Tight lower bounds for the size of epsilon-nets

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Abstract

According to a well known theorem of Haussler and Welzl (1987), any range space of bounded VC-dimension admits an ε-net of size $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. Using probabilistic techniques, Pach and Woeginger (1990) showed that there exist range spaces of VC-dimension 2, for which the above bound can be attained. The only known range spaces of small VC-dimension, in which the ranges are geometric objects in some Euclidean space and the size of the smallest ε-nets is superlinear in $\frac{1}{\varepsilon}$, were found by Alon (2010). In his examples, the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}g(\frac{1}{\varepsilon})\right)$, where g is an extremely slowly growing function, closely related to the inverse Ackermann function.

We show that there exist geometrically defined range spaces, already of VC-dimension 2, in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. We also construct range spaces induced by axis-parallel rectangles in the plane, in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$. By a theorem of Aronov, Ezra, and Sharir (2010), this bound is tight.

1 Introduction

Let X be a *finite* set and let $\mathcal{R}$ be a system of subsets of an underlying set which contains X. In computational geometry, the pair $(X,\mathcal{R})$ is usually called a *range space*. The elements of X and $\mathcal{R}$ are said to be the *points* and the *ranges* of the range space, respectively. Consider a subset $A \subseteq X$. It is called *shattered* if for every subset $B \subseteq A$, one can find a range $R_B \in \mathcal{R}$ with $R_B \cap A = B$. The size of the largest shattered subset of points, $A \subseteq X$, is said to be the *Vapnik-Chervonenkis dimension* (or *VC-dimension*) of the range space $(X,\mathcal{R})$.

In their seminal paper [VaC71], Vapnik and Chervonenkis proved that, from the point of view of random sampling, all range spaces whose VC-dimensions are bounded

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by a constant behave very nicely. In particular, for any $\varepsilon > 0$, a randomly selected "small" subset of X, whose number of elements depends only on the VC-dimension d and ε, will "hit" every range containing at least $\varepsilon|X|$ points of X, with large probability. A set of points in X with the property that every range $R \in \mathcal{R}$ with $|R \cap X| \geq \varepsilon|X|$ contains at least one of its elements is called an *ε-net* for the range space $(X, \mathcal{R})$. Note that these sets are often called *strong* ε-nets in the literature, to distinguish them from the so-called *weak ε-nets*, which may also contain points from $\cup\mathcal{R} \setminus X$, but must still hit all ranges that contain at least $\varepsilon|X|$ elements of X. In this paper, we will consider only strong ε-nets, apart from some remarks in the last section.

The ideas of Vapnik and Chervonenkis have been adapted by Haussler and Welzl [HaW87] to show that the minimum number $f = f_d(\varepsilon)$ such that every range space of VC-dimension d admits an ε-net of size at most f satisfies $f_d(\varepsilon) = O\left(\frac{d}{\varepsilon}\log\frac{d}{\varepsilon}\right)$. They asked whether the logarithmic factor can be removed in this formula. Pach and Woeginger [PaW90] proved that while $f_1(\varepsilon) = \max(2, \lceil\frac{1}{\varepsilon}\rceil - 1)$, the logarithmic factor is needed for every $d \geq 2$. Moreover, it was shown by Komlós et al. [KoPW92, PaA95]) that for any $d \geq 2$,

$$(d - 2 + \frac{1}{d+2} + o(1))\frac{1}{\varepsilon}\log\frac{1}{\varepsilon} \leq f_d(\varepsilon) \leq (d + o(1))\frac{1}{\varepsilon}\log\frac{1}{\varepsilon},$$

as ε tends to 0. (Here log denotes the natural logarithm.)

Haussler and Welzl discovered that the above results apply to many geometrically defined range spaces. Roughly speaking, the VC-dimension is bounded by a constant for any set of ranges with bounded *description complexity*, that is if the ranges can be described in terms of a bounded number of parameters. This observation has far reaching consequences. The construction of small epsilon-nets has become one of the most powerful general techniques in computational geometry (see [Ch00, EvRS05]).

In a number of basic geometric scenarios it was possible to improve on the above bounds. For instance, for any finite set of points in the plane, one can find an ϵ-net of size linear in $1/\varepsilon$, where the ranges are half-planes, translates of a convex polygon, disks or certain kind of pseudo-disks. Similar results hold in three-dimensional space for half-space ranges [PaW90, MaSW90, Ma92, PyR08]. We state two results here.

