A Carlitz type result for linearized polynomials

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Abstract

For an arbitrary q-polynomial f over $\mathbb{F}_{q^n}$ we study the problem of finding those q-polynomials g over $\mathbb{F}_{q^n}$ for which the image sets of $f(x)/x$ and $g(x)/x$ coincide. For $n \leq 5$ we provide sufficient and necessary conditions and then apply our result to study maximum scattered linear sets of $\mathrm{PG}(1, q^5)$.

1 Introduction

Let $\mathbb{F}_{q^n}$ denote the finite field of q^n elements where $q = p^h$ for some prime p. For $n > 1$ and $s \mid n$ the trace and norm over $\mathbb{F}_{q^s}$ of elements of $\mathbb{F}_{q^n}$ are defined as $\mathrm{Tr}_{q^n/q^s}(x) = x + x^{q^s} + \ldots + x^{q^{n-s}}$ and $\mathrm{N}_{q^n/q^s}(x) = x^{1+q^s+\ldots+q^{n-s}}$, respectively. When $s = 1$ then we will simply write $\mathrm{Tr}(x)$ and $\mathrm{N}(x)$. Every function $f\colon \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ can be given uniquely as a polynomial with coefficients in $\mathbb{F}_{q^n}$ and of degree at most $q^n - 1$. The function f is $\mathbb{F}_q$-linear if and only if it is represented by a *q-polynomial*, that is,

$$f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \tag{1}$$

with coefficients in $\mathbb{F}_{q^n}$. Such polynomials are also called *linearized*. If f is given as in (1), then its adjoint (w.r.t. the symmetric non-degenerate bilinear form defined by $\langle x, y\rangle = \mathrm{Tr}(xy)$) is

$$\hat{f}(x) := \sum_{i=0}^{n-1} a_i^{q^{n-i}} x^{q^{n-i}},$$

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i.e. $\mathrm{Tr}(xf(y)) = \mathrm{Tr}(y\hat{f}(x))$ for any $x, y \in \mathbb{F}_{q^n}$.

The aim of this paper is to study what can be said about two q-polynomials f and g over $\mathbb{F}_{q^n}$ if they satisfy

$$Im\left(\frac{f(x)}{x}\right) = Im\left(\frac{g(x)}{x}\right), \tag{2}$$

where by $Im\,(f(x)/x)$ we mean the image of the rational function $f(x)/x$, i.e. $\{f(x)/x \colon x \in \mathbb{F}_{q^n}^*\}$.

For a given q-polynomial f, the equality (2) clearly holds with $g(x) = f(\lambda x)/\lambda$ for each $\lambda \in \mathbb{F}_{q^n}^*$. It is less obvious that (2) holds also for $g(x) = \hat{f}(\lambda x)/\lambda$, see [3, Lemma 2.6] and the first part of [8, Section 3].

When one of the functions in (2) is a monomial then the answer to the question posed above follows from McConnel's generalization [24, Theorem 1] of a result due to Carlitz [7] (see also Bruen and Levinger [6]).

Theorem 1.1. *[24, Theorem 1] Let p denote a prime, $q = p^h$, and $1 < d$ a divisor of $q-1$. Also, let $F \colon \mathbb{F}_q \to \mathbb{F}_q$ be a function such that $F(0) = 0$ and $F(1) = 1$. Then*

$$(F(x) - F(y))^{\frac{q-1}{d}} = (x-y)^{\frac{q-1}{d}}$$

for all $x, y \in \mathbb{F}_q$ if and only if $F(x) = x^{p^j}$ for some $0 \leq j < h$ and $d \mid p^j - 1$.

Indeed, when the function F of Theorem 1.1 is $\mathbb{F}_q$-linear, we easily get the following corollary (see Section 2 for the proof, or [16, Corollary 1.4] for the case when q is an odd prime).

Corollary 1.2. *Let $g(x)$ and $f(x) = \alpha x^{q^k}$, $q = p^h$, be q-polynomials over $\mathbb{F}_{q^n}$ satisfying Condition (2). Denote $\gcd(k, n)$ by t. Then $g(x) = \beta x^{q^s}$ with $\gcd(s, n) = t$ for some β with $\mathrm{N}_{q^n/q^t}(\alpha) = \mathrm{N}_{q^n/q^t}(\beta)$.*

Another case for which we know a complete answer to our problem is when $f(x) = \mathrm{Tr}(x)$.

Theorem 1.3 ([8, Theorem 3.7])**.** *Let $f(x) = \mathrm{Tr}(x)$ and let $g(x)$ be a q-polynomial over $\mathbb{F}_{q^n}$ such that*

$$Im\,(f(x)/x) = Im\,(g(x)/x).$$

Then $g(x) = \mathrm{Tr}(\lambda x)/\lambda$ for some $\lambda \in \mathbb{F}_{q^n}^*$.

Note that in Theorem 1.3 we have $\hat{f}(x) = f(x)$ and the only solutions for g are $g(x) = f(\lambda x)/\lambda$, while in Corollary 1.2 we have (up to scalars) $\varphi(n)$ different solutions for g, where φ is the Euler's totient function.

The problem posed in (2) is also related to the study of the directions determined by an additive function. Indeed, when f is additive, then

$$Im\,(f(x)/x) = \left\{ \frac{f(x) - f(y)}{x - y} : x \neq y,\, x, y \in \mathbb{F}_{q^n} \right\},$$

is the *set of directions* determined by the graph of f, i.e. by the point set $\mathcal{G}_f := \{(x, f(x)) \colon x \in \mathbb{F}_{q^n}\} \subset \mathrm{AG}(2, q^n)$. Hence, in this setting, the problem posed in (2) corresponds to finding the $\mathbb{F}_q$-linear functions whose graph determines the same set of directions. The size of $Im\,(f(x)/x)$ (for any f, not necessarily additive) was studied extensively. When f is $\mathbb{F}_q$-linear the following result holds.

Result 1.4 ([1, 2]). *Let f be a q-polynomial over $\mathbb{F}_{q^n}$, with maximum field of linearity $\mathbb{F}_q$. Then*

$$q^{n-1} + 1 \leq |Im\,(f(x)/x)| \leq \frac{q^n - 1}{q - 1}.$$

The classical examples which show the sharpness of these bounds are the monomial functions x^{q^s}, with $\gcd(s, n) = 1$, and the $\mathrm{Tr}(x)$ function. However, these bounds are also achieved by other polynomials which are not "equivalent" to these examples (see Section 2 for more details).

Two $\mathbb{F}_q$-linear polynomials $f(x)$ and $h(x)$ of $\mathbb{F}_{q^n}[x]$ are *equivalent* if the two graphs $\mathcal{G}_f$ and $\mathcal{G}_h$ are equivalent under the action of the group $\Gamma\mathrm{L}(2, q^n)$, i.e. if there exists an element $\varphi \in \Gamma\mathrm{L}(2, q^n)$ such that $\mathcal{G}_f^{\varphi} = \mathcal{G}_h$. In such a case, we say that f and h are *equivalent* (via φ) and we write $h = f_\varphi$. It is easy to see that in this way we defined an equivalence relation on the set of q-polynomials over $\mathbb{F}_{q^n}$. If f and g are two q-polynomials such that $Im\,(f(x)/x) = Im\,(g(x)/x)$, then $Im\,(f_\varphi(x)/x) = Im\,(g_\varphi(x)/x)$ for any admissible $\varphi \in \Gamma\mathrm{L}(2, q^n)$ (see Proposition (2.6)). This means that the problem posed in (2) can be investigated up to equivalence.

For $n \leq 4$, the only solutions for g in Problem (2) are the trivial ones, i.e. either $g(x) = f(\lambda x)/x$ or $g(x) = \hat{f}(\lambda x)/x$ (cf. Theorem 2.8).

For the case $n = 5$, in Section 4, we prove the following main result.

Theorem 1.5. *Let $f(x)$ and $g(x)$ be two q-polynomials over $\mathbb{F}_{q^5}$, with maximum field of linearity $\mathbb{F}_q$, such that $Im\,(f(x)/x) = Im\,(g(x)/x)$. Then* *either there exists* $\varphi \in \Gamma\mathrm{L}(2,q^5)$ *such that* $f_\varphi(x) = \alpha x^{q^i}$ *and* $g_\varphi(x) = \beta x^{q^j}$ *with* $\mathrm{N}(\alpha) = \mathrm{N}(\beta)$ *for some* $i, j \in \{1,2,3,4\}$*, or there exists* $\lambda \in \mathbb{F}_{q^5}^*$ *such that* $g(x) = f(\lambda x)/\lambda$ *or* $g(x) = \hat{f}(\lambda x)/\lambda$*.*

Finally, the relation between $Im\,(f(x)/x)$ and the linear sets of rank n of the projective line $\mathrm{PG}(1,q^n)$ will be pointed out in Section 5. As an application of Theorem 1.5 we get a criterium of $\mathrm{P}\Gamma\mathrm{L}(2,q^5)$-equivalence for linear sets in $\mathrm{PG}(1,q^5)$ and this allows us to prove that the family of (maximum scattered) linear sets of rank n of size $(q^n-1)/(q-1)$ in $\mathrm{PG}(1,q^n)$ found by Sheekey in [27] contains members which are not-equivalent to the previously known linear sets of this size.

2 Background and preliminary results

Let us start this section by the following immediate corollary of Result 1.4.

Proposition 2.1. *If* $Im\,(f(x)/x) = Im\,(g(x)/x)$ *for two* q*-polynomials* f *and* g *over* $\mathbb{F}_{q^n}$*, then their maximum fields of linearity coincide.*

Proof. Let $\mathbb{F}_{q^m}$ and $\mathbb{F}_{q^k}$ be the maximum fields of linearity of f and g, respectively. Suppose to the contrary $m < k$. Then $|Im\,(g(x)/x)| \leq (q^n - 1)/(q^k - 1) < q^{n-k+1} + 1 \leq q^{n-m} + 1 \leq |Im\,(f(x)/x)|$, a contradiction by Corollary 1.2. □

Now we are able to prove Corollary 1.2.

