TOPOLOGICAL RIGIDITY OF COMPACT MANIFOLDS SUPPORTING SOBOLEV-TYPE INEQUALITIES

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Abstract. Let (M,g) be an n-dimensional $(n \geq 3)$ compact Riemannian manifold with $\text{Ric}_{(M,g)} \geq (n-1)g$. If (M,g) supports an AB-type critical Sobolev inequality with Sobolev constants close to the optimal ones corresponding to the standard unit sphere $(\mathbb{S}^n, g_0)$, we prove that (M,g) is topologically close to $(\mathbb{S}^n, g_0)$. Moreover, the Sobolev constants on (M,g) are precisely the optimal constants on the sphere $(\mathbb{S}^n, g_0)$ if and only if (M,g) is isometric to $(\mathbb{S}^n, g_0)$; in particular, the latter result answers a question of V.H. Nguyen.

1. Introduction

Let (M,g) be a smooth compact n-dimensional Riemannian manifold, $n \geq 3$. The general theory of Sobolev inequalities shows that there exist $A > 0$ and $B > 0$ such that

$$\left(\int_M |u|^{\frac{2n}{n-2}} \text{d}v_g\right)^{\frac{n-2}{n}} \leq A \int_M |\nabla_g u|^2 \text{d}v_g + B \int_M u^2 \text{d}v_g, \ \forall u \in H_1^2(M). \tag{1.1}$$

In fact, problem (1.1) is a part of the famous AB-program initiated by Aubin [1] concerning the optimality of the constants A and B; for a systematic presentation of this topic, see the monograph of Hebey [5, Chapters 4 & 5]. In particular, one can prove the existence of $B > 0$ such that (1.1) holds with $A = A_0 = \frac{4}{n(n-2)}\omega_n^{-\frac{2}{n}}$, cf. [5, Theorem 4.6], the latter value being the optimal Talenti constant in the Sobolev embedding $H_1^2(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$, $n \geq 3$, where $2^* = 2n/(n-2)$. Hereafter, $\omega_n = \text{Vol}_{g_0}(\mathbb{S}^n)$ denotes the volume of the standard unit sphere $(\mathbb{S}^n, g_0)$. If $u \equiv 1$ in (1.1), then we have $B \geq \text{Vol}_g(M)^{-\frac{2}{n}}$, where $\text{Vol}_g(S)$ denotes the volume of $S \subset M$ in (M,g). Moreover, if $n \geq 4$ then the validity of (1.1) with $A = A_0 = \frac{4}{n(n-2)}\omega_n^{-\frac{2}{n}}$ implies

$$B \geq \frac{1}{n(n-1)}\omega_n^{-\frac{2}{n}} \max_M \text{Scal}_{(M,g)},$$

where $\text{Scal}_{(M,g)}$ is the scalar curvature of (M,g), cf. [5, Proposition 5.1].

In the model case when $(M,g) = (\mathbb{S}^n, g_0)$ is the standard unit sphere of $\mathbb{R}^{n+1}$, Aubin [1] proved that the optimal values of A and B in (1.1) are

$$A_0 = \frac{4}{n(n-2)}\omega_n^{-\frac{2}{n}} \text{ and } B_0 = \omega_n^{-\frac{2}{n}}, \tag{1.2}$$

respectively; moreover, for every $\lambda > 1$, the function $u_\lambda(x) = (\lambda - \cos d_0(x))^{1-\frac{2}{n}}$, $x \in \mathbb{S}^n$, is extremal in (1.1), see also [5, Theorem 5.1]. Hereafter, $d_0(x) = d_{\mathbb{S}^n}(y_0, x)$, $x \in \mathbb{S}^n$, where $d_{\mathbb{S}^n}$ denotes the standard metric on $(\mathbb{S}^n, g_0)$ and the element $y_0 \in \mathbb{S}^n$ is arbitrarily fixed. Note however that on the quotients $M = \mathbb{S}^1(r) \times \mathbb{S}^2$ of $\mathbb{S}^3$ endowed with its natural

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metric g (with $r > 0$ sufficiently small) inequality (1.1) is not valid for $A = A_0$ and $B = \text{Vol}_g(M)^{-\frac{2}{n}}$, see [5, Proposition 5.7].

Let $B_M(x,\rho)$ and $B_{\mathbb{S}^n}(y,\rho)$ be the open geodesic balls with radius $\rho > 0$ and centers in $x \in M$ and $y \in \mathbb{S}^n$ in (M,g) and $(\mathbb{S}^n, g_0)$, respectively.