Theorem A. (Matoušek, Seidel, Welzl [MaSW90, Ma92]) *All range spaces $(X, \mathcal{R})$, where X is a finite set of points in $\mathbb{R}^3$ and $\mathcal{R}$ consists of half-spaces, admit ε-nets of size $O(1/\varepsilon)$.*

Theorem B. (Aronov, Ezra, Sharir [ArES10]) *All range spaces $(X, \mathcal{R})$, where X is a finite set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and $\mathcal{R}$ consists of axis-parallel rectangles (boxes), admit ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.*

Aronov et al. have also established a similar result for "fat" triangular ranges in the place of axis-parallel rectangles. For weak ε-nets, Ezra [Ez10] extended Theorem B to higher dimensions.

In algorithmic applications, it is often natural to consider the dual range space, in which the roles of points and ranges are swapped [BrG95, PaA95]. Given a finite family $\mathcal{R}$ of ranges in $\mathbb{R}^m$, the *dual range space induced* by them is defined as a set system (hypergraph) on the underlying set $\mathcal{R}$, consisting of the sets $\mathcal{R}_x := \{R \mid x \in R \in \mathcal{R}\}$, for all $x \in \mathbb{R}^m$. (Note that $\mathcal{R}_x$ and$\mathcal{R}_y$ may coincide for $x \neq y$.) It is easy to see that if the VC-dimension of the range space $(X, \mathcal{R})$ is less than d for every $X \subset \mathbb{R}^m$, then the VC-dimension of the dual range space induced by any subset of $\mathcal{R}$ is less than 2^d.

Clarkson and Varadarajan [ClV07] found a simple and beautiful connection between the complexity of the boundary of the union of n members of $\mathcal{R}$ and the size of the smallest epsilon-net in the dual range space. If the complexity of the boundary is $o(n \log n)$, then the dual range space admits ε-nets of size $o\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. This connection has been further explored and improved in [Va09, ArES10]. In particular, it was shown that dual range spaces of "fat" triangles in the plane admit ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\log\frac{1}{\varepsilon}\right)$.

In most range spaces $(X, \mathcal{R})$, one can find roughly $1/\varepsilon$ pairwise disjoint ranges $R \in \mathcal{R}$ such that the sets $R \cap X$ are of size at least $\varepsilon|X|$. In these cases, the size of any ε-net is $\Omega(1/\varepsilon)$. For the last two decades, "the prevailing conjecture" was that in "geometric scenarios," this bound is essentially tight: there always exists an ε-net of size $O(1/\varepsilon)$ (see, e.g., [MaSW90, ArES10]. This conjecture had to be revised after Alon [Al10] discovered some geometric range spaces of small VC-dimension, in which the ranges are straight lines, rectangles or infinite strips in the plane, and which do not admit ε-nets of size $O(1/\varepsilon)$. Alon's construction is based on the density version of the Hales-Jewett theorem [HaJ63], due to Furstenberg and Katznelson [FuK89, FuK91], and recently improved in [Po09]. However, his lower bound is only barely superlinear: $\Omega\left(\frac{1}{\varepsilon}g(\frac{1}{\varepsilon})\right)$, where g is an extremely slowly growing function, closely related to the inverse Ackermann function.

1.1 New lower bounds

The main aim of this note is to prove that the $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ general upper bound for the size of the smallest ε-nets in range spaces of bounded dimension is tight even in simple geometric scenarios.

Our first theorem claims that there exist dual range spaces induced by finite families of axis-parallel rectangles in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. More precisely, we have the following.

Theorem 1. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a dual range space Σ^* of VC-dimension 2, induced by n axis-parallel rectangles in $\mathbb{R}^2$,* *in which the minimum size of an ε-net is at least $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

From Theorem 1 it is not hard to deduce the following results for primal range spaces.

Theorem 2. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$ of VC-dimension 2, where X is a set of n points in $\mathbb{R}^4$, $\mathcal{R}$ consists of axis-parallel boxes with one of their vertices at the origin, and in which the size of the smallest ε-net is at least $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

Theorem 3. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$ of VC-dimension 2, where X is a set of n points in $\mathbb{R}^4$, $\mathcal{R}$ consists of half-spaces, and in which the size of the smallest ε-net is at least $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

Theorems 2 and 3 show that Theorems B and A cannot be generalized to 4-dimensional space. It also follows, by a standard duality argument, that there exist *dual* range spaces induced by half-spaces in $\mathbb{R}^4$, for which the size of the smallest ε-net is $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.