Proof. The maximum field of linearity of $f(x)$ is $\mathbb{F}_{q^t}$, thus, by Proposition 2.1, $g(x)$ has to be a q^t-polynomial as well. Then for $t > 1$ the result follows from the $t = 1$ case (after substituting q for q^t and n/t for n) and hence we can assume that $f(x)$ and $g(x)$ are strictly $\mathbb{F}_q$-linear. By (2), we note that $g(1) = \alpha z_0^{q^k-1}$, for some $z_0 \in \mathbb{F}_{q^n}^*$. Let $F(x) := g(x)/g(1)$, then F is a q-polynomial over $\mathbb{F}_{q^n}$, with $F(0) = 0$ and $F(1) = 1$. Also, from (2), for each $x \in \mathbb{F}_{q^n}^*$ there exists $z \in \mathbb{F}_{q^n}^*$ such that

$$\frac{F(x)}{x} = \left(\frac{z}{z_0}\right)^{q^k-1}.$$

This means that for each $x \in \mathbb{F}_{q^n}^*$ we get $\mathrm{N}\left(\frac{F(x)}{x}\right) = 1$. By Theorem 1.1 (applied to the q-polynomial F with $d = q-1 \mid q^n - 1$ and using the fact that F is additive) it follows that $F(x) = x^{p^j}$ for some $0 \leq j < nh$. Then

Proposition 2.1 yields $p^j = q^s$ with $\gcd(s,n) = 1$. We get the first part of the statement by putting $\beta = g(1)$. Then from the assumption (2) it is easy to deduce $\mathrm{N}(\alpha) = \mathrm{N}(\beta)$. □

We will use the following definition.

Definition 2.2. *Let f and g be two equivalent q-polynomials over $\mathbb{F}_{q^n}$ via the element $\varphi \in \Gamma\mathrm{L}(2,q^n)$ represented by the invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and with companion automorphism σ of $\mathbb{F}_{q^n}$. Then*

$$\left\{ \begin{pmatrix} x \\ g(x) \end{pmatrix} : x \in \mathbb{F}_{q^n} \right\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\sigma \\ f(x)^\sigma \end{pmatrix} : x \in \mathbb{F}_{q^n} \right\}. \tag{3}$$

Let

$$K_f^\varphi(x) = ax^\sigma + bf(x)^\sigma$$

and

$$H_f^\varphi(x) = cx^\sigma + df(x)^\sigma.$$

Proposition 2.3. *Let f and g be q-polynomials over $\mathbb{F}_{q^n}$ such that $g = f_\varphi$ for some $\varphi \in \Gamma\mathrm{L}(2,q^n)$. Then K_f^φ is invertible and $g(x) = H_f^\varphi((K_f^\varphi)^{-1}(x))$.*

Proof. It easily follows from (3). □

From (3) it is also clear that

$$Im\left(\frac{f_\varphi(x)}{x}\right) = \left\{ \frac{c + dz^\sigma}{a + bz^\sigma} : z \in Im\left(\frac{f(x)}{x}\right) \right\} \tag{4}$$

and hence

$$|Im\,(f_\varphi(x)/x)| = |Im\,(f(x)/x)|. \tag{5}$$

From Equations (5) and Result 1.4 the next result easily follows.

Proposition 2.4. *If two q-polynomials over $\mathbb{F}_{q^n}$ are equivalent, then their maximum fields of linearity coincide.* □

Note that $|Im\,(g(x)/x)| = |Im\,(f(x)/x)|$ does not imply the equivalence of f and g. In fact, in the last section we will list the known examples of q-polynomials f which are not equivalent to monomials but the size of $Im\,(f(x)/x)$ is maximal. To find such functions was also proposed in [16] and, as it was observed by Sheekey, they determine certain MRD-codes [27].

Let us give the following definition.

Definition 2.5. *An element $\varphi \in \Gamma\mathrm{L}(2,q^n)$ represented by the invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and with companion automorphism σ of $\mathbb{F}_{q^n}$ is said to be* admissible *w.r.t. a given q-polynomial f over $\mathbb{F}_{q^n}$ if either $b = 0$ or $-(a/b)^{\sigma^{-1}} \notin Im\,(f(x)/x)$.*

The following results will be useful later in the paper.

Proposition 2.6. *If $Im\,(f(x)/x) = Im\,(g(x)/x)$ for some q-polynomials over $\mathbb{F}_{q^n}$, then $Im\,(f_\varphi(x)/x) = Im\,(g_\varphi(x)/x)$ holds for each admissible $\varphi \in \Gamma\mathrm{L}(2,q^n)$.*

Proof. From $Im\,(f(x)/x) = Im\,(g(x)/x)$ it follows that any $\varphi \in \Gamma\mathrm{L}(2,q^n)$ admissible w.r.t. f is admissible w.r.t. g as well. Hence K_f^φ and K_g^φ are both invertible and we may construct f_φ and g_φ as indicated in Proposition 2.3. The statement now follows from Equation (4). □

Proposition 2.7. *Let f and g be q-polynomials over $\mathbb{F}_{q^n}$ and take some $\varphi \in \Gamma\mathrm{L}(2,q^n)$ with companion automorphism σ. Then $g_\varphi(x) = f_\varphi(\lambda^\sigma x)/\lambda^\sigma$ for some $\lambda \in \mathbb{F}_{q^n}^*$ if and only if $g(x) = f(\lambda x)/\lambda$.*

Proof. First we prove the "if" part. Since $g(x) = f(\lambda x)/\lambda = (\omega_{1/\lambda} \circ f \circ \omega_\lambda)(x)$, where ω_α denotes the scalar map $x \in \mathbb{F}_{q^n} \mapsto \alpha x \in \mathbb{F}_{q^n}$, direct computations show that $H_g^\varphi = \omega_{1/\lambda^\sigma} \circ H_f^\varphi \circ \omega_\lambda$ and $K_g^\varphi = \omega_{1/\lambda^\sigma} \circ K_f^\varphi \circ \omega_\lambda$. Then $g_\varphi = \omega_{1/\lambda^\sigma} \circ f_\varphi \circ \omega_{\lambda^\sigma}$ and the first part of the statement follows. The "only if" part follows from the "if" part applied to $g_\varphi(x) = f_\varphi(\lambda^\sigma x)/\lambda^\sigma$ and φ^{-1}; and from $(f_\varphi)_{\varphi^{-1}} = f$ and $(g_\varphi)_{\varphi^{-1}} = g$. □

Next we summarize what is known about Problem (2) for $n \leq 4$.

Theorem 2.8. *Suppose $Im\,(f(x)/x) = Im\,(g(x)/x)$ for some q-polynomials over $\mathbb{F}_{q^n}$, $n \leq 4$, with maximum field of linearity $\mathbb{F}_q$. Then there exist $\varphi \in \mathrm{GL}(2,q^n)$ and $\lambda \in \mathbb{F}_{q^n}^*$ such that the following holds.*

- *If $n = 2$ then $f_\varphi(x) = x^q$ and $g(x) = f(\lambda x)/\lambda$.*
- *If $n = 3$ then either*

$$f_\varphi(x) = \mathrm{Tr}(x) \text{ and } g(x) = f(\lambda x)/\lambda$$

  or

$$f_\varphi(x) = x^q \text{ and } g(x) = f(\lambda x)/\lambda \text{ or } g(x) = \hat{f}(\lambda x)/\lambda.$$

- *If $n = 4$ then $g(x) = f(\lambda x)/\lambda$ or $g(x) = \hat{f}(\lambda x)/\lambda$.*

Proof. In the $n=2$ case $f(x)=ax+bx^q$, $b\neq 0$. Let φ be represented by the matrix $\begin{pmatrix} 1 & 0 \\ -a/b & 1/b \end{pmatrix}$. Then $\varphi\in \mathrm{GL}(2,q^2)$ maps $f(x)$ to x^q. Then Proposition 2.6 and Corollary 1.2 give $g_\varphi(x)=f_\varphi(\mu x)/\mu$ and hence Proposition 2.7 gives $g(x)=f(\lambda x)/\lambda$ for some $\lambda\in\mathbb{F}_{q^n}$. If $n=3$ then according to [20, Theorem 5] and [8, Theorem 1.3] there exists $\varphi\in \mathrm{GL}(2,q^3)$ such that either $f_\varphi(x)=\mathrm{Tr}(x)$ or $f_\varphi(x)=x^q$. In the former case Proposition 2.6 and Theorem 1.3 give $g_\varphi(x)=f_\varphi(\mu x)/\mu$ and the assertion follows from Proposition 2.7. In the latter case Proposition 2.6 and Corollary 1.2 give $g_\varphi(x)=\alpha x^{q^i}$ where $i\in\{1,2\}$ and $\mathrm{N}(\alpha)=1$. If $i=1$, then $g_\varphi(x)=f_\varphi(\mu x)/\mu$ where $\mu^{q-1}=\alpha$ and the assertion follows from Proposition 2.7. Let now $i=2$ and denote by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the matrix of φ^{-1}. Also, let Δ denote the determinant of this matrix and recall that $f_\varphi(x)=x^q$, with $\varphi\in\mathrm{GL}(2,q^3)$. Then by Proposition 2.3

$$K_{f_\varphi}^{\varphi^{-1}}(x)=Ax+Bx^q$$

is invertible and its inverse is the map

$$\psi(x):=\frac{A^{q+q^2}x-A^{q^2}Bx^q+B^{1+q}x^{q^2}}{\mathrm{N}(A)+\mathrm{N}(B)}.$$

Also, by Proposition 2.3 we have

$$\left(f_\varphi\right)_{\varphi^{-1}}(x)=C\psi(x)+D\psi(x)^q,$$

which gives $f(x)=\left(f_\varphi\right)_{\varphi^{-1}}(x)$.

Using similar arguments, since $\mathrm{N}(\alpha)=1$, direct computations show

$$g(x)=(g_\varphi)_{\varphi^{-1}}(x)=\frac{(A^{q+q^2}C+B^{q+q^2}D)x-B^{q^2}\Delta\alpha^{q^2+1}x^q+A^q\Delta\alpha x^{q^2}}{\mathrm{N}(A)+\mathrm{N}(B)},$$

and hence $g(x)=\hat{f}(\lambda x)/\lambda$ for each $\lambda\in\mathbb{F}_{q^3}^*$ with $\lambda^{q-1}=\Delta^{1-q}/\alpha^q$.

The case $n=4$ is [8, Proposition 4.2]. □

Remark 2.9. Theorem 2.8 yields that there is a unique equivalence class of q-polynomials, with maximum field of linearity $\mathbb{F}_q$, when $n=2$. For $n=3$ there are two non-equivalent classes and they correspond to the classical examples: $\mathrm{Tr}(x)$ and x^q. Whereas, for $n=4$, from [8, Sec. 5.3] and [4, Table p. 54], there exist at least eight non-equivalent classes. The possible sizes for the sets of directions determined by these strictly $\mathbb{F}_q$-linear functions are q^3+1, q^3+q^2-q+1, q^3+q^2+1 and q^3+q^2+q+1 and each of them is determined by at least two non-equivalent q-polynomials. Also, by [13, Theorem 3.4], if f is a q-polynomial over $\mathbb{F}_{q^4}$ for which the set of directions is of maximum size then f is equivalent either to x^q or to $\delta x^q + x^{q^3}$, for some $\delta \in \mathbb{F}_{q^4}^*$ with $N(\delta) \neq 1$ (see [22]).

3 Preliminary results about $\mathrm{Tr}(x)$ and the monomial q-polynomials over $\mathbb{F}_{q^5}$

Let q be a power of a prime p. We will need the following results.