Our main result reads as follows:

Theorem 1.1. *Let (M,g) be an n-dimensional $(n \geq 3)$ compact Riemannian manifold with Ricci curvature $\text{Ric}_{(M,g)} \geq (n-1)g$ and assume that the Sobolev inequality (1.1) holds on (M,g) with some constants $A, B > 0$. Then the following assertions hold:*

(i) *$A \geq A_0$ and $B \geq B_0$, where A_0 and B_0 are from (1.2);*
(ii) *there exists $x_0 \in M$ such that for every $y_0 \in \mathbb{S}^n$ and $\rho \in [0,\pi]$,*

$$\text{Vol}_g(B_M(x_0,\rho)) \geq \min\left\{\frac{A_0}{A}, \frac{B_0}{B}\right\}^{\frac{n}{2}} \text{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0,\rho)). \tag{1.3}$$

Remark 1.1. Note that (1.3) is valid on the whole $[0,\infty)$. Indeed, since the Ricci curvature on (M,g) verifies $\text{Ric}_{(M,g)} \geq (n-1)g$, due to Bonnet-Myers theorem, the diameter $D_M := \text{diam}(M)$ of M is bounded from above by π; accordingly, for every $\rho \geq \pi$ one has $B_M(x_0,\rho) = M$ and $B_{\mathbb{S}^n}(y_0,\rho) = \mathbb{S}^n$, thus (1.3) can be extended beyond π.

Perelman [10] states that for every $n \geq 2$ there exists $\delta_n \in [0,1)$ such that if the n-dimensional compact Riemannian manifold (M,g) with Ricci curvature $\text{Ric}_{(M,g)} \geq (n-1)g$ verifies $\text{Vol}_g(M) \geq (1-\delta_n)\text{Vol}_{g_0}(\mathbb{S}^n)$, then M is homeomorphic to $\mathbb{S}^n$; this result has been improved by Cheeger and Colding [2, Theorem A.1.10] by replacing homeomorphic to diffeomorphic. The latter result, the equality case in Bishop-Gromov inequality and Theorem 1.1 imply the following topological rigidity for compact manifolds:

Corollary 1.1. *Under the same assumptions as in Theorem 1.1, if*

$$\max\left\{\frac{A}{A_0}, \frac{B}{B_0}\right\} \leq (1-\delta_n)^{-\frac{2}{n}},$$

then (M,g) is diffeomorphic to $(\mathbb{S}^n, g_0)$. Moreover, $A = A_0$ and $B = B_0$ if and only if (M,g) is isometric to $(\mathbb{S}^n, g_0)$.

Remark 1.2. The statement of Corollary 1.1 is in the spirit of the results of Ledoux [9] and do Carmo and Xia [4]. In these works certain Sobolev inequalities are considered on *non*-compact Riemannian manifolds with non-negative Ricci curvature, and the Riemannian manifold is isometric to the Euclidean space with the same dimension if and only if the Sobolev constants are precisely the Euclidean optimal constants. Further results in this direction can be found in the papers by Kristály [6, 7] and Kristály and Ohta [8]. Theorem 1.1 and Corollary 1.1 seem to be the first contributions within this topic in the setting of compact Riemannian manifolds, answering also a question of Nguyen [11].

2. Proofs

Proof of Theorem 1.1. (i) The validity of the Sobolev inequality (1.1) on (M,g) and a similar argument as in Hebey [5, Proposition 4.2] imply that $A \geq A_0$.

By Remark 1.1, we have $D_M :=\text{diam}(M) \leq \pi$. Since $\text{Ric}_{(M,g)} \geq (n-1)g$, by the Bishop-Gromov comparison principle we have that for every $x_0 \in M$ and $y_0 \in \mathbb{S}^n$, the function $\rho \mapsto \frac{\text{Vol}_g(B_M(x_0,\rho))}{\text{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0,\rho))}$ is non-increasing on $(0,\infty)$; in particular, we have

$$1 \geq \frac{\text{Vol}_g(B_M(x_0,\rho))}{\text{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0,\rho))} \geq \frac{\text{Vol}_g(B_M(x_0,\pi))}{\text{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0,\pi))} = \frac{\text{Vol}_g(M)}{\text{Vol}_{g_0}(\mathbb{S}^n)}, \ \forall \rho \in [0,\pi]. \tag{2.1}$$

Now, choosing $u \equiv 1$ in (1.1), it follows that

$$B \geq \mathrm{Vol}_g(M)^{-\frac{2}{n}} \geq \mathrm{Vol}_{g_0}(\mathbb{S}^n)^{-\frac{2}{n}} = \omega_n^{-\frac{2}{n}} = B_0.$$

(ii) If $D_M = \pi$, we have nothing to prove. Indeed, in this case (M, g) is isometric to $(\mathbb{S}^n, g_0)$, see Cheng [3] and Shiohama [12], i.e., $\mathrm{Vol}_g(M) = \mathrm{Vol}_{g_0}(\mathbb{S}^n)$ and (2.1) implies at once relation (1.3).