Our next result shows that Theorem B of Aronov, Ezra, and Sharir is tight.

Theorem 4. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$, where X is a set of n points in the plane, $\mathcal{R}$ consists of axis-parallel rectangles, and in which the size of the smallest ε-net is at least $C\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

The VC-dimension of the family of *all* axis-parallel rectangles in the plane is 4. However, it is easy to verify that the VC-dimension of the range spaces used for the proof of Theorem 4 is only at most 3. In the full version of this paper, we also outline a somewhat different approach to prove the existence of range spaces of VC-dimension 2 that satisfy the conditions in Theorem 4.

The proofs of Theorems 1 and 4 are based on two constructions from [PaT10] and [ChPS09], related to hypergraph coloring problems.

1.2 Organization

In Section 2, we present the proofs of Theorems 1, 2, and 3, based on an explicit construction of systems of axis-parallel rectangles, described in [PaT10]. Section 3 contains a similar proof of Theorem 4, based on randomized construction from Chen et al. [ChPS09]. In the final section, we make some concluding remarks and mention some open problems.

2 Boxes and half-spaces—Proofs of Theorems 1-3

Theorems 2 and 3 are corollaries of Theorem 1, so we start with the proof of Theorem 1. The proof is based on an explicit construction of systems of rectangles, presented in [PaT10]. In order to describe this construction, we have to introduce some notations.

For any two integers $c \geq 2$ and $k \geq 0$, let $[c] := \{0, 1, \ldots, c-1\}$ and let $[c]^k$ stand for the set of strings of length k over the alphabet $[c]$. For $x \in [c]^k$, let x_j denote the jth digit of x ($1 \leq j \leq k$), so that we have $x = x_1 \ldots x_k$. For notational convenience we write $x_0 = 0$. Expanding x as a c-ary fraction, we obtain a number $\overline{x} := \sum_{j=1}^{k} x_j / c^j$. Let θ denote the empty string so that $\overline{\theta} = 0$.

Let $c \geq 2$ and $d \geq 1$ be integers. For any $0 \leq k \leq d$, $u \in [c]^k$, and $v \in [c]^{d-k}$, define an open axis-parallel rectangle $R^k_{u,v}$ in the unit square, as follows:

$$R^k_{u,v} := (\overline{u}, \overline{u} + c^{-k}) \times (\overline{v}, \overline{v} + c^{k-d})$$

and consider the family

$$\mathcal{R} = \mathcal{R}(c,d) = \left\{ R^k_{u,v} \;\mid\; 0 \leq k \leq d,\ u \in [c]^k,\ v \in [c]^{d-k},\ u_k = v_{d-k} \right\}.$$

Clearly, we have $|\mathcal{R}| = (d+1)c^{d-1}$. Finally, let $\Sigma = \Sigma(c,d)$ be the (infinite) range space $(\mathbb{R}^2, \mathcal{R})$ and let $\Sigma^* = \Sigma^*(c,d)$ denote its dual. That is, the underlying set of Σ^* is $\mathcal{R} = \mathcal{R}(c,d)$, and its ranges (hyperedges) are all sets of the form $\{R \in \mathcal{R} \mid x \in R\}$ for some $x \in \mathbb{R}^2$.

The most important property of our construction is the following.

Lemma 2.1. *Let $d \geq 1$, $r \geq 2$, $c \geq 3$ and let $\Sigma^* = \Sigma^*(c,d)$ denote the dual range space defined above. If a subset $I \subseteq \mathcal{R}(c,d)$ contains no r-element range (hyperedge) of Σ^*, then we have*

$$|I| \leq (r-1)\frac{c-1}{c-2}c^{d-1}.$$

In [PaT10], we established the slightly weaker bound $|I| \leq \frac{c^{d-1}}{\frac{1}{r-1} - \frac{1}{c-1}}$, for any $c > r$. The main focus of that paper was the case $r = 2$, in which the two bounds coincide.