Proposition 3.1. *Let $f(x) = \sum_{i=0}^{4} a_i x^{q^i}$ and $g(x) = \mathrm{Tr}(x)$ be q-polynomials over $\mathbb{F}_{q^5}$. Then there is an element $\varphi \in \Gamma\mathrm{L}(2, q^5)$ such that $Im\,(f_\varphi(x)/x) = Im\,(g(x)/x)$ if and only if $a_1a_2a_3a_4 \neq 0$, $(a_1/a_2)^q = a_2/a_3$, $(a_2/a_3)^q = a_3/a_4$ and $\mathrm{N}(a_1) = \mathrm{N}(a_2)$.*

Proof. Let $\varphi \in \Gamma\mathrm{L}(2, q^5)$ such that $Im\,(f_\varphi(x)/x) = Im\,(g(x)/x)$. By Proposition 2.4, the maximum field of linearity of f is $\mathbb{F}_q$ and by Theorem 1.3 there exists $\lambda \in \mathbb{F}_{q^5}^*$ such that $f_\varphi(x) = \mathrm{Tr}(\lambda x)/\lambda$. This is equivalent to the existence of a, b, c, d, $ad - bc \neq 0$ and $\sigma : x \mapsto x^{p^h}$ such that

$$\left\{ \begin{pmatrix} y \\ \mathrm{Tr}(y) \end{pmatrix} : y \in \mathbb{F}_{q^5} \right\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\sigma \\ f(x)^\sigma \end{pmatrix} : x \in \mathbb{F}_{q^5} \right\}.$$

Then $cx^\sigma + df(x)^\sigma \in \mathbb{F}_q$ for each $x \in \mathbb{F}_{q^5}$. Let $z = x^\sigma$. Then

$$cz + d \sum_{i=0}^{4} a_i^\sigma z^{q^i} = c^q z^q + d^q \sum_{i=0}^{4} a_i^{\sigma q} z^{q^{i+1}},$$

for each z. As polynomials of z the left and right-hand sides of the above equation coincide modulo $z^{q^5} - z$ and hence comparing coefficients yield

$$c + da_0^\sigma = d^q a_4^{\sigma q},$$

$$da_1^\sigma = c^q + d^q a_0^{\sigma q},$$

$$da_{k+1}^\sigma = d^q a_k^{\sigma q},$$

for $k = 1, 2, 3$. If $d = 0$, then $c = 0$, a contradiction. Since $d \neq 0$, if one of a_1, a_2, a_3, a_4 is zero, then all of them are zero and hence f is $\mathbb{F}_{q^5}$-linear. This is not the case, so we have $a_1a_2a_3a_4 \neq 0$. Then the last three equations yield

$$\left(\frac{a_1}{a_2}\right)^q = \frac{a_2}{a_3},$$

$$\left(\frac{a_2}{a_3}\right)^q = \frac{a_3}{a_4},$$

and by taking the norm of both sides in $da_2^{\sigma} = d^q a_1^{\sigma q}$ we get $\mathrm{N}(a_1) = \mathrm{N}(a_2)$.

Now assume that the conditions of the assertion hold. It follows that $a_3 = a_2^{q+1}/a_1^q$ and $a_4 = a_3^{q+1}/a_2^q = a_2^{q^2+q+1}/a_1^{q^2+q}$. Let $\alpha_i = a_i/a_1$ for $i = 0,1,2,3,4$. Then $\alpha_1 = 1$, $\mathrm{N}(\alpha_2) = 1$, $\alpha_3 = \alpha_2^{q+1}$ and $\alpha_4 = \alpha_2^{1+q+q^2}$. We have $\alpha_2 = \lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^5}^*$. If

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 - \lambda^{1-q^4} a_0/a_1 & \lambda^{1-q^4}/a_1 \end{pmatrix},$$

then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} =$$

$$\begin{pmatrix} x \\ x + \lambda^{1-q^4} x^q + \lambda^{q-q^4} x^{q^2} + \lambda^{q^2-q^4} x^{q^3} + \lambda^{q^3-q^4} x^{q^4} \end{pmatrix} = \begin{pmatrix} x \\ \mathrm{Tr}(x\lambda^{q^4})/\lambda^{q^4} \end{pmatrix},$$

i.e. $f_\varphi(x) = \mathrm{Tr}(\lambda^{q^4} x)/\lambda^{q^4}$, where φ is defined by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. □

Proposition 3.2. *Let $f(x) = \sum_{i=0}^4 a_i x^{q^i}$, with $a_1a_2a_3a_4 \neq 0$. Then there is an element $\varphi \in \Gamma\mathrm{L}(2, q^5)$ such that $Im\,(f_\varphi(x)/x) = Im\,(x^q/x)$ if and only if one of the following holds:*

1. $(a_1/a_2)^q = a_2/a_3$, $(a_2/a_3)^q = a_3/a_4$ *and* $\mathrm{N}(a_1) \neq \mathrm{N}(a_2)$, *or*
2. $(a_4/a_1)^{q^2} = a_1/a_3$, $(a_1/a_2)^{q^2} = a_3/a_4$ *and* $\mathrm{N}(a_1) \neq \mathrm{N}(a_3)$.

In both cases, if the condition on the norms does not hold, then $Im\,(f_\varphi(x)/x) = Im\,(\mathrm{Tr}(x)/x)$.

Proof. We first note that the monomials x^{q^i} and $x^{q^{5-i}}$ are equivalent via the map $\psi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence, by Corollary 1.2, the statement holds if and only if there exist a, b, c, d, $ad - bc \neq 0$, $\sigma : x \mapsto x^{p^h}$ and $i \in \{1, 2\}$ such that

$$\left\{ \begin{pmatrix} y \\ y^{q^i} \end{pmatrix} : y \in \mathbb{F}_{q^5} \right\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\sigma \\ f(x)^\sigma \end{pmatrix} : x \in \mathbb{F}_{q^5} \right\}. \tag{6}$$

If Condition 1 holds then let $\alpha_j = a_j/a_1$ for $j = 0,1,2,3,4$. So $\alpha_1 = 1$, $\mathrm{N}(\alpha_2) \neq 1$, $\alpha_3 = \alpha_2^{q+1}$, $\alpha_4 = \alpha_2^{1+q+q^2}$ and it turns out that

$$\begin{pmatrix} 1 & \alpha_2^{q^4} \\ \alpha_2^{1+q+q^2+q^3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha_0 & 1/a_1 \end{pmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & \alpha_2^{q^4} \\ \alpha_2^{1+q+q^2+q^3} & 1 \end{pmatrix} \begin{pmatrix} x \\ x^q + \alpha_2 x^{q^2} + \alpha_3 x^{q^3} + \alpha_4 x^{q^4} \end{pmatrix}.$$

Hence (6) is satisfied with $i = 1$, $\sigma : x \mapsto x$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \alpha_2^{q^4} \\ \alpha_2^{1+q+q^2+q^3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha_0 & 1/a_1 \end{pmatrix}.$$

If Condition (2) holds then let $\alpha_j = a_j/a_3$ for $j = 0, 1, 2, 3, 4$. So $\alpha_3 = 1$, $\mathrm{N}(\alpha_1) \neq 1$, $\alpha_2 = \alpha_1^{1+q+q^3}$, $\alpha_4 = \alpha_1^{1+q^3}$ and (6) is satisfied with $i = 2$, $\sigma : x \mapsto x$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_1^{1+q+q^3+q^4} & 1 \\ 1 & \alpha_1^{q^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha_0 & 1/a_3 \end{pmatrix}.$$

Suppose now that (6) holds and put $z = x^\sigma$. Then

$$(za + b\sum_{j=0}^{4} a_j^\sigma z^{q^j})^{q^i} = cz + d\sum_{j=0}^{4} a_j^\sigma z^{q^j}$$

for each $z \in \mathbb{F}_{q^5}$ and hence, as polynomials in z, the left-hand side and right-hand side of the above equation coincide modulo $z^{q^5} - z$. The coefficients of z, z^{q^i} and z^{q^k} with $i \in \{1, 2\}$ and $k \in \{1, 2, 3, 4\} \setminus \{i\}$ give

$$b^{q^i} a_{-i}^{\sigma q^i} = c + d a_0^\sigma,$$

$$a^{q^i} + b^{q^i} a_0^{\sigma q^i} = d a_i^\sigma,$$

$$b^{q^i} a_{k-i}^{\sigma q^i} = d a_k^\sigma,$$

respectively, where the indices are considered modulo 5. Note that $db \neq 0$ since otherwise also $a = c = 0$ and hence $ad - bc = 0$. With $\{r, s, t\} = \{1, 2, 3, 4\} \setminus \{i\}$, the last three equations yield:

$$\left(\frac{a_{r-i}}{a_{s-i}}\right)^{q^i} = \frac{a_r}{a_s},$$

$$\left(\frac{a_{s-i}}{a_{t-i}}\right)^{q^i} = \frac{a_s}{a_t}.$$

First assume $i = 1$. Then we have

$$\left(\frac{a_1}{a_2}\right)^q = \frac{a_2}{a_3} \quad \text{and} \quad \left(\frac{a_2}{a_3}\right)^q = \frac{a_3}{a_4}.$$

If $\mathrm{N}(a_1) = \mathrm{N}(a_2)$, from Proposition 3.1 and Equation (5) it follows that $|Im\,(x^q/x)| = |Im\,(\mathrm{Tr}(x)/x)|$. Since $|Im\,(x^q/x)| = (q^n-1)/(q-1)$ and $|Im\,(\mathrm{Tr}(x)/x)| = q^{n-1}+1$, we get a contradiction.

Now assume $i = 2$. Then we have $(a_4/a_1)^{q^2} = a_1/a_3$ and

$$\left(\frac{a_1}{a_2}\right)^{q^2} = \frac{a_3}{a_4}. \tag{7}$$

Multiplying these two equations yields $a_4^{q^2+1} = a_1 a_2^{q^2}$ and hence

$$a_2 = a_1^{1+q+q^3}/a_3^{q^3+q}. \tag{8}$$

By (7) this implies

$$a_4 = a_1^{q^3+1}/a_3^{q^3}. \tag{9}$$

If $\mathrm{N}(a_1) = \mathrm{N}(a_3)$, then also $\mathrm{N}(a_1) = \mathrm{N}(a_2) = \mathrm{N}(a_3) = \mathrm{N}(a_4)$. We show that in this case $Im\,(f_\varphi(x)/x) = Im\,(\mathrm{Tr}(x)/x)$, so we must have $\mathrm{N}(a_1) \neq \mathrm{N}(a_3)$. According to Proposition 3.1 it is enough to show $(a_1/a_2)^q = a_2/a_3$ and $(a_2/a_3)^q = a_3/a_4$. By (7) we have $(a_1/a_2)^q = (a_3/a_4)^{q^4}$, which equals a_2/a_3 if and only if $(a_2/a_3)^q = a_3/a_4$, i.e. $a_3^{1+q} = a_4 a_2^q$. Taking into account (8) and (9), this equality follows from $\mathrm{N}(a_1) = \mathrm{N}(a_3)$. □

4 Proof of the main theorem

In this section we prove Theorem 1.5. In order to do this, we use the following two results and the technique developed in [8].