Accordingly, we assume that $D_M < \pi$. Fix $x_0, \tilde{x}_0 \in M$ such that $d_g(x_0, \tilde{x}_0) = D_M$, and $y_0 \in \mathbb{S}^n$. Let $\mathrm{d}v_g$ and $\mathrm{d}v_{g_0}$ be the canonical volume forms on (M, g) and $(\mathbb{S}^n, g_0)$, respectively. Let $f, s : (1, \infty) \to \mathbb{R}$ be the functions defined as

$$f(\lambda) = \int_M (\lambda - \cos d_g)^{2-n} \mathrm{d}v_g \quad \text{and} \quad s(\lambda) = \int_{\mathbb{S}^n} (\lambda - \cos d_0)^{2-n} \mathrm{d}v_{\mathbb{S}^n}, \quad \lambda > 1, \tag{2.2}$$

where $d_g = d_g(x_0, \cdot)$ ans $d_0 = d_{\mathbb{S}^n}(y_0, \cdot)$. It is easily seen that both functions f and s are well-defined and smooth on $(1, \infty)$.

The proof will be provided in several steps.

Step 1 (local behavior of f and s around 1). We claim that

$$\liminf_{\lambda \to 1^+} \frac{f(\lambda) - \lambda f'(\lambda)}{s(\lambda) - \lambda s'(\lambda)} \geq 1. \tag{2.3}$$

By the layer cake representation of functions and a change of variables, we have that

$$\begin{aligned}
I(\lambda) &:= f(\lambda) - \lambda f'(\lambda) \\
&= \int_M (\lambda - \cos d_g)^{1-n}((n-1)\lambda - \cos d_g) \mathrm{d}v_g \\
&= \int_0^\infty \mathrm{Vol}_g(\{x \in M : (\lambda - \cos d_g)^{1-n}((n-1)\lambda - \cos d_g) > t\}) \mathrm{d}t \\
&= (n-2) \int_0^{D_M} \mathrm{Vol}_g(B_M(x_0, \rho))(\lambda - \cos\rho)^{-n}(n\lambda - \cos\rho) \sin\rho \mathrm{d}\rho \\
&\quad + \mathrm{Vol}_g(M)(\lambda - \cos D_M)^{1-n}((n-1)\lambda - \cos D_M).
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
J(\lambda) &:= s(\lambda) - \lambda s'(\lambda) \\
&= (n-2) \int_0^{\pi} \mathrm{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0, \rho))(\lambda - \cos\rho)^{-n}(n\lambda - \cos\rho) \sin\rho \mathrm{d}\rho \\
&\quad + \mathrm{Vol}_{g_0}(\mathbb{S}^n)(\lambda + 1)^{1-n}((n-1)\lambda + 1).
\end{aligned}$$

Fix $\varepsilon > 0$ arbitrarily. Then the local behavior of the geodesic balls both on (M, g) and $(\mathbb{S}^n, g_0)$ implies that there exits $\delta = \delta_\varepsilon > 0$ sufficiently small such that for every $\rho \in (0, \delta)$,

$$\mathrm{Vol}_g(B_M(x_0, \rho)) \geq (1 - \varepsilon)\tilde{\omega}_n \rho^n$$

and

$$\mathrm{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0, \rho)) \leq (1 + \varepsilon)\tilde{\omega}_n \rho^n,$$

where $\tilde{\omega}_n$ denotes the volume of the n-dimensional unit ball in $\mathbb{R}^n$. Therefore, the above estimates give that

$$\frac{I(\lambda)}{J(\lambda)} \geq \frac{(1-\varepsilon)(n-2)\tilde{\omega}_n \displaystyle\int_0^\delta (\lambda - \cos\rho)^{-n}(n\lambda - \cos\rho)\rho^n \sin\rho \mathrm{d}\rho}{(1+\varepsilon)(n-2)\tilde{\omega}_n \displaystyle\int_0^\delta (\lambda - \cos\rho)^{-n}(n\lambda - \cos\rho)\rho^n \sin\rho \mathrm{d}\rho + \tilde{s}(\lambda, \delta, n)}, \tag{2.4}$$