Proof of Lemma 2.1. To explain the proof, first we have to sketch the original argument from [PaT10]. Two distinct rectangles $R, R' \in \mathcal{R}$ are called *siblings* if $R = R^k_{u,v}$, $R' = R^k_{u',v'}$, where u and u' differ only in their last digit, and the same is true for v and v'. Clearly, for $1 \leq k < d$, the rectangles of the form $R^k_{u,v} \in \mathcal{R}$ fall into groups, each consisting of c siblings. For $k = 0$ and $k = d$, $R^k_{u,v}$ has no sibling. A rectangle $R \in \mathcal{R}$ is called *bad* if $R \notin I$, but for each of its $c-1$ siblings we have $R' \in I$. Let B denote the set of bad rectangles.

Using the assumption that $I \subseteq \mathcal{R}$ contains no r-element range of Σ^*, we proved in [PaT10] that

$$|I| \le (r-1)|B| + (r-1)c^{d-1}. \tag{1}$$

Comparing (1) to the trivial inequality $|B| \le |I|/(c-1)$, we obtain the weaker bound $|I| \le \frac{c^{d-1}}{\frac{1}{r-1}-\frac{1}{c-1}}$.

Now we choose a different strategy to deal with bad rectangles. For every $R \in B$, we pick one of the $c-1$ siblings of R and remove it from the set I. Since the resulting set $I' \subseteq I$ contains no r-element range in Σ^*, we can apply inequality (1) to I'. By the construction of I', the corresponding set B' of bad rectangles in $\mathcal{R}$ is empty, so that we obtain

$$|I'| \le (r-1)c^{d-1}.$$

Comparing this inequality to $|I'| = |I| - |B| \ge \frac{c-2}{c-1}|I|$, the lemma follows. □

Lemma 2.2. *Both Σ and Σ^* have VC-dimension 2.*

Before turning to the proof, we have to introduce a partial order on the family of axis-parallel rectangles in the plane. For any two axis-parallel rectangles R and R', we write $R \prec R'$ if the orthogonal projection of R on the x-axis is contained in the orthogonal projection of R' on the x-axis, and the orthogonal projection of R on the y-axis contains the orthogonal projection of R' on the y-axis. Obviously, this is a partial order.

Proof of Lemma 2.2. Clearly, we have VC-dim(Σ^*), VC-dim(Σ) ≥ 2.

Observe first that no rectangle in $\mathcal{R}$ contains a vertex of any other rectangle in its interior. This implies that any two intersecting rectangles in $\mathcal{R}$ are comparable by $\prec$.

Assume for contradiction that Σ or Σ^* has VC-dimension 3 or more. In either case, the existence of a shattered 3-element set would mean that there are three distinct points p_1, p_2, and p_3 in the plane and three rectangles $R_1, R_2, R_3 \in \mathcal{R}$ with $\{p_1, p_2, p_3\} \setminus R_i = \{p_i\}$ for $i = 1, 2, 3$. The rectangles R_i pairwise intersect, and hence must be linearly ordered by $\prec$. Suppose without loss of generality $R_1 \prec R_2 \prec R_3$. Then $R_1 \cap R_3 \subseteq R_2$, contradicting our assumption that p_2 is contained in the left-hand side but not in the right. □

Proof of Theorem 1. Let $0 < \varepsilon < 2^{-6}$, and set $r = \lceil \log \frac{1}{\varepsilon}/6 \rceil \ge 2$, $c = 4$, and $d = 3r-4$, where log denotes the binary logarithm. Consider the dual range space $\Sigma^* = \Sigma^*(c, d)$. The number of rectangles in this range space is $|\mathcal{R}| = |\mathcal{R}(c, d)| = (d+1)c^{d-1}$, and by Lemma 2.2, the VC-dimension of Σ^* is 2.

Let $\mathcal{S} \subseteq \mathcal{R}$ be an ε-net in Σ^*, that is, a set of rectangles with the property that any point of the plane which is covered by at least $\varepsilon|\mathcal{R}|$ members of $\mathcal{R}$ is contained in an element of $\mathcal{S}$. Notice that with our choice of parameters we have $\varepsilon|\mathcal{R}| < r$, hence the rectangles in $\mathcal{R} \setminus \mathcal{S}$ cannot induce any r-element range (hyperedge) in Σ^*. Applying Lemma 2.1 with $I = \mathcal{R} \setminus \mathcal{S}$, we obtain that

$$|\mathcal{R} \setminus \mathcal{S}| \leq (r-1)\frac{c-1}{c-2}c^{d-1} = \frac{(d+1)c^{d-1}}{2} = \frac{|\mathcal{R}|}{2}.$$