Lemma 4.1 ([8, Lemma 3.4]). *Let f and g be two linearized polynomials over $\mathbb{F}_{q^n}$. If $Im\,(f(x)/x) = Im\,(g(x)/x)$, then for each positive integer d the following holds*

$$\sum_{x\in\mathbb{F}_{q^n}^*} \left(\frac{f(x)}{x}\right)^d = \sum_{x\in\mathbb{F}_{q^n}^*} \left(\frac{g(x)}{x}\right)^d.$$

Lemma 4.2 (see for example [8, Lemma 3.5]). *For any prime power q and integer d we have $\sum_{x\in\mathbb{F}_q^*} x^d = -1$ if $q-1 \mid d$ and $\sum_{x\in\mathbb{F}_q^*} x^d = 0$ otherwise.*

Proposition 4.3. *Let $f(x) = \sum_{i=0}^{4} a_i x^{q^i}$ and $g(x) = \sum_{i=0}^{4} b_i x^{q^i}$ be two q-polynomials over $\mathbb{F}_{q^5}$ such that $Im\,(f(x)/x) = Im\,(g(x)/x)$. Then the following relations hold between the coefficients of f and g:*

$$a_0 = b_0, \tag{10}$$

$$a_1 a_4^q = b_1 b_4^q, \tag{11}$$

$$a_2 a_3^{q^2} = b_2 b_3^{q^2}, \tag{12}$$

$$a_1^{q+1} a_3^{q^2} + a_2 a_4^{q+q^2} = b_1^{q+1} b_3^{q^2} + b_2 b_4^{q+q^2}, \tag{13}$$

$$a_1 a_2^{q+q^3} + a_3^{1+q^3} a_4^q = b_1 b_2^{q+q^3} + b_3^{1+q^3} b_4^q, \tag{14}$$

$$\begin{aligned} &a_1^{1+q+q^2} a_2^{q^3} + a_2^{1+q} a_3^{q^2+q^3} + a_1^q a_3^{1+q^2+q^3} + a_1^{q^2} a_2 a_3^{q^3} a_4^q + a_2^{1+q+q^3} a_4^{q^2} + \\ &a_1^q a_2^{q^3} a_3 a_4^{q^2} + a_1 a_2^q a_3^{q^2} a_4^{q^3} + a_1^{1+q^2} a_4^{q+q^3} + a_3 a_4^{q+q^2+q^3} = \\ &b_1^{1+q+q^2} b_2^{q^3} + b_2^{1+q} b_3^{q^2+q^3} + b_1^q b_3^{1+q^2+q^3} + b_1^{q^2} b_2 b_3^{q^3} b_4^q + b_2^{1+q+q^3} b_4^{q^2} + \\ &b_1^q b_2^{q^3} b_3 b_4^{q^2} + b_1 b_2^q b_3^{q^2} b_4^{q^3} + b_1^{1+q^2} b_4^{q+q^3} + b_3 b_4^{q+q^2+q^3}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\mathrm{N}(a_1) + \mathrm{N}(a_2) + \mathrm{N}(a_3) + \mathrm{N}(a_4) + \mathrm{Tr}(a_1^q a_2^{q^2+q^3+q^4} a_3 + a_1^{q+q^3} a_2^{q^4} a_3^{1+q^2} + \\ &a_1^{q+q^2} a_2^{q^3+q^4} a_4 + a_1^{q+q^2+q^4} a_3^{q^3} a_4 + a_2^q a_3^{q^2+q^3+q^4} a_4 + a_1^{q^2} a_3^{q^3+q^4} a_4^{1+q} + \\ &a_2^{q+q^3} a_3^{q^4} a_4^{1+q^2} + a_1^{q^2} a_2^{q^4} a_4^{1+q+q^3}) = \\ &\mathrm{N}(b_1) + \mathrm{N}(b_2) + \mathrm{N}(b_3) + \mathrm{N}(b_4) + \mathrm{Tr}(b_1^q b_2^{q^2+q^3+q^4} b_3 + b_1^{q+q^3} b_2^{q^4} b_3^{1+q^2} + \\ &b_1^{q+q^2} b_2^{q^3+q^4} b_4 + b_1^{q+q^2+q^4} b_3^{q^3} b_4 + b_2^q b_3^{q^2+q^3+q^4} b_4 + b_1^{q^2} b_3^{q^3+q^4} b_4^{1+q} + \\ &b_2^{q+q^3} b_3^{q^4} b_4^{1+q^2} + b_1^{q^2} b_2^{q^4} b_4^{1+q+q^3}). \end{aligned} \tag{16}$$

Proof. Equations (10)–(14) follow from [8, Lemma 3.6]. To prove (15) we will use Lemma 4.1 with $d = q^3 + q^2 + q + 1$. This gives us

$$\sum_{1 \le i,j,m,n \le 4} a_i a_j^q a_m^{q^2} a_n^{q^3} \sum_{x \in \mathbb{F}_{q^5}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+2} - q^2 + q^{n+3} - q^3} =$$

$$\sum_{1 \le i,j,m \le 4} b_i b_j^q b_m^{q^2} b_n^{q^3} \sum_{x \in \mathbb{F}_{q^5}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+2} - q^2 + q^{n+3} - q^3}.$$

By Lemma 4.2 we have $\sum_{x \in \mathbb{F}_{q^5}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+2} - q^2 + q^{n+3} - q^3} = -1$ if and only if

$$q^i + q^{j+1} + q^{m+2} + q^{n+3} \equiv 1 + q + q^2 + q^3 \pmod{q^5 - 1}, \tag{17}$$

and zero otherwise. Suppose that the former case holds. The right-hand side of (17) is smaller than the left-hand side, thus

$$q^i + q^{j+1} + q^{m+2} + q^{n+3} = 1 + q + q^2 + q^3 + k(q^5 - 1),$$

for some positive integer k. We have $q^i + q^{j+1} + q^{m+2} + q^{n+3} \le q^4 + q^5 + q^6 + q^7 < 1 + q + q^2 + q^3 + (q^2 + q + 2)(q^5 - 1)$ and hence $k \le q^2 + q + 1$. If $i = 1$, then $q^2 \mid 1 - k$ and hence $k = 1$, $j = m = 1$ and $n = 2$, or $k = q^2 + 1$, $n = 4$ and either $j = 2$ and $m = 3$, or $j = 4$ and $m = 1$. If $i > 1$, then q^2 divides $q + 1 - k$ and hence $k = q + 1$, or $k = q^2 + q + 1$. In the former case $i = j = n = 2$ and $m = 4$, or $i = j = 2$ and $n = m = 3$, or $i = 3$, $j = 1$, $m = 4$ and $n = 2$, or $i = 3$, $j = 1$ and $m = n = 3$, or $m = 1$, $i = 2$, $j = 4$ and $n = 3$. In the latter case $i = 3$ and $n = m = j = 4$. Then (15) follows.

To prove (16) we follow the previous approach with $d = q^4+q^3+q^2+q+1$. We obtain

$$\sum a_i a_j^q a_m^{q^2} a_n^{q^3} a_r^{q^4} = \sum b_i b_j^q b_m^{q^2} b_n^{q^3} b_r^{q^4},$$

where the summation is on the quintuples (i, j, m, n, r) with elements taken from $\{1, 2, 3, 4\}$ such that $L_{i,j,m,n,r} := (q^i - 1) + (q^{j+1} - q) + (q^{m+2} - q^2) + (q^{n+3} - q^3) + (q^{r+4} - q^4)$ is divisible by $q^5 - 1$. Then

$$L_{i,j,m,n,r} \equiv K_{i,j',m',n',r'} \pmod{q^5 - 1},$$

where

$$K_{i,j',m',n',r'} = (q^i - 1) + (q^{j'} - q) + (q^{m'} - q^2) + (q^{n'} - q^3) + (q^{r'} - q^4),$$

such that $j' \equiv j + 1$, $m' \equiv m + 2$, $n' \equiv n + 3$, $r' \equiv r + 4 \pmod 5$ with

$$j' \in \{0, 2, 3, 4\}, \quad m' \in \{0, 1, 3, 4\}, \quad n' \in \{0, 1, 2, 4\}, \quad r' \in \{0, 1, 2, 3\}. \tag{18}$$

For $q = 2$ and $q = 3$ we can determine by computer those quintuples (i, j', m', n', r') for which $K_{i,j',m',n',r'}$ is divisible by $q^5 - 1$ and hence (16) follows. So we may assume $q > 3$. Then

$$3 - q^2 - q^3 - q^4 = (q - 1) + (1 - q) + (1 - q^2) + (1 - q^3) + (1 - q^4) \le$$

$$K_{i,j',m',n',r'} \le$$

$$(q^4 - 1) + (q^4 - q) + (q^4 - q^2) + (q^4 - q^3) + (q^3 - q^4) = 3q^4 - 1 - q - q^2,$$

and hence $L_{i,j,m,n,r}$ is divisible by $q^5 - 1$ if and only if $K_{i,j',m',n',r'} = 0$. It follows that

$$q^i + q^{j'} + q^{m'} + q^{n'} + q^{r'} = 1 + q + q^2 + q^3 + q^4. \tag{19}$$

For $h \in \{0,1,2,3,4\}$ let c_h denote the number of elements in the multiset $\{i, j', m', n', r'\}$ which equals h. So $\sum_{h=0}^{4} c_h q^h = 1+q+q^2+q^3+q^4$ for some $0 \leq c_h \leq 5$ with $\sum_{h=0}^{4} c_h = 5$. We cannot have $c_0 = 5$ since $q > 1$. If $c_i = 5$ for some $1 \leq i \leq 4$ then the left hand side of (19) is not congruent to 1 modulo q, a contradiction. It follows that $c_h \leq 4$. Thus for $q > 3$ (19) holds if and only if $c_h = 1$ for $h = 0,1,2,3,4$ and we have to find those quintuples (i, j', m', n', r') for which $i \in \{1,2,3,4\}$, $\{i, j', m', n', r'\} = \{0,1,2,3,4\}$ and (18) are satisfied. This can be done by computer and the 44 solutions yield (16). $\square$

Proof of Theorem 1.5

Since f has maximum field of linearity $\mathbb{F}_q$, we cannot have $a_1 = a_2 = a_3 = a_4 = 0$. If three of $\{a_1, a_2, a_3, a_4\}$ are zeros, then $f(x) = a_0 x + a_i x^{q^i}$, for some $i \in \{1,2,3,4\}$. Hence with φ represented by $\begin{pmatrix} 1 & 0 \\ -a_0/a_i & 1/a_i \end{pmatrix}$ we have $f_\varphi(x) = x^{q^i}$. Then Proposition 2.6 and Corollary 1.2 give $g_\varphi(x) = \beta x^{q^j}$ where $\mathrm{N}(\beta) = 1$ and $j \in \{1,2,3,4\}$. Now, we distinguish three main cases according to the number of zeros among $\{a_1, a_2, a_3, a_4\}$.

