})$, as desired.

Indeed, consider the i-th selected point p_i. Suppose without loss of generality that p lies directly below p_i. Let K_k and K_{k+1} be the inner and outer circles bounding the annulus that contains p_i. Trace rays z_1 and z_2 from p_i tangent to K_{k-1}, touching K_{k-1} at points q_1 and q_2. See Figure 5 (bottom right).

Every point p_j, $j < i$, that is *not* contained in the same annulus as p_i lies below these rays. Moreover, the angle β that these rays make with the horizontal is $\Theta(n^{-1/3})$: Consider, for example, the ray z_1. The triangle pp_iq_1 is right-angled, with angle $\measuredangle p_ipq_1 = \beta$. We have $pq_1 = \Theta(n^{1/2})$ and $pp_i = pq_1 + \Theta(n^{-1/6})$. It follows that $p_iq_1 = \Theta(n^{1/6})$, and so $\beta \approx \tan \beta = p_iq_1/pq_1 = \Theta(n^{-1/3})$.

Now suppose that p_{i-1} lies in the same annulus as p_i. If the constant C^* in the definition of α is chosen large enough, then p_{i-1} must have a smaller y-coordinate than p_i. (In the worst case, p_i lies near the bottom-left corner of its sector and p_{i-1} lies near the top-right corner of its sector.)

Thus, p_i contributes external angle $\tau_i \geq \beta = \Omega(n^{-1/3})$, as claimed. □

4 The "Worst" Dense Set: the Grid

In this section, we assume that n is a square number and G_n denotes an $n^{1/2}$ by $n^{1/2}$ integer grid. Note that G_n is a $\sqrt{2}$-dense set consisting of n points.

As we mentioned in the Introduction, in the special case where $P = \varepsilon G_n$, Theorem 4 has a simple proof. For $\mathcal{D}(\varepsilon G_n)$, one can produce a stacking order with large visible perimeter using the following greedy algorithm (which can also be applied to any other point set P): Set $P_n = G_n$, and select a vertex of $\mathrm{conv}(P_n)$ whose external angle is maximum. Let this vertex be p_n, the last element in the desired order f_{greedy}. Repeat the same step for the set $P_{n-1} = P_n \setminus \{p_n\}$, and continue in this fashion until the first element p_1 gets defined.

By Jarnik's theorem [Ja25], every convex polygon has $O(n^{1/3})$ vertices in G_n. Therefore, at each step, the greedy algorithm selects a point p_i that makes an external angle $\tau_i = \Omega(n^{-1/3})$. Hence, $\sum \tau_i = \Omega(n^{2/3})$ for the order f_{greedy}. Lemma 7 completes the proof.

Now we turn to Theorem 5.

Theorem 5. *Let n be a perfect square and let G_n denote an $n^{1/2}$ by $n^{1/2}$ uniform grid in the plane. For any stacking order f, we have*

$$\lim_{\varepsilon \to 0} \mathrm{visible}(\mathcal{D}(\varepsilon G_n), f) = O(n^{2/3}).$$

Our proof is an improved version of the proof of Theorem 2. There we were concerned with how the *perimeter* of the convex hull grows as we add the points of our set one by one

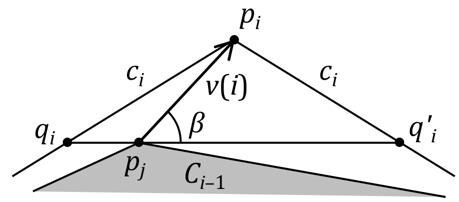

Figure 6: The triangle $p_i q_i q_i'$ is the largest isosceles triangle at point p_i that does not intersect the interior of C_{i-1}.

as prescribed by the stacking order. As is well known, the perimeter of a convex set in the plane is the integral of its width in all directions (this is known as Cauchy's theorem; see *e.g.* [PaA95], Theorem 16.15). The proof of Theorem 5 is very similar, but we deal with the widths in different directions in a non-uniform way. The width in a direction close to the direction of a short grid vector is more important in the analysis than widths in other directions.

