# ON A CRITICAL KIRCHHOFF-TYPE PROBLEM

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Abstract. In the present paper we study a Kirchhoff type problem involving the critical Sobolev exponent. We give sufficient conditions for the sequentially weakly lower semicontinuity and the Palais Smale property of the energy functional associated to the problem.

## 1. Introduction

In the present paper we deal with the following Kirchhoff type problem involving a critical term

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\Delta_p u = |u|^{p^*-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{P})$$

where $\Omega$ is an open connected set of $\mathbb{R}^N$ with smooth boundary, $1 < p < N$, $p^* = \frac{pN}{N-p}$ is the critical Sobolev exponent, $M : [0,+\infty[\to [0,+\infty[$ is a continuous function.

The nonlocal operator $M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\Delta_p u$ generalizes the term $\left(a + b\int_\Omega |\nabla u|^2\right)\Delta u$ of the Kirchhoff equation, proposed in 1883 as a model for describing the transversal oscillation of a strectched strings ( [10]).

Nonlocal problems received wide attention after the pioneering work of Lions ( [13]), where a functional analysis approach was implemented to study problems arising in the theory of evolutionary boundary value problems of mathematical physics. After the work of Alves, Correa and Figueiredo ( [1]), the existence and multiplicity of solutions of Kirchhoff type problems with critical nonlinearities in bounded or unbounded domains (or even in the whole space) have been studied by a number of authors by employing different techniques as variational methods, genus theory, the Nehari manifold, the Ljusternik–Schnirelmann category theory (see for instance [2–4,6,8] and the references therein). We mention also the recent works [9, 17], where an application of the Lions' Concentration Compactness principle ( [15]) ensures the Palais Smale condition of the energy functional, a key property for the application of the classical Mountain Pass Theorem.

In the present paper, generalizing the recent work [5], we claim to show that the interaction between the Kirchhoff and the critical term leads to some variational properties of the energy functional as the sequentially weakly lower semicontinuity and the Palais Smale condition.

An application to a Kirchhoff type problem on exterior domains is given. More precisely, we combine our results with a recent minimax theory by Ricceri ( [20]) to prove the existence of two solutions for a suitable perturbation of $(\mathcal{P})$.

Before stating our results, let us introduce some notations. Let $\widehat{M} : [0,+\infty[\to [0,+\infty[$ be the primitive of the function $M$, defined by

$$\widehat{M}(t) = \int_0^t M(s)\,\mathrm{d}s.$$

We endow the Sobolev and the Lebesgue spaces $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$ $(1 \le q \le p^*)$ with the classical norms

$$\|u\| = \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_q = \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$

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respectively, and denote by $S_N$ the embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, i.e.

$$\|u\|_{p^*}^p \leq S_N^{-1}\|u\|^p, \quad \text{for every} \quad u \in W_0^{1,p}(\Omega).$$

Let $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional associated to the problem $(\mathcal{P})$, defined by

$$\mathcal{E}(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{1}{p^*}\|u\|_{p^*}^{p^*}, \quad \text{for every} \quad u \in W_0^{1,p}(\Omega),$$

whose derivative at $u \in W_0^{1,p}(\Omega)$ is given by

$$\mathcal{E}'(u)(v) = M(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla v \nabla u \,\mathrm{d}x - \int_\Omega |u|^{p^*-2}uv\,\mathrm{d}x, \quad \text{for every} \quad v \in W_0^{1,p}(\Omega).$$

Define the constant

$$c_p = \begin{cases} (2^{p-1}-1)^{\frac{p^*}{p}} \frac{p}{p^*} S_N^{-\frac{p^*}{p}}, & \text{if} \;\; p \geq 2 \\ \\ 2^{2p^*-1-\frac{p^*}{p}} \frac{p}{p^*} S_N^{-\frac{p^*}{p}}, & \text{if} \;\; 1 < p < 2 \end{cases} \tag{1.1}$$

To prove the lower semicontinuity property we require the following assumptions on $\widehat{M}$:

*i)* $\widehat{M}(t+s) \geq \widehat{M}(t) + \widehat{M}(s)$ for every $t, s \in [0, +\infty[$;

*ii)* $\displaystyle\inf_{t>0} \frac{\widehat{M}(t)}{t^{\frac{p^*}{p}}} \geq c_p$.