Two zeros among $\{a_1, a_2, a_3, a_4\}$

Applying Proposition 4.3 we obtain $a_0 = b_0$. The two non-zero coefficients can be chosen in six different ways, however the cases $a_1 a_2 \neq 0$ and $a_1 a_3 \neq 0$ correspond to $a_3 a_4 \neq 0$ and $a_2 a_4 \neq 0$, respectively, since $Im\,(f(x)/x) = Im\,(\hat{f}(x)/x)$. Thus, after possibly interchanging f with $\hat{f}$, we may consider only four cases.

First let $\boxed{f(x) = a_0 x + a_1 x^q + a_4 x^{q^4},\ a_1 a_4 \neq 0}$.

Applying Proposition 4.3 we obtain $0 = b_2 b_3^{q^2}$. Since $b_1 b_4 \neq 0$, from (13) we get $b_2 = b_3 = 0$ and hence (16) gives

$$\mathrm{N}(a_1) + \mathrm{N}(a_4) = \mathrm{N}(b_1) + \mathrm{N}(b_4).$$

Also, from (11) we have $\mathrm{N}(a_1)\,\mathrm{N}(a_4) = \mathrm{N}(b_1)\,\mathrm{N}(b_4)$. It follows that either $\mathrm{N}(a_1) = \mathrm{N}(b_1)$ and $\mathrm{N}(a_4) = \mathrm{N}(b_4)$, or $\mathrm{N}(a_1) = \mathrm{N}(b_4)$ and $\mathrm{N}(a_4) = \mathrm{N}(b_1)$. In the first case $b_1 = a_1 \lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^5}^*$ and by (11) we get $g(x) = f(\lambda x)/\lambda$. In the latter case $b_1 = a_4^q \lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^5}^*$ and by (11) we get $g(x) = \hat{f}(\lambda x)/\lambda$.

Now consider $\boxed{f(x)=a_1x^q+a_3x^{q^3},\, a_1a_3\neq 0}$.

Applying Proposition 4.3 and arguing as above we have either $b_2=b_4=0$ or $b_1=b_3=0$. In the first case from (15) we obtain

$$a_1^q a_3^{1+q^2+q^3}=b_1^q b_3^{1+q^2+q^3}$$

and together with (13) this yields $\mathrm{N}(a_1)=\mathrm{N}(b_1)$ and $\mathrm{N}(a_3)=\mathrm{N}(b_3)$. In this case $g(x)=f(\lambda x)/\lambda$ for some $\lambda\in\mathbb{F}_{q^5}^*$. If $b_1=b_3=0$, then in $\hat{g}(x)$ the coefficients of x^{q^2} and x^{q^4} are zeros thus applying the result obtained in the former case we get $\lambda\hat{g}(x)=f(\lambda x)$ and hence after substituting $y=\lambda x$ and taking the adjoints of both sides we obtain $g(y)=\hat{f}(\mu y)/\mu$, where $\mu=\lambda^{-1}$.

The cases $\boxed{f(x)=a_1x^q+a_2x^{q^2}}$ and $\boxed{f(x)=a_2x^{q^2}+a_3x^{q^3}}$ can be handled in a similar way, applying Equations (11)–(16) of Proposition 4.3.

One zero among $\{a_1,a_2,a_3,a_4\}$

Since $Im\,(f(x)/x)=Im\,(\hat{f}(x)/x)$, we may assume $a_1=0$ or $a_2=0$.

First suppose $a_1=0$. Then by (11) either $b_1=0$ or $b_4=0$. In the former case putting together Equations (12), (13), (14) we get $\mathrm{N}(a_i)=\mathrm{N}(b_i)$ for $i\in\{2,3,4\}$ and hence there exists $\lambda\in\mathbb{F}_{q^5}^*$ such that $g(x)=f(\lambda x)/\lambda$. If $a_1=b_4=0$, then in $\hat{g}(x)$ the coefficient of x^q is zero thus applying the previous result we get $g(x)=\hat{f}(\mu x)/\mu$, where $\mu=\lambda^{-1}$.

Now suppose $a_2=0$. Then by (12) either $b_2=0$ or $b_3=0$. Using the same approach but applying (11), (13) and (14) we obtain $g(x)=f(\lambda x)/\lambda$ or $g(x)=\hat{f}(\lambda x)/\lambda$.

Case $a_1a_2a_3a_4\neq 0$

We will apply (10)-(15) of Proposition 4.3. Note that Equations (11) and (12) yield $a_1a_2a_3a_4\neq 0\Leftrightarrow b_1b_2b_3b_4\neq 0$. Multiplying (13) by a_2 and applying (12) yield

$$a_2^2a_4^{q+q^2}-a_2(b_1^{q+1}b_3^{q^2}+b_2b_4^{q+q^2})+a_1^{q+1}b_3^{q^2}b_2=0.$$

Taking (11) into account, this is equivalent to

$$(a_2a_4^{q+q^2}-b_1^{q+1}b_3^{q^2})(a_2a_4^{q+q^2}-b_2b_4^{q+q^2})=0.$$

Multiplying (14) by a_1 and applying (11) yield

$$a_1^2 a_2^{q+q^3} - a_1(b_1 b_2^{q+q^3} + b_3^{1+q^3} b_4^q) + a_3^{1+q^3} b_4^q b_1 = 0.$$

Taking (12) into account, this is equivalent to

$$(a_1 a_2^{q+q^3} - b_1 b_2^{q+q^3})(a_1 a_2^{q+q^3} - b_3^{1+q^3} b_4^q) = 0.$$

We distinguish four cases:

Case 1. $a_2 a_4^{q+q^2} = b_1^{q+1} b_3^{q^2}$ and $a_1 a_2^{q+q^3} = b_1 b_2^{q+q^3}$,

Case 2. $a_2 a_4^{q+q^2} = b_1^{q+1} b_3^{q^2}$ and $a_1 a_2^{q+q^3} = b_3^{1+q^3} b_4^q$,

Case 3. $a_2 a_4^{q+q^2} = b_2 b_4^{q+q^2}$ and $a_1 a_2^{q+q^3} = b_1 b_2^{q+q^3}$,

Case 4. $a_2 a_4^{q+q^2} = b_2 b_4^{q+q^2}$ and $a_1 a_2^{q+q^3} = b_3^{1+q^3} b_4^q$.

We show that these four cases produce the relations:

$$\mathrm{N}\left(\frac{b_1}{a_4}\right) = \frac{a_1 a_2^{q+q^3}}{a_4^q a_3^{q^3+1}} = \frac{b_1 b_2^{q+q^3}}{b_4^q b_3^{q^3+1}}, \tag{20}$$

$$\mathrm{N}\left(\frac{b_1}{a_4}\right) = 1, \tag{21}$$

$$\mathrm{N}\left(\frac{b_1}{a_1}\right) = 1, \tag{22}$$

$$\mathrm{N}\left(\frac{b_1}{a_1}\right) = \frac{a_3^{q^3+1} a_4^q}{a_1 a_2^{q+q^3}} = \frac{b_1 b_2^{q+q^3}}{b_3^{q^3+1} b_4^q}, \tag{23}$$

respectively.

To see (20) observe that from $a_2 a_4^{q+q^2} = b_1^{q+1} b_3^{q^2}$ and (11) we get

$$\mathrm{N}\left(\frac{b_1}{a_4}\right) = \left(\frac{b_1^{q+1}}{a_4^{q+q^2}}\right)^{q^2+1} \frac{b_1^{q^4}}{a_4} = \left(\frac{a_2^{q^2+1}}{b_3^{q^2+q^4}}\right) \frac{a_1^{q^4}}{b_4} = \frac{a_1 a_2^{q+q^3}}{b_4^q b_3^{q^3+1}}, \tag{24}$$

where the last equation follows from $\mathrm{N}(b_1/a_4)^q = \mathrm{N}(b_1/a_4)$. Hence by $a_1 a_2^{q+q^3} = b_1 b_2^{q+q^3}$ and (14) we get (20).

Equation (21) immediately follows from (24) taking $a_1 a_2^{q+q^3} = b_3^{1+q^3} b_4^q$ into account.

Now we show (23). By (11), we get

$$\mathrm{N}\left(\frac{b_1}{a_1}\right) = \mathrm{N}\left(\frac{a_4}{b_4}\right) = \left(\frac{a_4}{b_4}\right)^{q+q^2}\left(\left(\frac{a_4}{b_4}\right)^{q+q^2}\right)^{q^2}\left(\frac{a_4}{b_4}\right). \tag{25}$$

Since $\mathrm{N}(b_1/a_1)^q = \mathrm{N}(b_1/a_1)$, by $a_2a_4^{q+q^2} = b_2b_4^{q+q^2}$, the previous equation becomes

$$\mathrm{N}\left(\frac{b_1}{a_1}\right) = \frac{b_2^{q^3+q}}{a_2^{q^3+q}}\frac{a_4^q}{b_4^q} \tag{26}$$

and taking $a_1a_2^{q+q^3} = b_3^{1+q^3}b_4^q$ and (12) into account we get (23).

Equation (22) immediately follows from (26) taking $a_1a_2^{q+q^3} = b_1b_2^{q+q^3}$ and (11) into account.

- In Case 3 by (22) we get $b_1 = a_1\lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^5}^*$ and by (11) and (12) we have $g(x) = f(\lambda x)/\lambda$.
- Analogously, in Case 2 $g(x) = \hat{f}(\lambda x)/\lambda$.
- Case 4 is just Case 3 after replacing g by $\hat{g}$ since $Im\,(g(x)/x) = Im\,(\hat{g}(x)/x)$.

This allows us to restrict ourself to Case 1.