Proof of Theorem 5. Let $G_n = \{p_1, \ldots, p_n\}$ be an enumeration of the points of G_n according to a given stacking order, and let τ_i denote the corresponding external angles, as defined in Lemma 7. According to the lemma, we need to prove that $\sum_{i=1}^{n} \tau_i = O(n^{2/3})$. Let us partition this sum into several parts, and bound the contribution of each part separately.

Let $[n] = \{1, \ldots, n\}$. We start with the small angles. Let

$$I_0 = \{i \in [n] \mid \tau_i < n^{-1/3}\}.$$

Clearly, we have

$$\sum_{i \in I_0} \tau_i < n \cdot n^{-1/3} = n^{2/3}.$$

As in the proof of Theorem 2, let $C_i = \operatorname{conv}\{p_1, \ldots, p_i\}$ and denote the perimeter of C_i by $\operatorname{per}(i)$. Since G_n is an $n^{1/2} \times n^{1/2}$ integer grid, we have $\operatorname{per}(n) = 4(n^{1/2} - 1)$. Consider only those indices $i > 1$ that do not belong to I_0. For these indices, we have $\tau_i > 0$, so that p_i must be a vertex of C_i. For each such point p_i, let c_i denote the smallest number satisfying the following condition: the segment connecting the points q_i and q_i' that lie on the boundary of C_i at distance c_i from p_i, intersects C_{i-1}. (In the case where C_i is a segment, we have $q_i = q_i' \in C_{i-1}$.) Note that the segment $q_i q_i'$ contains a point p_j with $1 \le j < i$. See Figure 6.

In the proof of Theorem 2, we argued that $\operatorname{per}(i) - \operatorname{per}(i-1) \ge \tau_i^2/5$. Now the same argument gives that $\operatorname{per}(i) - \operatorname{per}(i-1) > c_i \tau_i^2/5$. Let

$$I_1 = \{i \in [n] \setminus (I_0 \cup \{1\}) \mid c_i \tau_i > n^{-1/6}\}.$$

For $i \in I_1$, we have $\operatorname{per}(i) - \operatorname{per}(i-1) \ge \tau_i n^{-1/6}/5$. Since $\operatorname{per}(i)$ is monotone in i, we conclude that

$$\sum_{i \in I_1} \tau_i \le 5n^{1/6}(\operatorname{per}(n) - \operatorname{per}(1)) < 20n^{2/3}.$$

Let

$$I_2 = [n] \setminus (\{1\} \cup I_0 \cup I_1).$$

To bound the angles τ_i for indices $i \in I_2$, we need a charging scheme and we need to consider the growth of the width of C_i in some specific directions. The *width* of a planar set in a given direction is the diameter of the orthogonal projection of the set to a line in this direction. Let us associate the directions in the plane with the points of the unit circle K. We identify opposite points of this circle as the widths of the same set in opposite directions are the same. This makes the total length of K become π. We define a set of arcs along K as follows. For any non-zero grid vector v from the integer grid and for any integer $\ell \geq 0$, let $V_{v,\ell}$ denote the arc of length $2^{-\ell}$ symmetric around the direction of v. For any direction $\alpha \in K$, let $\rho_i(\alpha)$ denote the width of C_i in the direction *orthogonal* to α (*i.e.*, where the corresponding projection is parallel to α).