Our first result allows $\Omega$ to be an unbounded subset of $\mathbb{R}^N$, due to a general inequality valid for $p \geq 2$.

**Theorem 1.1.** *Let $\Omega$ be an open connected set with smooth boundary, $p \geq 2$.*
*If i) and ii) hold, then $\mathcal{E}$ is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$.*

The same conclusion holds for $1 < p < 2$, but in our proof we need the boundedness of $\Omega$. It remains an open question if the property still holds for a general domain $\Omega$.

**Theorem 1.2.** *Let $\Omega$ be a bounded open connected set with smooth boundary, $1 < p < 2$.*
*If i) and ii) hold, then $\mathcal{E}$ is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$.*

In order to ensure the Palais–Smale property we need the following condition on $M$:

*iii)* $\displaystyle\inf_{t>0} \frac{M(t)}{t^{\frac{p^*}{p}-1}} > S_N^{-\frac{p^*}{p}}$.

**Theorem 1.3.** *Let $\Omega$ be an open connected set with smooth boundary, $p > 1$.*
*If iii) holds, then $\mathcal{E}$ satisfies the Palais–Smale property in $W_0^{1,p}(\Omega)$.*

## 2. Proof of Theorems 1.1 and 1.2

Fix $u \in W_0^{1,p}(\Omega)$ and let $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Assume by contradiction that

$$l \equiv \liminf_{n\to\infty} \mathcal{E}(u_n) < \mathcal{E}(u). \tag{2.1}$$

Then, there exists a subsequence $\{u_{n_j}\}$ such that

$$\lim_{j\to\infty} \mathcal{E}(u_{n_j}) = l.$$

Since $\mathcal{E}$ is strongly continuous, $\{u_{n_j}\}$ can not be strongly convergent to $u$ in $W_0^{1,p}(\Omega)$, that is $\limsup_{j\to\infty} \|u_{n_j} - u\| = L > 0$. Thus, there exists a a subsequence (still denoted by $\{u_{n_j}\}$), such that

$$\lim_{j\to\infty} \|u_{n_j} - u\| = L > 0.$$

The above limit implies also that

$$\lim_{j\to\infty} \|u_{n_j}\| > \|u\|. \tag{2.2}$$

We have,

$$\mathcal{E}(u_{n_j}) - \mathcal{E}(u) = \frac{1}{p}\left(\widehat{M}(\|u_{n_j}\|^p) - \widehat{M}(\|u\|^p)\right) - \frac{1}{p^*}\left(\|u_{n_j}\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*}\right).$$

Let us recall that the Brézis-Lieb lemma implies

$$\|u_{n_j}\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*} = \|u_{n_j} - u\|_{p^*}^{p^*} + o(1). \tag{2.3}$$

We distinguish now the two cases $p \geq 2$ and $1 < p < 2$.

**End of proof of Theorem 1.1:** $p \geq 2$. From [12, Lemma 2], the following inequality holds true:

$$|x|^p - |y|^p \geq \frac{1}{2^{p-1}-1}|x-y|^p + p\langle x-y, |y|^{p-2}y\rangle \qquad \text{for every } x, y \in \mathbb{R}^N. \tag{2.4}$$

Thus,

$$\|u_{n_j}\|^p \geq \|u\|^p + \frac{1}{2^{p-1}-1}\|u_{n_j} - u\|^p + p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_{n_j} - u)\,\mathrm{d}x.$$