where

$$\begin{aligned}\tilde{s}(\lambda,\delta,n) &= (n-2)\int_\delta^\pi \mathrm{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0,\rho))(\lambda-\cos\rho)^{-n}(n\lambda-\cos\rho)\sin\rho \mathrm{d}\rho \\ &\quad +\mathrm{Vol}_{g_0}(\mathbb{S}^n)(\lambda+1)^{1-n}((n-1)\lambda+1).\end{aligned}$$

Note first that $\tilde{s}(\lambda,\delta,n)=O(1)$ as $\lambda\to 1$. Now, we show that

$$\lim_{\lambda\to 1}\int_0^\delta (\lambda-\cos\rho)^{-n}(n\lambda-\cos\rho)\rho^n\sin\rho \mathrm{d}\rho=+\infty. \tag{2.5}$$

Since $\cos\rho>1-\rho^2$, $n\lambda-\cos\rho\geq n-1$ and $\sin\rho\geq\frac{2}{\pi}\rho$ for every $\rho\in(0,\delta)$ and $\lambda>1$, it suffices to prove that

$$\lim_{\lambda\to 1}\int_0^\delta \frac{\rho^{n+1}}{(\lambda-1+\rho^2)^n}\mathrm{d}\rho=+\infty.$$

In order to check the latter limit, by changes of variables one has

$$\begin{aligned}\lim_{\lambda\to 1}\left((\lambda-1)^{\frac{n}{2}-1}\int_0^\delta \frac{\rho^{n+1}}{(\lambda-1+\rho^2)^n}\mathrm{d}\rho\right) &= \lim_{\lambda\to 1}\int_0^{\delta/\sqrt{\lambda-1}}\frac{\tau^{n+1}}{(1+\tau^2)^n}\mathrm{d}\tau \qquad [\rho=\sqrt{\lambda-1}\tau]\\ &= \int_0^\infty \frac{\tau^{n+1}}{(1+\tau^2)^n}\mathrm{d}\tau\\ &= \frac{1}{2}\int_0^1 \theta^{\frac{n}{2}}(1-\theta)^{\frac{n}{2}-2}\mathrm{d}\theta \qquad \left[\tau=\sqrt{\frac{\theta}{1-\theta}}\right]\\ &= \frac{1}{2}\,\mathsf{B}\left(\frac{n}{2}+1,\frac{n}{2}-1\right).\end{aligned}$$

Step 2 (ODE vs. ODI; global comparison of f and s). Due to Aubin [1], the extremal function in (1.1) when $(M,g)=(\mathbb{S}^n,g_0)$ is $u_\lambda(x)=(\lambda-\cos d_0)^{1-\frac{n}{2}}$ for every $\lambda>1$. Thus, inserting u_λ into (1.1) when $(M,g)=(\mathbb{S}^n,g_0)$ and using the notation in (2.2), we have the following ODE:

$$\left[\frac{s''(\lambda)}{(n-2)(n-1)}\right]^{\frac{2}{2^*}}=\frac{2}{n}\omega_n^{-\frac{2}{n}}\left[\frac{1-\lambda^2}{2(n-1)}s''(\lambda)-\lambda s'(\lambda)+s(\lambda)\right],\ \lambda>1. \tag{2.6}$$

Let $K_0=\frac{2}{n}\omega_n^{-\frac{2}{n}}$ and $C=K_0\max\left\{\frac{A}{A_0},\frac{B}{B_0}\right\}$. Without loss of generality, we may assume that $A>A_0$; indeed, since $A\geq A_0$, we may take $A=A_0+\varepsilon$ for $\varepsilon>0$ sufficiently small. Since $B\geq \mathrm{Vol}_g(M)^{-\frac{2}{n}}\geq B_0$, it turns out that $C>K_0$. By introducing the function

$$H(\lambda)=\left(\frac{K_0}{C}\right)^{\frac{n}{2}}J(s)=\left(\frac{K_0}{C}\right)^{\frac{n}{2}}(s(\lambda)-\lambda s'(\lambda)),$$

one has $H'(\lambda)=-\lambda\left(\frac{K_0}{C}\right)^{\frac{n}{2}}s''(\lambda)$, therefore $s''(\lambda)=-\frac{H'(\lambda)}{\lambda}\left(\frac{K_0}{C}\right)^{-\frac{n}{2}}$. This means that the second order ODE (2.6) is equivalent to the following first order ODE:

$$\left[-\frac{H'(\lambda)}{\lambda(n-2)(n-1)}\right]^{\frac{2}{2^*}}=C\left[\frac{\lambda^2-1}{2\lambda(n-1)}H'(\lambda)+H(\lambda)\right],\ \lambda>1. \tag{2.7}$$