The perimeter $\mathrm{per}(i)$ is equal to the integral of $\rho_i(\alpha)$ along the circle K (note that after the identification of opposite points the length of K became π). We have $\rho_i(\alpha) = \rho_{i-1}(\alpha)$, unless the direction α is tangent to C_i at the vertex p_i. Let U_i denote the arc of directions where such a tangency occurs. Clearly, the length of U_i is τ_i, and for any arc V that contains U_i, we have

$$\int_V (\rho_i(\alpha) - \rho_{i-1}(\alpha))d\alpha = \mathrm{per}(i) - \mathrm{per}(i-1) \geq c_i\tau_i^2/5.$$

For each index $i \in I_2$, choose a grid point p_j on the segment q_iq_i'. (Recall that the points q_i and q_i' are at distance c_i from p_i, and that there is always a grid point between them.) We *charge* the index i to the pair $(v(i), \ell(i))$, where $v(i)$ is the grid vector pointing from p_j to p_i and $\ell(i)$ is the largest integer such that $V_{v(i),\ell(i)}$ contains U_i. Notice that $|v(i)| \leq c_i$. Denote by $I_2(v, \ell)$ the set of indices $i \in I_2$ that are charged to the pair (v, ℓ).

Note that U_i is symmetric around the direction of the segment q_iq_i'. For the angle β between this direction and the direction of $v(i)$ we have $|v(i)| \sin\beta = c_i \sin(\tau_i/2)$ (refer again to Figure 6). This implies $\beta < c_i\tau_i/|v(i)|$, and hence $2^{-\ell(i)} < 4\beta + 2\tau_i < 6c_i\tau_i/|v(i)|$. Finally, we also have

$$\int_{V_{v(i),\ell(i)}} (\rho_i(\alpha) - \rho_{i-1}(\alpha))d\alpha \geq c_i\tau_i^2/5 > 2^{-\ell(i)}|v(i)|\tau_i/30.$$

Let $s(v, \ell) = \sum_{i \in I_2(v,\ell)} \tau_i$. The integral $\int_{V_{v,\ell}} \rho_i(\alpha)d\alpha$ is monotone in i and grows by at least $2^{-\ell}|v|\tau_i/30$ at every $i \in I_2(v, \ell)$. We have $\rho_1(\alpha) = 0$ and $\rho_n(\alpha) < (2n)^{1/2}$, so that the final integral satisfies $\int_{V_{v,\ell}} \rho_n(\alpha)d\alpha \leq 2^{-\ell}(2n)^{1/2}$. Therefore, $\sum_{i \in I_2(v,\ell)} \left(2^{-\ell}|v|\tau_i/30\right) \leq 2^{-\ell}(2n)^{1/2}$, which implies that $s(v, \ell) \leq 30\sqrt{2}n^{1/2}/|v|$.

Consider the set of all pairs (v, ℓ) such that there is an index $i \in I_2$ charged to them. We have $c_i\tau_i \leq n^{-1/6}$, $\tau_i \geq n^{-1/3}$ and $|v| \leq c_i$, which implies that $|v| \leq n^{1/6}$. We proved that $2^{-\ell} < 6c_i\tau_i/|v| \leq 6n^{-1/6}/|v|$. On the other hand, we also have $2^{-\ell} \geq 2\tau_i \geq 2n^{-1/3}$. Thus, for any given grid vector v, there are at most $\log(6n^{1/6}/|v|)$ possible values of ℓ, where log denotes the binary logarithm.

Hence,

$$\sum_{i \in I_2} \tau_i = \sum_{v,\ell} s(v, \ell) \leq \sum_{|v| \leq n^{1/6}} \frac{30\sqrt{2}n^{1/2}}{|v|} \log \frac{6n^{1/6}}{|v|}.$$

To evaluate this sum, we note that the number of grid vectors v satisfying $2^k \leq |v| < 2^{k+1}$ is $\Theta(2^{2k})$. Thus,

$$\sum_{i \in I_2} \tau_i = O\left(n^{1/2} \sum_{k=0}^{\log n^{1/6}} (\log n^{1/6} - k)2^k\right) = O(n^{1/2}n^{1/6}) = O(n^{2/3}).$$

In conclusion, we have

$$\sum_{i=1}^{n} \tau_i = \tau_1 + \sum_{i \in I_0} \tau_i + \sum_{i \in I_1} \tau_i + \sum_{i \in I_2} \tau_i = 2\pi + O(n^{2/3}) + O(n^{2/3}) + O(n^{2/3}) = O(n^{2/3}),$$

completing the proof of the theorem. □

5 Collections of disks with bounded overlap

In this section, we prove Theorem 6.