This yields that

$$|\mathcal{S}| \geq \frac{|\mathcal{R}|}{2} = \frac{1}{\varepsilon} \cdot \frac{r}{2} \geq \frac{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}{12}.$$

So far our examples may appear quite special, because for every ε, we have defined only one particular space Σ^*, consisting of $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ rectangles. However, from this small example we can easily construct arbitrarily large ones, as follows. Keep r and the corresponding ε fixed, and choose a large integer t. Replace each rectangle $R \in \mathcal{R}$ by a chain of rectangles $R_1 \prec R_2 \prec \cdots \prec R_t$, where $\prec$ denotes the ordering relation defined after Lemma 2.2, and each R_i differs only very little from R. Note that the dual range space Σ^*, as well as the corresponding primal space have VC-dimension 2 by Lemma 2.2. It is not difficult to see that if the difference between (the coordinates of) the new rectangles R_i and the original rectangle $R \in \mathcal{R}$ is small enough, then the VC-dimension of the dual range space Σ_t^* induced by the resulting family of rectangles $\mathcal{R}_t$, as well as the VC-dimension of the "primal" space $\Sigma_t = (\mathbb{R}^2, \mathcal{R}_t)$, remains 2.

We have $|\mathcal{R}_t| = t|\mathcal{R}|$, and the size of the smallest ε-net for Σ_t^* is at least as large as it was in Σ^*. Suppose to the contrary that there is a smaller set $\mathcal{S}'$ of rectangles in $\mathcal{R}_t$ that form an ε-net in Σ_t^*. Let $\mathcal{S}''$ be the set of rectangles in $\mathcal{R}$ that were replaced by the elements of $\mathcal{S}'$. Since $|\mathcal{S}''| \leq |\mathcal{S}'|$, the rectangles in $\mathcal{S}''$ do not form an ε-net in Σ^*. Thus, there is a point in the plane contained in at least $\varepsilon|\mathcal{R}|$ elements of $\mathcal{R}$, which is not covered by any element of $\mathcal{S}''$. We can choose such a point lying not too close to the boundaries of the rectangles in $\mathcal{R}$, and then it is contained in at least $t\varepsilon|\mathcal{R}| = \varepsilon|\mathcal{R}_t|$ elements of $\mathcal{R}_t$, none of which belongs to $\mathcal{S}'$, a contradiction. $\square$

Proof of Theorem 2. The statement follows from Theorem 1 by a standard duality argument. We assume without loss of generality the the rectangles are closed and lie in the first quadrant of the plane. We assign to each rectangle $R = [x_1, x_2] \times [y_1, y_2]$ the point $p(R) = (x_1, 1/x_2, y_1, 1/y_2) \in \mathbb{R}^4$. Now a point $q = (a, b)$ of the first quadrant lies in R if and only if $x_1 \leq a \leq x_2$ and $y_1 \leq b \leq y_2$, that is, if and only if the point $p(R)$ is contained in the 4-dimensional box

$$B(q) = [0, a] \times [0, 1/a] \times [0, b] \times [0, 1/b]. \quad \square$$

Theorem 3 is an immediate corollary of Theorem 2 and the following lemma.

Lemma 2.3. *Let P be a finite set of points in the positive orthant of $\mathbb{R}^d$. To each $p \in P$, we can assign a point p' in the positive orthant of $\mathbb{R}^d$ so that the set $P' = \{p' \mid p \in P\}$ satisfies the following condition.*

For any axis-parallel box $B \subset \mathbb{R}^d$ that contains the origin, there is a half-space $H_B \subset \mathbb{R}^d$ which contains the origin and for which

$$\{p' \mid p \in B \cap P\} = P' \cap H(B).$$

Proof. Let $x_1, x_2, \ldots, x_d$ denote the orthogonal coordinates in $\mathbb{R}^d$. Observe that from the point of view of intersections with axis-parallel boxes, the actual values of the coordinates do not matter: we need to know only the order of the x_i-coordinates of the points of P for each i. For every i $(1 \leq i \leq d)$, let $0 < \xi_{i,1} < \xi_{i,2} < \xi_{i,3} < \ldots$ denote the sequence of different values of the x_i-coordinates of the elements of P. Every such sequence is of length at most $|P|$. By rescaling the coordinates if necessary, we can assume that $\xi_{i,j+1}/\xi_{i,j} > d$ holds for every i and j.