Now, if we replace $w_\lambda(x)=(\lambda-\cos d_g)^{1-\frac{n}{2}}$ for every $\lambda>1$ into (1.1) and we explore the eikonal equation $|\nabla_g d_g|=1$ valid a.e. on M, we obtain

$$\left[\int_M(\lambda-\cos d_g)^{-n}\mathrm{d}v_g\right]^{\frac{2}{2^*}}\leq A\int_M(\lambda-\cos d_g)^{-n}\sin^2 d_g\mathrm{d}v_g+B\int_M(\lambda-\cos d_g)^{2-n}\mathrm{d}v_g.$$

By using the notation in (2.2), the latter inequality can be rewritten into

$$\left[\frac{f''(\lambda)}{(n-2)(n-1)}\right]^{\frac{2}{2^*}} \leq K_0\left[\frac{A}{A_0}\frac{1-\lambda^2}{2(n-1)}f''(\lambda) - \frac{A}{A_0}\lambda f'(\lambda) + \left(\frac{2-n}{2}\frac{A}{A_0} + \frac{n}{2}\frac{B}{B_0}\right)f(\lambda)\right],$$

for every $\lambda > 1$. Since

$$\frac{1-\lambda^2}{2(n-1)}f''(\lambda) - \lambda f'(\lambda) + \frac{2-n}{2}f(\lambda) = \frac{n-2}{2}\int_M (\lambda - \cos d_g)^{-n}\sin^2 d_g dv_g \geq 0,$$

and $C = K_0 \max\left\{\frac{A}{A_0}, \frac{B}{B_0}\right\}$, the latter inequality implies that

$$\left[\frac{f''(\lambda)}{(n-2)(n-1)}\right]^{\frac{2}{2^*}} \leq C\left[\frac{1-\lambda^2}{2(n-1)}f''(\lambda) - \lambda f'(\lambda) + f(\lambda)\right], \quad \lambda > 1.$$

Since $I(\lambda) = f(\lambda) - \lambda f'(\lambda)$, we get the following first order ordinary differential inequality:

$$\left[-\frac{I'(\lambda)}{\lambda(n-2)(n-1)}\right]^{\frac{2}{2^*}} \leq C\left[\frac{\lambda^2-1}{2\lambda(n-1)}I'(\lambda) + I(\lambda)\right]. \tag{2.8}$$

We claim that

$$I(\lambda) \geq H(\lambda), \ \forall \lambda > 1. \tag{2.9}$$

First of all, by (2.3) we clearly have that

$$\liminf_{\lambda\to 1^+}\frac{I(\lambda)}{H(\lambda)} = \liminf_{\lambda\to 1^+}\frac{f(\lambda)-\lambda f'(\lambda)}{\left(\frac{K_0}{C}\right)^{\frac{n}{2}}(s(\lambda)-\lambda s'(\lambda))} \geq \left(\frac{C}{K_0}\right)^{\frac{n}{2}} > 1.$$

Thus, for sufficiently small $\delta_0 > 0$ one has

$$I(\lambda) \geq H(\lambda), \ \forall \lambda \in (1, \delta_0 + 1).$$

Assume by contradiction that $I(\lambda_0) < H(\lambda_0)$ for some $\lambda_0 > 1$. Clearly, $\lambda_0 > 1 + \delta_0$. Let us define

$$\lambda_s := \sup\{\lambda < \lambda_0 : I(\lambda) = H(\lambda)\} < \lambda_0.$$

Thus for any $\lambda \in [\lambda_s, \lambda_0]$ we have $I(\lambda) \leq H(\lambda)$. It is also clear that

$$-\frac{I'(\lambda)}{\lambda(n-2)(n-1)} = \frac{f''(\lambda)}{(n-2)(n-1)} > 0$$

and

$$-\frac{H'(\lambda)}{\lambda(n-2)(n-1)} = \frac{s''(\lambda)}{(n-2)(n-1)} > 0.$$