Taking (11) and (12) into account, it will be useful to express a_1, a_2, a_3 as follows:

$$a_1 = \frac{b_1b_4^q}{a_4^q}, \quad a_2 = \frac{b_1^{q+1}b_3^{q^2}}{a_4^{q+q^2}}, \quad a_3 = \frac{b_2^{q^3}b_4^{1+q^4}}{a_1^{q^3+q^4}}. \tag{27}$$

We are going to simplify (15). Using Equations (27) and (11) it is easy to see that $a_2^{1+q}a_3^{q^2+q^3} = b_2^{1+q}b_3^{q^2+q^3}$, $a_1^{1+q^2}a_4^{q+q^3} = b_1^{1+q^2}b_4^{q+q^3}$, $a_1^{q^2}a_2a_3^{q^3}a_4^q = b_1b_2^qb_3^{q^2}b_4^{q^3}$, $a_1^qa_2^{q^3}a_3a_4^{q^2} = b_1^qb_2^{q^3}b_3b_4^{q^2}$, $a_1a_2^qa_3^{q^2}a_4^{q^3} = b_1^{q^2}b_2b_3^{q^3}b_4^q$ and hence

$$a_1^{1+q+q^2}a_2^{q^3} + a_1^qa_3^{1+q^2+q^3} + a_2^{1+q+q^3}a_4^{q^2} + a_3a_4^{q+q^2+q^3} = \tag{28}$$

$$b_1^{1+q+q^2}b_2^{q^3} + b_1^qb_3^{1+q^2+q^3} + b_2^{1+q+q^3}b_4^{q^2} + b_3b_4^{q+q^2+q^3}.$$

The following equations can be proved applying (11), (12) and (27):

$$\mathrm{N}\left(\frac{b_1}{a_4}\right)b_3b_4^{q+q^2+q^3} = a_2^{q^3}a_1^{1+q+q^2}, \tag{29}$$

$$\mathrm{N}\left(\frac{a_4}{b_1}\right) b_4^{q^2} b_2^{1+q+q^3} = a_1^q a_3^{1+q^2+q^3}, \tag{30}$$

$$\mathrm{N}\left(\frac{b_1}{a_4}\right) b_1^q b_3^{1+q^2+q^3} = a_2^{1+q+q^3} a_4^{q^2}, \tag{31}$$

$$\mathrm{N}\left(\frac{a_4}{b_1}\right) b_2^{q^3} b_1^{1+q+q^2} = a_3 a_4^{q+q^2+q^3}. \tag{32}$$

Then (28) can be written as

$$(\mathrm{N}(b_1/a_4)-1)(b_3 b_4^{q+q^2+q^3} + b_1^q b_3^{1+q^2+q^3}) = \frac{\mathrm{N}(b_1/a_4)-1}{\mathrm{N}(b_1/a_4)}(b_4^{q^2} b_2^{1+q+q^3} + b_2^{q^3} b_1^{1+q+q^2}).$$

If $\mathrm{N}(b_1/a_4) = 1$, then (24) equals 1 and hence $a_1 a_2^{q+q^3} = b_4^q b_3^{q^3+1}$ which means that we are in Case 2. Then again $g(x) = \hat{f}(\lambda x)/\lambda$.

Otherwise dividing by $\mathrm{N}(b_1/a_4)-1$ and substituting $\mathrm{N}(b_1/a_4) = b_1 b_2^{q+q^3}/b_4^q b_3^{q^3+1}$ we obtain

$$b_1 b_2^{q+q^3}(b_3 b_4^{q+q^2+q^3} + b_1^q b_3^{1+q^2+q^3}) = b_4^q b_3^{q^3+1}(b_4^{q^2} b_2^{1+q+q^3} + b_2^{q^3} b_1^{1+q+q^2}).$$

Substituting $\mathrm{N}(b_1/a_4) b_4^q b_3^{q^3+1}/b_2^{q+q^3}$ for b_1 and using the fact that $\mathrm{N}(b_1/a_4) \in \mathbb{F}_q$ we obtain

$$\left(1 - \mathrm{N}\left(\frac{b_1}{a_4}\right)^2 \mathrm{N}\left(\frac{b_3}{b_2}\right)\right)\left(\mathrm{N}\left(\frac{b_1}{a_4}\right) b_4^{q+q^3} b_3 - b_2^{1+q+q^3}\right) = 0.$$

This gives us two possibilities:

$$\mathrm{N}\left(\frac{b_1}{a_4}\right) b_4^{q+q^3} b_3 = b_2^{1+q+q^3}, \tag{33}$$

or

$$\mathrm{N}\left(\frac{b_2}{b_3}\right) = \mathrm{N}\left(\frac{b_1}{a_4}\right)^2. \tag{34}$$

First consider the case when (34) holds.

We show $\mathrm{N}(a_1) = \mathrm{N}(b_1)$, that is, (22). We have $a_2 a_4^{q+q^2} = b_1^{q+1} b_3^{q^2}$ from (27) and hence $\mathrm{N}(a_2)\,\mathrm{N}(a_4)^2 = \mathrm{N}(b_1)^2\,\mathrm{N}(b_3)$. It follows that

$$\mathrm{N}\left(\frac{b_1}{a_4}\right)^2 = \mathrm{N}\left(\frac{a_2}{b_3}\right).$$

Combining this with (34) we obtain $\mathrm{N}(b_2) = \mathrm{N}(a_2)$. Then $\mathrm{N}(b_1) = \mathrm{N}(a_1)$ follows from $a_1 a_2^{q+q^3} = b_1 b_2^{q+q^3}$ since we are in Case 1.

From now on we can suppose that (33) holds.
Then (20) yields

$$\left(\frac{b_1}{b_2}\right)^{q^2} = \frac{b_3}{b_4}. \tag{35}$$

Multiplying both sides of (33) by $b_4^{q^2}$ and applying (29) gives

$$a_2^{q^3} a_1^{1+q+q^2} = b_2^{1+q+q^3} b_4^{q^2}. \tag{36}$$

Then multiplying (29) by (30) and taking (36) into account we obtain

$$a_1^q a_3^{1+q^2+q^3} = b_3 b_4^{q+q^2+q^3}. \tag{37}$$

Multiplying (31) and (32) yield

$$(b_1^q b_3^{1+q^2+q^3})(b_2^{q^3} b_1^{1+q+q^2}) = (a_2^{1+q+q^3} a_4^{q^2})(a_3 a_4^{q+q^2+q^3}).$$

On the other hand, from (28), and taking (36) and (37) into account, it follows that

$$b_1^q b_3^{1+q^2+q^3} + b_2^{q^3} b_1^{1+q+q^2} = a_2^{1+q+q^3} a_4^{q^2} + a_3 a_4^{q+q^2+q^3}.$$

Hence $b_1^q b_3^{1+q^2+q^3} = a_2^{1+q+q^3} a_4^{q^2}$ and $b_2^{q^3} b_1^{1+q+q^2} = a_3 a_4^{q+q^2+q^3}$, or $b_1^q b_3^{1+q^2+q^3} = a_3 a_4^{q+q^2+q^3}$ and $b_2^{q^3} b_1^{1+q+q^2} = a_2^{1+q+q^3} a_4^{q^2}$. In the former case (31) yields $\mathrm{N}(b_1/a_4) = 1$, which is (21). In the latter case (20) and (32) gives

$$\frac{b_4^q b_3^{q^3+1}}{b_1 b_2^{q+q^3}} b_2^{q^3} b_1^{1+q+q^2} = \mathrm{N}(a_4/b_1) b_2^{q^3} b_1^{1+q+q^2} = b_1^q b_3^{1+q^2+q^3},$$

and hence

$$\frac{b_4}{b_2} = \left(\frac{b_3}{b_1}\right)^q. \tag{38}$$

Equation (35) is equivalent to

$$b_4 b_1^{q^2} = b_3 b_2^{q^2}, \tag{39}$$

while (38) is equivalent to

$$b_4 b_1^q = b_3^q b_2.$$

Dividing these two equations by each other yield

$$b_2^{q^2-1} = b_3^{q-1} b_1^{q^2-q}.$$

It follows that there exists $\lambda \in \mathbb{F}_q^*$ such that

$$b_2^{q+1} = \lambda b_3 b_1^q, \tag{40}$$

thus

$$b_3 = b_2^{q+1}/(b_1^q \lambda) \tag{41}$$

and by (39)

$$b_4 = b_2^{1+q+q^2}/(b_1^{q+q^2}\lambda). \tag{42}$$

Then (20) can be written as

$$\mathrm{N}\left(\frac{b_1}{a_4}\right) = \frac{b_1 b_2^{q+q^3}}{b_4^q b_3^{q^3+1}} = \mathrm{N}\left(\frac{b_1}{b_2}\right)\lambda^3,$$

and hence

$$\mathrm{N}\left(\frac{b_2}{a_4}\right) = \lambda^3. \tag{43}$$

By (11), (43) and (42) we get

$$\mathrm{N}(a_1) = \mathrm{N}(b_2)^2/(\mathrm{N}(b_1)\lambda^2). \tag{44}$$

By (27), (41) and (43) we have

$$\mathrm{N}(a_2) = \mathrm{N}(b_1)\lambda. \tag{45}$$

By (27), (44) and (42) we get

$$\mathrm{N}(a_3) = \mathrm{N}(b_2)^3/(\mathrm{N}(b_1)^2\lambda^6), \tag{46}$$

and by (43) we have

$$\mathrm{N}(a_4) = \mathrm{N}(b_2)/\lambda^3. \tag{47}$$

Before we go further, we simplify (16) and prove

$$\mathrm{N}(a_1) + \mathrm{N}(a_2) + \mathrm{N}(a_3) + \mathrm{N}(a_4) = \mathrm{N}(b_1) + \mathrm{N}(b_2) + \mathrm{N}(b_3) + \mathrm{N}(b_4). \tag{48}$$

It is enough to show

$$\mathrm{Tr}(\overbrace{a_1^q a_2^{q^2+q^3+q^4}}^{A_1} a_3 + \overbrace{a_1^{q+q^3} a_2^{q^4} a_3^{1+q^2}}^{A_2} + \overbrace{a_1^{q+q^2} a_2^{q^3+q^4}}^{A_3} a_4 + \overbrace{a_1^{q+q^2+q^4} a_3^{q^3}}^{A_4} a_4 +$$

$$\overbrace{a_2^q a_3^{q^2+q^3+q^4} a_4}^{A_5} + \overbrace{a_1^{q^2} a_3^{q^3+q^4} a_4^{1+q}}^{A_6} + \overbrace{a_2^{q+q^3} a_3^{q^4} a_4^{1+q^2}}^{A_7} + \overbrace{a_1^{q^2} a_2^{q^4} a_4^{1+q+q^3}}^{A_8}) =$$

$$\mathrm{Tr}(\overbrace{b_1^q b_2^{q^2+q^3+q^4} b_3}^{B_1} + \overbrace{b_1^{q+q^3} b_2^{q^4} b_3^{1+q^2}}^{B_7} + \overbrace{b_1^{q+q^2} b_2^{q^3+q^4} b_4}^{B_3} + \overbrace{b_1^{q+q^2+q^4} b_3^{q^3} b_4}^{B_8} +$$

$$\overbrace{b_2^q b_3^{q^2+q^3+q^4} b_4}^{B_5} + \overbrace{b_1^{q^2} b_3^{q^3+q^4} b_4^{1+q}}^{B_6} + \overbrace{b_2^{q+q^3} b_3^{q^4} b_4^{1+q^2}}^{B_2} + \overbrace{b_1^{q^2} b_2^{q^4} b_4^{1+q+q^3}}^{B_4}),$$

which can be done by proving $\mathrm{Tr}(A_i) = \mathrm{Tr}(B_i)$ for $i = 1, 2, \ldots, 8$. Expressing a_3 with a_4 in (27), and using (11) as well, we get $a_3 = b_2^{q^3} a_4^{q^4+1} / b_1^{q^3+q^4}$. Then a_1, a_2, a_3 can be eliminated in all of the A_i, $i \in \{1, 2 \ldots, 8\}$. It turns out that this procedure eliminates also a_4 when $i \in \{2, 4, 7, 8\}$ and we obtain $A_2 = B_2^{q^2}$, $A_4 = B_4^{q^2}$, $A_7 = B_7^{q^3}$ and $A_8 = B_8^{q^2}$. In each of the other cases what remains is $\mathrm{N}(a_4)$ times an expression in b_1, b_2, b_3, b_4. Then by using (20) we can also eliminate $\mathrm{N}(a_4)$ and hence A_i can be expressed in terms of b_1, b_2, b_3, b_4. This gives $A_1 = B_1$ and $A_5 = B_5$. Applying also (35) and (38) we obtain $A_3 = B_3^{q^2}$ and $A_6 = B_6$.