Consider now an axis-parallel box B, which contains the origin and intersects P in at least one element. We can shrink B if necessary, without changing its intersection with P, so that we can suppose without loss of generality that B is of the form

$$B = [0, b_1] \times [0, b_2] \times \ldots \times [0, b_d],$$

where each b_i is equal to ξ_{ij_i} for a suitable j_i.

We claim that $B \cap P$ is equal to the intersection of P with the half-space $H(B)$ defined by

$$\frac{x_1}{b_1} + \frac{x_2}{b_2} + \ldots + \frac{x_d}{b_d} \leq d.$$

For every point in B, each term of the above sum is at most 1, so that we have $B \subset H(B)$, and hence $B \cap P \subseteq H(B) \cap P$. Suppose now that p is a point of P that does not belong to B. Then one of its coordinates, $x_i(p)$, say, is more than d times larger than b_i. Therefore, the i-th term in the above sum is already larger than d, which implies that $p \notin H(B)$. $\Box$

3 Proof of Theorem 4

Theorem 4 is an easy consequence of the following result on a set of randomly selected points in the unit square. A similar property of random point sets with respect to axis-parallel rectangles was established in Chen et al. [ChPS09] (see Theorem 9). In their setting, r was a constant, $\varepsilon = r/n$, and it was shown that every ε-net contains all but a very small fraction of point set. Here we allow r to slowly tend to infinity.

Lemma 3.1. *Let $n > 2$, $r = \lceil \log\log n/5 \rceil$ be integers, where* log *stands for the binary logarithm, and let and $\varepsilon = r/n$. Let X be a set of n randomly and uniformly selected points in the unit square, and let $\mathcal{R}$ denote the family of all axis-parallel rectangles of the form $[j/2^t, (j+1)/2^t) \times [a, b]$, where j, t are nonnegative integers, and $a < b$ are reals.*

Then, with probability tending to 1, the range space $(X, \mathcal{R})$ does not admit an ε-net of size at most $n/2$.

Proof. We write $[n]$ to denote the index set $\{1, \dots, n\}$. Let us choose the y-coordinates of our random points p_i first, and then enumerate them in the increasing order of their y-coordinates. That is, let $p_i = (x_i, y_i)$, where the numbers $y_1 < y_2 < \cdots < y_n$ are fixed and the x_i-s are chosen uniformly and independently from $[0, 1]$. Finally, let $X = \{p_i \mid i \in [n]\}$.

Fix a subset $I \subseteq [n]$ of size at most $n/2$, and let $S_I = \{p_i \mid i \in I\}$. We will prove that the probability that S_I is an ε-net for the range space $(X, \mathcal{R})$ is very small.

We write each x_i as an infinite binary fraction $0.d_i^{(1)} d_i^{(2)} \dots$. That is, $x_i = \sum_{i=1}^{\infty} d_i^{(t)}$, where $d_i^{(t)} = 0$ or 1. The t-th *truncation* of x_i, denoted by $x_i^{(t)}$, is the finite binary fraction $0.d_i^{(1)} d_i^{(2)} \dots d_i^{(t-1)}$. In particular, we have $x_i^{(1)} = 0$.

Choosing x_i uniformly at random can be achieved by selecting all of its binary digits $d_i^{(t)}$ uniformly and independently. This will be done in stages. At stage t, we choose $d_i^{(t)}$ for all i.

Consider now stage t of our selection process for a fixed t, $1 \le t \le \log(n/r) - 1$. Before the selections are made, $x_i^{(t)}$ has been fixed for all i. For every $x \in [0, 1]$, define

$$H_x = H_x^{(t)} = \{1 \le i \le n \mid x_i^{(t)} = x\}.$$

The sets H_x form partition $[n]$ into at most 2^{t-1} nonempty parts.

For each $x \in [0, 1]$, divide H_x into as many pairwise disjoint *intervals* as possible, each containing r elements not in I. More precisely, select $\lfloor |H_x \setminus I|/r| \rfloor$ pairwise disjoint sets $H_{x,j}$ of the form $H_{x,j} = \{i \in H_x \mid a_{x,j} \le i \le b_{x,j}\}$ with $|H_{x,j} \setminus I| = r$.