Since $\{u_{n_j}\}$ weakly converges to $u$,

$$\frac{1}{2^{p-1}-1}\|u_{n_j} - u\|^p + p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_{n_j} - u)\,\mathrm{d}x \to \frac{1}{2^{p-1}-1}L^p > 0,$$

and for $j$ large enough, one has $\frac{1}{2^{p-1}-1}\|u_{n_j} - u\|^p + p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_{n_j} - u)\,\mathrm{d}x > 0$. Employing the monotonicity of $\widehat{M}$ ensured by $i)$, and (2.3), together with $i)$ and $ii)$ we obtain

$$\begin{aligned}
\mathcal{E}(u_{n_j}) - \mathcal{E}(u) =& \frac{1}{p}\widehat{M}\left(\|u\|^p + \frac{1}{2^{p-1}-1}\|u_{n_j} - u\|^p + p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_{n_j} - u)\,\mathrm{d}x\right) - \frac{1}{p}\widehat{M}(\|u\|^p) \\
& - \frac{1}{p^*}\|u_{n_j} - u\|_{p^*}^{p^*} + o(1) \\
\overset{i)}{\geq}& \frac{1}{p}\widehat{M}\left(\frac{1}{2^{p-1}-1}\|u_{n_j} - u\|^p + p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_{n_j} - u)\,\mathrm{d}x\right) \\
& - \frac{S_N^{-\frac{p^*}{p}}}{p^*}\|u_{n_j} - u\|^{p^*} + o(1) \\
\overset{ii)}{\geq}& \frac{c_p}{p}\left(\frac{1}{2^{p-1}-1}\|u_{n_j} - u\|^p + p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_{n_j} - u)\,\mathrm{d}x\right)^{\frac{p^*}{p}} \\
& - \frac{S_N^{-\frac{p^*}{p}}}{p^*}\|u_{n_j} - u\|^{p^*} + o(1).
\end{aligned}$$

Passing to the limit in the above estimate, one has that

$$l - \mathcal{E}(u) \geq \frac{S_N^{-\frac{p^*}{p}}}{p^*}L^{p^*} - \frac{S_N^{-\frac{p^*}{p}}}{p^*}L^{p^*} = 0,$$

which contradicts (2.1).

**End of proof of Theorem 1.2:** $1 < p < 2$. For $k \geq 1$, we consider the following two auxiliary functions $T_k, R_k : \mathbb{R} \to \mathbb{R}$ given by

$$T_k(s) = \begin{cases} -k, & \text{if } s < -k \\ s, & \text{if } -k \leq s \leq k \\ k, & \text{if } s > k, \end{cases}$$

and $R_k = \mathrm{Id}_\mathbb{R} - T_k$, i.e.,

$$R_k(s) = \begin{cases} s+k, & \text{if } s > -k \\ 0, & \text{if } -k \leq s \leq k \\ s-k, & \text{if } s > k, \end{cases}$$

so, for every $v \in W_0^{1,p}(\Omega)$

$$\|v\|^p = \|T_k(v)\|^p + \|R_k(v)\|^p, \tag{2.5}$$

and

$$\lim_{k\to\infty} \|T_k(v)\|^p = \|v\|^p \qquad \text{and} \qquad \lim_{k\to\infty} \|R_k(v)\| = 0.$$

Also, for every $k \in \mathbb{N}$ (see [16])

$$\liminf_{n\to\infty} \|T_k(u_n)\|^p \geq \|T_k(u)\|^p. \tag{2.6}$$

From (2.5) and the elementary inequality

$$\|R_k(u_n)\|^p \geq \frac{1}{2^{p-1}}\|R_k(u_n)-R_k(u)\|^p - \|R_k(u)\|^p,$$

we obtain that

$$\begin{aligned}\|u_{n_j}\|^p - \|u\|^p =& \|T_k(u_{n_j})\|^p + \|R_k(u_{n_j})\|^p - \|T_k(u)\|^p - \|R_k(u)\|^p \\ \geq& \|T_k(u_{n_j})\|^p - \|T_k(u)\|^p + \frac{1}{2^{p-1}}\|R_k(u_n)-R_k(u)\|^p - 2\|R_k(u)\|^p.\end{aligned}$$