Theorem 6. *Let $\mathcal{D}$ be a collection of n unit disks in which at most c disks have a point in common. Then there exists a stacking order f for which*

$$\text{visible}(\mathcal{D}, f) = \Omega(v(c)n/c),$$

where $v(c)$ is given in (1). This bound is worst-case asymptotically tight.

Note that Lemma 7 is not relevant in this case, since we cannot contract the set of centers of $\mathcal{D}$.

Proof. Partition the plane into an infinite grid of axis-parallel square cells of side-length 4, where the position of the grid is chosen uniformly at random. For each unit disk, the probability that it belongs entirely to a single cell is 1/4. Thus, we can fix the grid in such a way that at least $n/4$ disks lie entirely in a cell. Let k_i be the number of disks entirely contained in cell i. By area considerations, we have $k_i \leq (16/\pi)c$.

For each cell i, we independently select a stacking order that achieves visible perimeter at least $v(k_i)$; then we place all the remaining disks behind them. Thus, our stacking order achieves visible perimeter at least $\sum_i v(k_i)$.

For any n-element point set $\mathcal{D}$, we can take an rn-element point set $\mathcal{D}'$ as the union of r pairwise disjoint translates of $\mathcal{D}$. We clearly have $\max_f \text{visible}(\mathcal{D}', f) \leq r \max_f \text{visible}(\mathcal{D}, f)$. This implies that $v(rn) \leq rv(n)$. Let $r_i = \lceil c/k_i \rceil \leq (16/\pi)c/k_i$, and we have $v(c) \leq v(r_i k_i) \leq r_i v(k_i)$, thus

$$v(k_i) \geq v(c)/r_i \geq \frac{\pi v(c)}{16c} k_i.$$

Since $\sum k_i \geq n/4$, the claimed bound follows.

To show that this bound is worst-case asymptotically tight, take the union of $\lceil n/c \rceil$ worst-case sets of c disks far from each other. □

6 Concluding remarks

A. The greedy algorithm described at the beginning of Section 4 was first considered by Cabello *et al.* (unpublished) in the context of maximizing the *minimum* visible perimeter of a single disk. They showed that the order f_{greedy} is always optimal for this purpose. Unfortunately, this stacking order is *not* always optimal with respect to the total visible perimeter. Indeed, let n be a perfect square and consider the set of points $\{p_i \mid 1 \leq i \leq n\}$, where the polar coordinates of p_i are $(r_i, \theta_i) = (e^{bi}, 2\pi i/n^{1/2})$ with $b > 0$ sufficiently small. This point set is obtained as the intersection of $n^{1/2}$ equally-spaced rays emanating from the origin, and $n^{1/2}$ "rounds" of a very tight logarithmic spiral centered at the origin. The greedy algorithm produces the stacking order indicated by the indices, so it takes the points of P outwards along the spiral. The contribution τ_i is equal for every point p_i with $i \geq n^{1/2}$ and tends to $2\pi/n^{1/2}$ as b goes to zero, making $\sum \tau_i = \Theta(n^{1/2})$ if b is small enough. However, taking the points ray by ray in a cyclic order, going outwards along each ray, the contribution τ_i is a constant for the first half of the points, making $\sum \tau_i = \Theta(n)$.

B. Theorem 4 can be generalized to point sets satisfying weaker density conditions. Indeed, let P be a set of n points in the plane with diameter D and minimum distance d. A randomized construction, similar to the one used in the proof Theorem 4, guarantees the existence of a stacking order f such that visible$(\mathcal{D}(P), f) = \Omega(n/(D/d)^{2/3})$. This beats the $\Omega(n^{1/2})$ bound mentioned in the Introduction as long as $D/d = o(n^{3/4})$.

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