For a given x, out of the at least $n/2$ indices in $[n] \setminus I$, there are fewer than r that do not belong to any interval of H_x. Using our assumption $t \le \log(n/r) - 1$, the total number of indices in $[n] \setminus I$ that belong to some interval $H_{x,j}$ over all x and j is larger than $n - |I| - 2^{t-1} r \ge n/4$. Since each interval contains precisely r such indices, the number of intervals is larger than $n/(4r)$.

We call an interval $H_{x,j}$ *bad* if its size is at least $4r$, otherwise is called *good*. Any bad interval contains at least $3r$ elements of I, so the number of bad intervals is at most $|I|/3r \le n/(6r)$. Consequently, the number of good intervals is at least the total number of intervals minus $n/(6r)$, which is larger than $n/(4r) - n/(6r) = n/(12r)$.

Let $G_{x,j}$ be a good interval. With probability $2^{-|G_{x,j}|} > 2^{-4r}$ we have $d_i^{(t)} = 0$ for all $i \in G_{x,j} \setminus I$ but $d_i^{(t)} = 1$ for all $i \in G_{x,j} \cap I$. If this happens, we say that the interval

$G_{x,j}$ *fails.* If $G_{x,j}$ fails, then for the rectangle $R = [x, x+2^{-t}) \times [y_{a_{x,j}}, y_{b_{x,j}}]$ we have $R \cap X = \{p_i \mid i \in G_{x,j} \setminus I\}$. That is, in this case we have $|R \cap X| = r = \varepsilon n$ and $R \cap S_I = \emptyset$, showing that S_I is not an ε-net for $(X, \mathcal{R})$.

Notice that at a fixed stage t ($1 \le t \le \log(n/r) - 1$), all the at least $n/(12r)$ good intervals fail independently, each with probability larger than 2^{-4r}. We say that S_I *survives stage* t if none of the intervals fail. We have

$$\text{Prob}[S_I \text{ survives stage } t] < (1 - 2^{-4r})^{n/(12r)} < 2^{-n/(12r2^{4r})}.$$

This inequality holds independently of what happened at the earlier stages, so that

$$\text{Prob}[S_I \text{ is an } \varepsilon\text{-net for } (X, \mathcal{R})] <$$

$$\text{Prob}[S_I \text{ survives all stages } t \le \log(n/r) - 1)] < 2^{-(\log(n/r)-2)n/(12r2^{4r})}.$$

There are fewer than 2^n choices for a set I with $|I| \le n/2$. By the union bound, this yields that

$$\text{Prob}[(x, \mathcal{R}) \text{ admits an } \varepsilon\text{-net of size} \le n/2] < 2^{n-(\log(n/r)-2)n/(12r2^{4r})}.$$

The right-hand side of this inequality tends to 0, as $n \to \infty$. □

Proof of Theorem 4. Consider the random range space $(X, \mathcal{R})$ described in Lemma 3.1, where n is so large that the probability that $(X, \mathcal{R})$ does not admit an ε-net of size at most $n/2$ is positive. Fix an n-element set X with this property. Then the minimum size of an ε-net for $(X, \mathcal{R})$ is larger than $\frac{n}{2} = \frac{1}{\varepsilon} \cdot \frac{r}{2} > \frac{1}{\varepsilon} \log\log\frac{1}{\varepsilon}/10$.

Once we have one example of a range space $\Sigma = (X, \mathcal{R})$ that admits no small ε-net for a given value of ε, we can create arbitrarily large examples with the same property, by replacing each point $p \in X$ with t new points, very close to p. (The same trick was applied in [Al10] and in the proof of Theorem 1.) This completes the proof of Theorem 4. □

The VC-dimension of the random range space we considered is 3. However, we can also construct a range space of VC-dimension 2, meeting the requirements of Theorem 4.

4 Concluding remarks

1. It was shown in [PaW90] that any range space $(X, \mathcal{R})$, where X is a finite point set in the plane and $\mathcal{R}$ consists of half-planes, admits ε-nets of size at most $\lceil 2/\varepsilon \rceil - 1$, and that this bound is tight up to an additive constant at most 1. The corresponding result on the line is almost trivial. Consequently, Theorem A holds in any dimension $d \le 3$, and our Theorem 4 shows that it is false for $d > 3$.

The epsilon-net problem for half-spaces (containing the origin) is self-dual. That is, any *dual* range space induced by half-spaces in $\mathbb{R}^d$ admits an ε-net of size $O(1/\varepsilon)$ if $d \leq 3$, and this statement is false whenever $d > 3$.