Moreover we point out that

$$\begin{aligned}\|u_{n_j}-u\|_{p^*}^{p^*} &= \int_\Omega |T_k(u_{n_j}) - T_k(u) + R_k(u_{n_j}) - R_k(u)|^{p^*}\,\mathrm{d}x \\ &\leq 2^{p^*-1}\int_\Omega |T_k(u_{n_j}) - T_k(u)|^{p^*}\,\mathrm{d}x + 2^{p^*-1}\int_\Omega |R_k(u_{n_j}) - R_k(u)|^{p^*}\,\mathrm{d}x,\end{aligned}$$

and, since $\Omega$ is bounded, we can apply Lebesgue dominated convergence theorem to get, for every $k \in \mathbb{N}$,

$$\lim_{j\to\infty}\int_\Omega |T_k(u_{n_j}) - T_k(u)|^{p^*}\,\mathrm{d}x = 0$$

and

$$\|u_{n_j}-u\|_{p^*}^{p^*} \leq 2^{p^*-1}S_N^{-\frac{p^*}{p}}\|R_k(u_{n_j}) - R_k(u)\|^{p^*} + o_j(1),$$

where $o_j(1) \to 0$ as $j \to \infty$.
We consider two subcases:

a) $\lim\limits_{k\to\infty}\lim\limits_{j\to\infty}\|R_k(u_{n_j}) - R_k(u)\| = 0$. In such case, we have that

$$\begin{aligned}\lim_j \|u_{n_j}-u\|_{p^*}^{p^*} =& \lim_k\lim_j \|u_{n_j}-u\|_{p^*}^{p^*} \\ \leq& 2^{p^*-1}S_N^{-\frac{p^*}{p}}\lim_k\lim_j\|R_k(u_{n_j}) - R_k(u)\|^{p^*} = 0.\end{aligned}$$

Thus, together with (2.3) and assumption $i)$ we get

$$\begin{aligned}\mathcal{E}(u_{n_j}) - \mathcal{E}(u) =& \frac{1}{p}\widehat{M}\left(\|u_{n_j}\|^p - \|u\|^p + \|u\|^p\right) - \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{1}{p^*}\|u_{n_j}-u\|_{p^*}^{p^*} + o_j(1) \\ \geq& \frac{1}{p}\widehat{M}\left(\|u_{n_j}\|^p - \|u\|^p\right) - \frac{1}{p^*}\|u_{n_j}-u\|_{p^*}^{p^*} + o_j(1),\end{aligned}$$

and passing to the limit,

$$l - \mathcal{E}(u) \geq 0,$$

which contradicts (2.1).

b) $\lim\limits_{k\to\infty}\lim\limits_{j\to\infty}\|R_k(u_{n_j}) - R_k(u)\| > 0$. Thus, there exist $\alpha > 0$ and $\bar{k} \in \mathbb{N}$ with the property: for every $k > \bar{k}$ there exists $j_k \in \mathbb{N}$ such that for $j > j_k$ one has

$$\|R_k(u_{n_j}) - R_k(u)\| > \alpha.$$

Choosing eventually bigger $\bar{k}$ and for $k > \bar{k}$, bigger $j_k$ one can assume that

$$\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p + \frac{1}{2^{p-1}}\|R_k(u_{n_j}) - R_k(u)\|^p - 2\|R_k(u)\|^p > 0,$$

so that putting things together, and using (2.3), assumptions $i)$ and $ii)$, for $k > \bar{k}$ and $j > j_k$,