Let $x = \mathrm{N}(b_2/b_1)$. Multiplying both sides of (48) by $\lambda^6/\mathrm{N}(b_1)$, taking into account (44), (45), (46) and (47) for the left hand side and (41) and (42) for the right hand side we get the following equation

$$x^2\lambda^4 + \lambda^7 + x^3 + x\lambda^3 = \lambda^6 + x\lambda^6 + x^2\lambda + \lambda x^3.$$

After rearranging we get:

$$(1 - \lambda)(x - \lambda)(x - \lambda^2)(x - \lambda^3) = 0.$$

First suppose $\boxed{\lambda \neq 1}$, then we have three possibilities.

If

$$x = \lambda,$$

in which case $N(b_2) = N(a_2)$ follows from (45). Since $\gcd(q - 1, q^5 - 1) = \gcd(q^2 - 1, q^5 - 1)$, in $\mathbb{F}_{q^5}^*$ the set of $(q - 1)$-th powers is the same as the set of $(q^2 - 1)$-th powers and hence there exists and element $\nu \in \mathbb{F}_{q^5}^*$ such that $b_2 = \nu^{q^2-1} a_2$. Therefore, since we are in Case 1, from $a_1 a_2^{q+q^3} = b_1 b_2^{q+q^3}$ we obtain $b_1 = \nu^{q-1} a_1$. Equations (11) and (12) give $g(x) = f(\nu x)/\nu$.

If

$$x = \lambda^3,$$

then $N(a_4) = N(b_1)$ follows from (47). Then (24) equals 1 and hence $a_1 a_2^{q+q^3} = b_4^q b_3^{q^3+1}$ which means that we are in Case 2, thus $g(x) = \hat{f}(\mu x)/\mu$.

If

$$x = \lambda^2,$$

then we show that there exists $\varphi \in \Gamma\mathrm{L}(2, q^5)$ such that either $Im\,(g_\varphi(x)/x) = Im\,(x^q/x)$ or $Im\,(g_\varphi(x)/x) = Im\,(\mathrm{Tr}(x)/x)$. In the former case by Proposition 2.6 and Corollary 1.2 we get $f_\varphi(x) = \alpha x^{q^i}$ and $g_\varphi(x) = \beta x^{q^j}$ for some $i, j \in \{1, 2, 3, 4\}$, with $\mathrm{N}(\alpha) = \mathrm{N}(\beta) = 1$. In the latter case, by Theorem 1.3 and by Propositions 2.6 and 2.7, there exists $\mu \in \mathbb{F}_{q^5}^*$ such that $g(x) = f(\mu x)/\mu$.

According to Proposition 3.2 part 2, it is enough to show

$$(b_4/b_1)^{q^2} = b_1/b_3, \quad (b_1/b_2)^{q^2} = b_3/b_4.$$

(Note that there is no need to confirm $\mathrm{N}(b_1) \neq \mathrm{N}(b_3)$ since otherwise the result follows from the last part of Proposition 3.2 and from Theorem 1.3.) The second equation is just (35), thus it is enough to prove the first one.

First we show

$$b_2 b_3^{q+q^3} = b_1^{1+q+q^3}. \tag{49}$$

From (40) we have

$$\mathrm{N}\left(\frac{b_2}{b_1}\right) = \lambda^2 = \left(\frac{b_2^{q+1}}{b_3 b_1^q}\right)^2,$$

and hence after rearranging

$$\frac{b_2^{q^2+q^3+q^4} b_3}{b_1^{1+q^2+q^3+q^4}} = \frac{b_2^{q+1}}{b_3 b_1^q}.$$

On the right-hand side we have λ, which is in $\mathbb{F}_q$, thus, after taking q-th powers on the left and q^3-th powers on the right, the following also holds

$$\frac{b_2^{q^3+q^4+1} b_3^q}{b_1^{q+q^3+q^4+1}} = \frac{b_2^{q^3+q^4}}{b_3^{q^3} b_1^{q^4}}.$$

After rearranging we obtain (49).

Now we show that $(b_4/b_1)^{q^2} = b_1/b_3$ is equivalent to (49). Expressing b_4 from (35) we get

$$(b_4/b_1)^{q^2} = b_1/b_3 \Leftrightarrow b_3^{1+q^2} b_2^{q^4} = b_1^{1+q^2+q^4},$$

where the equation on the right-hand side is just the q^4-th power of (49).

Finally, consider the case $\boxed{\lambda = 1}$. Then $b_3 = b_2^{q+1}/b_1^q$, $b_4 = b_2^{1+q+q^2}/b_1^{q+q^2}$ and it follows from Proposition 3.2 that there exists $\varphi \in \Gamma\mathrm{L}(2, q^5)$ such that either $Im\,(g_\varphi(x)/x) = Im\,(x^q/x)$ or $Im\,(g_\varphi(x)/x) = Im\,(\mathrm{Tr}(x)/x)$. As above, the assertion follows either from Proposition 2.6 and Corollary 1.2 or from Theorem 1.3 and by Propositions 2.6 and 2.7.

This finishes the proof when $\prod_{i=1}^4 a_i b_i \neq 0$. □

5 New maximum scattered linear sets of $\mathrm{PG}(1, q^5)$

A point set L of a line $\Lambda = \mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$ is said to be an $\mathbb{F}_q$-*linear set* of Λ of rank n if it is defined by the non-zero vectors of an n-dimensional $\mathbb{F}_q$-vector subspace U of the two-dimensional $\mathbb{F}_{q^n}$-vector space W, i.e.

$$L = L_U := \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

One of the most natural questions about linear sets is their equivalence. Two linear sets L_U and L_V of $\mathrm{PG}(1, q^n)$ are said to be $\mathrm{P\Gamma L}$-*equivalent* (or simply *equivalent*) if there is an element in $\mathrm{P\Gamma L}(2, q^n)$ mapping L_U to L_V. In the applications it is crucial to have methods to decide whether two linear sets are equivalent or not. This can be a difficult problem and some results in this direction can be found in [8, 11]. If L_U and L_V are two equivalent $\mathbb{F}_q$-linear sets of rank n in $\mathrm{PG}(1, q^n)$ and φ is an element of $\Gamma\mathrm{L}(2, q^n)$ which induces a collineation mapping L_U to L_V, then $L_{U^\varphi} = L_V$. Hence the first step to face with the equivalence problem for linear sets is to determine which $\mathbb{F}_q$-subspaces can define the same linear set.

For any q-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ over $\mathbb{F}_{q^n}$, the graph $\mathcal{G}_f = \{(x, f(x)) \colon x \in \mathbb{F}_{q^n}\}$ is an $\mathbb{F}_q$-vector subspace of the 2-dimensional vector space $V = \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ and the point set

$$L_f := L_{\mathcal{G}_f} = \{\langle (x, f(x)) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$$

is an $\mathbb{F}_q$-linear set of rank n of $\mathrm{PG}(1, q^n)$. In this context, the problem posed in (2) corresponds to find all $\mathbb{F}_q$-subspaces of V of rank n (cf. [8, Proposition 2.3]) defining the linear set L_f. The maximum field of linearity of f is the maximum field of linearity of L_f, and it is well-defined (cf. Proposition 2.1 and [8, Proposition 2.3]). Also, by the Introduction, for any q-polynomial f over $\mathbb{F}_{q^n}$, the linear sets L_f, L_{f_λ} (with $f_\lambda(x) := f(\lambda x)/\lambda$ for each $\lambda \in \mathbb{F}_{q^n}^*$) and $L_{\hat{f}}$ coincide (cf. [3, Lemma 2.6] and the first part of [8, Section 3]). If f

and g are two equivalent q-polynomials over $\mathbb{F}_{q^n}$, i.e. $\mathcal{G}_f$ an $\mathcal{G}_g$ are equivalent w.r.t. the action of the group $\Gamma\mathrm{L}(2,q^n)$, then the corresponding $\mathbb{F}_q$-linear sets L_f and L_g of $\mathrm{PG}(1,q^n)$ are $\mathrm{P}\Gamma\mathrm{L}(2,q^n)$-equivalent. The converse does not hold (see [11] and [8] for further details).

The relation between the problem posed in (2) and the equivalence problem of linear sets of the projective line is summarized in the following result.

Proposition 5.1. *Let L_f and L_g be two $\mathbb{F}_q$-linear sets of rank n of $\mathrm{PG}(1,q^n)$. Then L_f and L_g are $\mathrm{P}\Gamma\mathrm{L}(2,q^n)$-equivalent if and only if there exists an element $\varphi \in \Gamma\mathrm{L}(2,q^n)$ such that $Im\,(f_\varphi(x)/x) = Im\,(g(x)/x)$.* □

Linear sets of rank n of $\mathrm{PG}(1,q^n)$ have size at most $(q^n-1)/(q-1)$. A linear set L_U of rank n whose size achieves this bound is called *maximum scattered*. For applications of these objects we refer to [26] and [18].

Definition 5.2 ([15, 21])**.** *A maximum scattered $\mathbb{F}_q$-linear set L_U of rank n in $\mathrm{PG}(1,q^n)$ is of* pseudoregulus type *if it is $\mathrm{P}\Gamma\mathrm{L}(2,q^n)$-equivalent to L_f with $f(x)=x^q$ or, equivalently, if there exists an element $\varphi \in \mathrm{GL}(2,q^n)$ such that*

$$L_{U^\varphi} = \{\langle (x,x^q)\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}.$$

By Proposition 5.1 and Corollary 1.2, it follows

Proposition 5.3. *An $\mathbb{F}_q$-linear set L_f of rank n of $\mathrm{PG}(1,q^n)$ is of pseudoregulus type if and only if $f(x)$ is equivalent to x^{q^i} for some i with $\gcd(i,n)=1$.* □

For the proof of the previous result see also [19].