2. Recall that a *weak ε-net* for a range space $(X, \mathcal{R})$ is a set of elements of $\cup_{R\in\mathcal{R}} R$ (not necessarily in X) such that every range $R \in \mathcal{R}$ with $|R \cap X| \geq \varepsilon|X|$ contains at least one of them. In [Ez10], Ezra proved that if X is any finite set of points in $\mathbb{R}^d$ and $\mathcal{R}$ consists of all axis-parallel boxes, then $(X, \mathcal{R})$ admits a weak ε-net of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$. This implies that our Theorem 2 cannot be strengthened by requiring that the constructed range spaces do not admit *weak* ε-nets of size smaller than $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$, provided that $\varepsilon > 0$ is sufficiently small.

It is easy to see that the analogue of Theorem 3 is also false for *weak* ε-nets instead of strong ones. Indeed, any finite system of half-spaces in $\mathbb{R}^d$ can be hit by $d+1$ points, so that in (primal or dual) half-space range spaces there always exist weak ε-nets of size $O(1)$.

However, we have been unable to decide whether the analogue of Theorem 4 holds for weak ε-nets in place of strong ones.

3. Let X be a finite or infinite set and let $\mathcal{R}$ be a family of "ranges" of a certain type in $\mathbb{R}^d$ (e.g., lines, balls, half-spaces, axis-parallel boxes). We say that a subfamily $\mathcal{S} \subset \mathcal{R}$ forms a *k-fold covering* of X if every point of X belongs to at least k members of $\mathcal{S}$. It is an old problem in discrete geometry to decide whether every k-fold covering selected from a family $\mathcal{R}$ can be decomposed into two or more coverings [PaTT09]. For example, it was shown by Gibson and Varadarajan [GiV09] that every k-fold covering of the plane with translates of a convex polygon can be decomposed into $\Omega(k)$ coverings.

There is an intimate relationship between epsilon-net problems and problems about decomposition of multiple coverings. If we know that every k-fold covering $\mathcal{S} \subset \mathcal{R}$ with $|\mathcal{S}| = n$ splits into at least ck coverings for some absolute constant $c > 0$, then one of these coverings contains at most $n/(ck)$ sets. Setting $k = \varepsilon n$, we find a covering consisting of at most $1/(c\varepsilon)$ members of $\mathcal{S}$. This means that the *dual* range space Σ^* induced by the members of $\mathcal{S}$ admits an ε-net of size $O(1/\varepsilon)$. Therefore, if the dual range space does not always admit an ε-net of size $O(1/\varepsilon)$, then it cannot be true that every k-fold covering with ranges from $\mathcal{R}$ splits into $\Omega(k)$ coverings.

In particular, Alon [Al10] proved that there are n-element point sets $X \subset \mathbb{R}^2$ and straight-line ranges that do not admit ε-nets of size $O(1/\varepsilon)$. The standard duality between points and lines preserves incidences. Switching to the dual, we obtain dual range spaces induced by sets of n lines in the plane that do not admit ε-nets of size $O(1/\varepsilon)$. According to the argument in the previous paragraph, this implies that it cannot be true that every k-fold covering of a finite set of points in $\mathbb{R}^2$ with straight lines splits into $\Omega(k)$ coverings. This consequence of Alon's theorem had been proved earlier, using the Hales-Jewett theorem [PaTT09]. Alon [Al10] proved that the same example also disproves that all range spaces consisting of straight-line ranges in the plane admit ε-nets of size $O(1/\varepsilon)$.

4. If in the proof of Theorems 1, 2, and 3, we replace Lemma 2.1 by the weaker inequality $|I| \leq \frac{c^{d-1}}{\frac{1}{r-1}-\frac{1}{c-1}}$, established in [PaT10] for every $c > r$, we obtain slightly weaker versions of Theorems 1, 2 and 3, with $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}/\log\log\frac{1}{\varepsilon}\right)$ lower bounds on the sizes of the corresponding ε-nets. In a similar manner, if we replace Lemma 3.1 by a slightly weaker statement (Theorem 9) in [ChPS09], we obtain a weaker version of Theorem 4, with an $\Omega\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}/\log\log\log\frac{1}{\varepsilon}\right)$ bound on the size of the smallest ε-net.

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