$$\begin{aligned}
\mathcal{E}(u_{n_j}) - \mathcal{E}(u) =& \frac{1}{p}\widehat{M}\left(\|u_{n_j}\|^p - \|u\|^p + \|u\|^p\right) - \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{1}{p^*}\|u_{n_j} - u\|_{p^*}^{p^*} + o_j(1) \\
\overset{i)}{\geq}& \frac{1}{p}\widehat{M}\left(\|u_{n_j}\|^p - \|u\|^p\right) - \frac{1}{p^*}\|u_{n_j} - u\|_{p^*}^{p^*} + o_j(1) \\
\geq& \frac{1}{p}\widehat{M}\left(\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p + \frac{1}{2^{p-1}}\|R_k(u_{n_j}) - R_k(u)\|^p - 2\|R_k(u)\|^p\right) \\
&- \frac{2^{p^*-1}}{p^*}\|R_k(u_{n_j}) - R_k(u)\|_{p^*}^{p^*} + o_j(1) \\
\overset{ii)}{\geq}& \frac{c_p}{p}\left(\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p + \frac{1}{2^{p-1}}\|R_k(u_{n_j}) - R_k(u)\|^p - 2\|R_k(u)\|^p\right)^{\frac{p^*}{p}} \\
&- S_N^{-\frac{p^*}{p}}\frac{2^{p^*-1}}{p^*}\|R_k(u_{n_j}) - R_k(u)\|^{p^*} + o_j(1) \\
\geq& \frac{c_p\|R_k(u_{n_j}) - R_k(u)\|^{p^*}}{p}\left[\left(\frac{\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p}{\|R_k(u_{n_j}) - R_k(u)\|^p} + \frac{1}{2^{p-1}} - \frac{2\|R_k(u)\|^p}{\alpha^p}\right)^{\frac{p^*}{p}}\right] \\
&- S_N^{-\frac{p^*}{p}}\frac{2^{p^*-1}}{p^*}\|R_k(u_{n_j}) - R_k(u)\|^{p^*} + o_j(1). \qquad (2.7)
\end{aligned}$$

Since for fixed $k > \bar{k}$, by (2.6),

$$\liminf_{j\to\infty}\frac{\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p}{\|R_k(u_{n_j}) - R_k(u)\|^p} \geq 0 \qquad \text{and} \qquad \lim_{j\to\infty}\frac{o_j(1)}{\|R_k(u_{n_j}) - R_k(u)\|^{p^*}} = 0,$$

we have

$$\begin{aligned}
&\liminf_{k\to\infty}\liminf_{j\to\infty}\left[\left(\frac{\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p}{\|R_k(u_{n_j}) - R_k(u)\|^p} + \frac{1}{2^{p-1}} - \frac{2\|R_k(u)\|^p}{\alpha^p}\right)^{\frac{p^*}{p}} - S_N^{-\frac{p^*}{p}}\frac{2^{p^*-1}p}{p^*c_p}\right] \\
&\geq \liminf_{k\to\infty}\left(\frac{1}{2^{p-1}} - \frac{2\|R_k(u)\|^p}{\alpha^p}\right)^{\frac{p^*}{p}} - S_N^{-\frac{p^*}{p}}\frac{2^{p^*-1}p}{p^*c_p} \\
&\overset{(1.1)}{\geq}\left(\frac{1}{2^{p^*-\frac{p^*}{p}}} - S_N^{-\frac{p^*}{p}}\frac{2^{p^*-1}p}{p^*c_p}\right) = 0
\end{aligned}$$

Thus, for every $\varepsilon > 0$ there exist $\tilde{k} \in \mathbb{N}$ such that for every $k > \tilde{k}$ there exists $j_k \in \mathbb{N}$ such that for $j > j_k$ one has

$$\left[\left(\frac{\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p}{\|R_k(u_{n_j}) - R_k(u)\|^p} + \frac{1}{2^{p-1}} - \frac{2\|R_k(u)\|^p}{\alpha^p}\right)^{\frac{p^*}{p}} - S_N^{-\frac{p^*}{p}}\frac{2^{p^*-1}p}{p^*c_p}\right] > -\varepsilon.$$

Moreover, by the boundedness of $\{u_{n_j}\}$ it follows that there exists a constant $M > 0$ such that $\|R_k(u_{n_j}) - R_k(u)\|^{p^*} \leq M$, so that we obtain

$$\frac{c_p\|R_k(u_{n_j}) - R_k(u)\|^{p^*}}{p}\left[\left(\frac{\|T_k(u_{n_j})\|^p - \|T_k(u)\|^p}{\|R_k(u_{n_j}) - R_k(u)\|^p} + \frac{1}{2^{p-1}} - \frac{2\|R_k(u)\|^p}{\alpha^p}\right)^{\frac{p^*}{p}} - S_N^{-\frac{p^*}{p}}\frac{2^{p^*-1}p}{p^*c_p}\right] > -\varepsilon M\frac{c_p}{p}.$$

This means that passing to the lim inf in the right hand side of (2.7), we obtain

$$l - \mathcal{E}(u) = \liminf_{k\to\infty}\lim_{j\to\infty}(\mathcal{E}(u_{n_j}) - \mathcal{E}(u)) \geq 0,$$

which contradicts (2.1).