Let us define the increasing function $\varphi_\lambda : (0,\infty) \to \mathbb{R}$ by

$$\varphi_\lambda(t) = t^{\frac{2}{2^*}} + \frac{(n-2)}{2}C(\lambda^2-1)t.$$

By relations (2.7), (2.8) and the definition of φ_λ, for every $\lambda \in [\lambda_s, \lambda_0]$ we have that

$$\begin{aligned}\varphi_\lambda\left(-\frac{I'(\lambda)}{\lambda(n-2)(n-1)}\right) &= \left(-\frac{I'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{2}{2^*}} + \frac{(n-2)}{2}C(\lambda^2-1)\left(-\frac{I'(\lambda)}{\lambda(n-2)(n-1)}\right)\\ &\leq CI(\lambda)\\ &\leq CH(\lambda)\\ &= \varphi_\lambda\left(-\frac{H'(\lambda)}{\lambda(n-2)(n-1)}\right).\end{aligned}$$

Therefore, the monotonicity of φ_λ implies

$$I'(\lambda) \geq H'(\lambda),\ \forall \lambda \in [\lambda_s, \lambda_0].$$

In particular $\lambda \mapsto I(\lambda) - H(\lambda)$ is non-decreasing on the interval $[\lambda_s, \lambda_0]$. Consequently, we have

$$0 = I(\lambda_s) - H(\lambda_s) \leq I(\lambda_0) - H(\lambda_0) < 0,$$

a contradiction, which shows the validity of (2.9).

Step 3 (proving (1.3)). Due to (2.1), the claim is concluded once we prove

$$\frac{\mathrm{Vol}_g(M)}{\mathrm{Vol}_{g_0}(\mathbb{S}^n)} \geq \min\left\{\frac{A_0}{A}, \frac{B_0}{B}\right\}^{\frac{n}{2}}. \tag{2.10}$$

Note that relation (2.9) is equivalent to

$$(n-2)\int_0^{D_M} \mathrm{Vol}_g(B_M(x_0,\rho))\frac{n\lambda - \cos\rho}{(\lambda-\cos\rho)^n}\sin\rho \mathrm{d}\rho + \mathrm{Vol}_g(M)\frac{(n-1)\lambda - \cos D_M}{(\lambda - \cos D_M)^{n-1}}$$

$$\geq \left(\frac{K_0}{C}\right)^{\frac{n}{2}}\left[(n-2)\int_0^{\pi} \mathrm{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0,\rho))\frac{n\lambda - \cos\rho}{(\lambda-\cos\rho)^n}\sin\rho \mathrm{d}\rho + \mathrm{Vol}_{g_0}(\mathbb{S}^n)\frac{(n-1)\lambda + 1}{(\lambda+1)^{n-1}}\right],$$

for every $\lambda > 1$.

Let us multiply the above inequality by λ^{n-2} and take the limit when $\lambda \to \infty$; the Lebesgue dominance theorem implies that both integrals tend to 0, remaining

$$\mathrm{Vol}_g(M) \geq \left(\frac{K_0}{C}\right)^{\frac{n}{2}}\mathrm{Vol}_{g_0}(\mathbb{S}^n).$$

Since $C = K_0 \max\left\{\frac{A}{A_0}, \frac{B}{B_0}\right\}$, the latter relation implies (2.10) at once, which concludes the proof of (1.3). $\square$

Proof of Corollary 1.1. Since $\max\left\{\frac{A}{A_0}, \frac{B}{B_0}\right\} \leq (1-\delta_n)^{-\frac{2}{n}}$, by the quantitative volume estimate (1.3) it follows that

$$\mathrm{Vol}_g(M) \geq (1-\delta_n)\mathrm{Vol}_{g_0}(\mathbb{S}^n).$$

The statement follows by Cheeger and Colding [2].

If (M,g) is isometric to $(\mathbb{S}^n, g_0)$, it is clear that $A = A_0$ and $B = B_0$, due to Aubin [1]. Conversely, when $A = A_0$ and $B = B_0$, we apply (1.3) and (2.1) in order to obtain $\mathrm{Vol}_{g_0}(B_{\mathbb{S}^n}(y_0,\rho)) = \mathrm{Vol}_g(B_M(x_0,\rho))$ for every $\rho \in [0,\pi]$ (in fact, for every $\rho \in [0,\infty)$). Now, the equality in the Bishop-Gromov comparison principle implies that (M,g) is isometric to $(\mathbb{S}^n, g_0)$. $\square$

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