The known pairwise non-equivalent families of q-polynomials over $\mathbb{F}_{q^n}$ which define maximum scattered linear sets of rank n in $\mathrm{PG}(1,q^n)$ are

1. $f_s(x) = x^{q^s}$, $1 \leq s \leq n-1$, $\gcd(s,n)=1$ ([5, 12]),
2. $g_{s,\delta}(x) = \delta x^{q^s} + x^{q^{n-s}}$, $n \geq 4$, $\mathrm{N}_{q^n/q}(\delta) \notin \{0,1\}$ [1], $\gcd(s,n)=1$ ([22] for $s=1$, [27, 23] for $s \neq 1$),
3. $h_{s,\delta}(x) = \delta x^{q^s} + x^{q^{s+n/2}}$, $n \in \{6,8\}$, $\gcd(s,n/2)=1$, $\mathrm{N}_{q^n/q^{n/2}}(\delta) \notin \{0,1\}$, for the precise conditions on δ and q see [9, Theorems 7.1 and 7.2] [2],

[1] This condition implies $q \neq 2$.

[2] Also here $q > 2$, otherwise the linear set defined by $h_{s,\delta}$ is never scattered.

4. $k_b(x) = x^q + x^{q^3} + bx^{q^5}$, $n = 6$, with $b^2 + b = 1$, $q \equiv 0, \pm 1 \pmod 5$ ([10]).

Remark 5.4. *All the previous polynomials in cases 2.,3.,4. above are examples of functions which are not equivalent to monomials but the set of directions determined by their graph has size $(q^n-1)/(q-1)$, i.e. the corresponding linear sets are maximum scattered. The existence of such linearized polynomials is briefly discussed also in [16, p. 132].*

For $n = 2$ the maximum scattered $\mathbb{F}_q$-linear sets coincide with the Baer sublines. For $n = 3$ the maximum scattered linear sets are all of pseudoregulus type and the corresponding q-polynomials are all $\mathrm{GL}(2, q^3)$-equivalent to x^q (cf. [20]). For $n = 4$ there are two families of maximum scattered linear sets. More precisely, if L_f is a maximum scattered linear set of rank 4 of $\mathrm{PG}(1, q^4)$, with maximum field of linearity $\mathbb{F}_q$, then there exists $\varphi \in \mathrm{GL}(2, q^4)$ such that either $f_\varphi(x) = x^q$ or $f_\varphi(x) = \delta x^q + x^{q^3}$, for some $\delta \in \mathbb{F}^*_{q^4}$ with $\mathrm{N}_{q^4/q}(\delta) \notin \{0, 1\}$ (cf. [13]). It is easy to see that $L_{f_1} = L_{f_s}$ for any s with $\gcd(s, n) = 1$, and f_i is equivalent to f_j if and only if $j \in \{i, n-i\}$. Also, the graph of $g_{s,\delta}$ is $\mathrm{GL}(2, q^n)$-equivalent to the graph of $g_{n-s,\delta^{-1}}$.

In [22, Theorem 3] Lunardon and Polverino proved that $L_{g_{1,\delta}}$ and L_{f_1} are not $\mathrm{P\Gamma L}(2, q^n)$-equivalent when $q > 3$, $n \geq 4$. This was extended also for $q = 3$ [10, Theorem 3.4]. Also in [10], it has been proven that for $n = 6, 8$ the linear sets L_{f_1}, $L_{g_{s,\delta}}$, $L_{h_{s',\delta'}}$ and L_{k_b} are pairwise non-equivalent for any choice of $s, s', \delta, \delta', b$.

In this section we prove that one can find for each $q > 2$ a suitable δ such that $L_{g_{2,\delta}}$ of $\mathrm{PG}(1, q^5)$ is not equivalent to the linear sets $L_{g_{1,\mu}}$ of $\mathrm{PG}(1, q^5)$ for each $\mu \in \mathbb{F}^*_{q^5}$, with $\mathrm{N}_{q^5/q}(\mu) \notin \{0, 1\}$. In order to do this, we first reformulate Theorem 1.5 as follows.

Theorem 5.5 (Theorem 1.5)**.** *Let $f(x)$ and $g(x)$ be two q-polynomials over $\mathbb{F}_{q^5}$ such that $L_f = L_g$. Then either $L_f = L_g$ is of pseudoregulus type or there exists some $\lambda \in \mathbb{F}^*_{q^5}$ such that $g(x) = f(\lambda x)/\lambda$ or $g(x) = \hat{f}(\lambda x)/\lambda$ holds.*

From [27, Theorem 8] and [23, Theorem 4.4] it follows that the family of $\mathbb{F}_q$-subspaces $U_{g_{s,\delta}}$, $s \notin \{1, n-1\}$, $\gcd(s, n) = 1$, contains members which are not $\Gamma\mathrm{L}$-equivalent to the previously known $\mathbb{F}_q$-subspaces defining maximum scattered linear sets of $\mathrm{PG}(1, q^n)$. Our next result shows that the corresponding family $L_{g_{s,\delta}}$ of linear sets contains (at least for $n = 5$) examples which are not $\mathrm{P\Gamma L}$-equivalent to the previously known maximum scattered linear sets.

Theorem 5.6. *Let $g_{2,\delta}(x) = \delta x^{q^2} + x^{q^3}$ for some $\delta \in \mathbb{F}_{q^5}^*$ with $\mathrm{N}(\delta)^5 \neq 1$. Then $L_{g_{2,\delta}}$ is not $\mathrm{P\Gamma L}(2, q^5)$-equivalent to any linear set $L_{g_{1,\mu}}$ and hence it is a new maximum scattered linear set.*

Proof. Suppose, contrary to our claim, that $L_{g_{2,\delta}}$ is $\mathrm{P\Gamma L}(2, q^5)$-equivalent to a linear set $L_{g_{1,\mu}}$. From Proposition 5.1 and Theorem 5.5, taking into account that $L_{g_{1,\mu}}$ is not of pseudoregulus type, it follows that there exist $\varphi \in \Gamma\mathrm{L}(2, q^5)$ and $\lambda \in \mathbb{F}_{q^5}^*$ such that either $(g_{2,\delta})_\varphi(x) = g_{1,\mu}(\lambda x)/\lambda$ or $(g_{2,\delta})_\varphi(x) = \hat{g}_{1,\mu}(\lambda x)/\lambda$. This is equivalent to say that there exist α, β, A, B, C, $D \in \mathbb{F}_{q^5}$ with $AD - BC \neq 0$ and a field automorphism τ of $\mathbb{F}_{q^5}$ such that

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x^\tau \\ g_{2,\delta}(x)^\tau \end{pmatrix} : x \in \mathbb{F}_{q^5} \right\} = \left\{ \begin{pmatrix} z \\ \alpha z^q + \beta z^{q^4} \end{pmatrix} : z \in \mathbb{F}_{q^5} \right\},$$

where $N(\alpha) \neq N(\beta)$ and $\alpha\beta \neq 0$. We may substitute x^τ by y, then

$$\alpha(Ay + B\delta^\tau y^{q^2} + By^{q^3})^q + \beta(Ay + B\delta^\tau y^{q^2} + By^{q^3})^{q^4} =$$

$$Cy + D\delta^\tau y^{q^2} + Dy^{q^3}$$

for each $y \in \mathbb{F}_{q^5}$. Comparing coefficients yields $C = 0$ and

$$\alpha A^q + \beta B^{q^4} \delta^{q^4 \tau} = 0, \tag{50}$$

$$\beta B^{q^4} = D\delta^\tau, \tag{51}$$

$$\alpha B^q \delta^{q\tau} = D, \tag{52}$$

$$\alpha B^q + \beta A^{q^4} = 0. \tag{53}$$

Conditions (51) and (52) give

$$B^{q^4 - q} = \delta^{(q+1)\tau} \alpha / \beta. \tag{54}$$

On the other hand from (53) we get $A^q = -B^{q^3} \alpha^{q^2} / \beta^{q^2}$ and substituting this into (50) we have

$$B^{q^3 - q^4} = \delta^{q^4 \tau} \beta^{q^2+1} / \alpha^{q^2+1}. \tag{55}$$

Equations (54) and (55) give $N(\beta/\alpha) = N(\delta)^{2\tau}$ and $N(\alpha/\beta)^2 = N(\delta)^\tau$, respectively. It follows that $N(\delta)^{5\tau} = 1$ and hence $N(\delta)^5 = 1$, a contradiction. □

Open problems

We conclude the paper by the following open problems.

1. Is it true also for $n > 5$ that for any pair of q-polynomials $f(x)$ and $g(x)$ of $\mathbb{F}_{q^n}[x]$, with maximum field of linearity $\mathbb{F}_q$, if $Im\,(f(x)/x) = Im\,(g(x)/x)$ then either there exists $\varphi \in \Gamma\mathrm{L}(2, q^n)$ such that $f_\varphi(x) = \alpha x^{q^i}$ and $g_\varphi(x) = \beta x^{q^j}$ with $\mathrm{N}(\alpha) = \mathrm{N}(\beta)$ and $\gcd(i, n) = \gcd(j, n) = 1$, or there exists $\lambda \in \mathbb{F}^*_{q^n}$ such that $g(x) = f(\lambda x)/\lambda$ or $g(x) = \hat{f}(\lambda x)/\lambda$?

2. Is it possible, at least for small values of $n > 4$, to classify, up to equivalence, the q-polynomials $f(x) \in \mathbb{F}_{q^n}[x]$ such that $|Im\,(f(x)/x)| = (q^n - 1)/(q - 1)$? Find new examples!

3. Is it possible, at least for small values of n, to classify, up to equivalence, the q-polynomials $f(x) \in \mathbb{F}_{q^n}[x]$ such that $|Im\,(f(x)/x)| = q^{n-1} + 1$? Find new examples!

4. Is it possible, at least for small values of n, to classify, up to equivalence, the q-polynomials $f(x) \in \mathbb{F}_{q^n}[x]$ such that in the multiset $\{f(x)/x \colon x \in \mathbb{F}^*_{q^n}\}$ there is a unique element which is represented more than $q - 1$ times? In this case the linear set L_f is an *i-club* of rank n and when $q = 2$, then such linear sets correspond to translation KM-arcs cf. [14] (a KM-arc, or $(q + t, t)$-arc of type $(0, 2, t)$, is a set of $q + t$ points of $\mathrm{PG}(2, 2^n)$, such that each line meet the point set in 0,2 or in t points, cf. [17]). Find new examples!

5. Determine the equivalence classes of the set of q-polynomials in $\mathbb{F}_{q^4}[x]$.

6. Determine, at least for small values of n, all the possible sizes of $Im\,(f(x)/x)$ where $f(x) \in \mathbb{F}_{q^n}[x]$ is a q-polynomial.

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