The proof is complete. □

**Remark 2.1.** From the proof of the above theorems, it immediately follows that for $\mu \geq 1$, the functional $\mathcal{E}_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}_\mu(u) = \frac{\mu}{p}\widehat{M}(\|u\|^p) - \frac{1}{p^*}\|u\|_{p^*}^{p^*} \quad \text{for every} \quad u \in W_0^{1,p}(\Omega),$$

is sequentially weakly lower semicontinuous in $W_0^{1,p}(\Omega)$.

**Remark 2.2.** A comparison with [5, Lemma 2.1] is in order.
The above result extends to more general Kirchhoff operator the result of [5, Lemma 2.1] where the simple case $M(t) = a + bt$ when $p = 2$ is considered. The sequential weak lower semicontinuity of $\mathcal{E}$ in such case is proved when (see also the proof of [5, Lemma 2.1])

- $N = 4$, $a \geq 0, b > 0$ with $b \geq C_1(N)$,
- $N > 4$, $a > 0, b > 0$ with $a^{\frac{N-4}{2}} b \geq C_1(N)$,

where

$$C_1(N) = \begin{cases} \dfrac{4(N-4)^{\frac{N-4}{2}}}{N^{\frac{N-2}{2}} S_N^{\frac{N}{2}}} & N > 4 \\ S_4^{-2} & N = 4, \end{cases}$$

Notice that when $p = 2$, assumption $ii)$ in Theorem 1.1 reads as

$$\inf_{t>0} \frac{\widehat{M}(t)}{t^{\frac{2^*}{2}}} \geq \frac{2}{2^*} S_N^{-\frac{2^*}{2}},$$

and it is fulfilled by the particular case investigated in [5].
Notice also that, with respect to [5], our result holds also for $N = 3$.

**Remark 2.3.** When $p \geq 2$ we have a "better" estimate of the constant $c_p$. This reason is due to the fact that inequality (2.4) is no longer valid when $1 < p < 2$ and we employ more rough estimates which carry along some extra constants.

**Remark 2.4.** When $p = 2$, assumption $ii)$ is equivalent to the following sign property for $\mathcal{E}$:

$$\mathcal{E}(u) \geq 0 \quad \text{for every} \quad u \in W_0^{1,2}(\Omega).$$

If $ii)$ does not hold, there exists $\bar{t} > 0$ such that $\frac{1}{2}\widehat{M}(\bar{t}) < \frac{S_N^{-\frac{2^*}{2}}}{2^*}\bar{t}^{\frac{2^*}{2}}$. Let $\{u_n\}$ be a minimizing sequence for $S_N$. Then, it is weakly convergent and it has a subsequence $\{u_{n_k}\}$ strongly converging. Let $c > 0$ such that $c \lim_{k\to\infty} \|u_{n_k}\| = \sqrt{\bar{t}}$. Then,

$$\begin{aligned} \lim_{k\to\infty} \mathcal{E}(cu_{n_k}) &= \lim_{k\to\infty} \left[\frac{1}{2}\widehat{M}(\|cu_{n_k}\|^2) - \frac{1}{2^*}\|cu_{n_k}\|_{2^*}^{2^*}\right] \\ &= \lim_{k\to\infty} \left[\frac{1}{2}\widehat{M}(\|cu_{n_k}\|^2) - \frac{S_N^{-\frac{2^*}{2}}}{2^*}\|cu_{n_k}\|^{2^*}\right] \\ &= \frac{1}{2}\widehat{M}(\bar{t}) - \frac{S_N^{-\frac{2^*}{2}}}{2^*}\bar{t}^{\frac{2^*}{2}} < 0. \end{aligned}$$

The reverse implication follows from the Sobolev embedding. It is clear that we can not expect such equivalence for $p > 2$ since inequality (2.4) is far from being optimal.

## 3. Proof of Theorem 1.3

Let $\{u_n\}$ be a Palais Smale sequence for $\mathcal{E}$, that is

$$\begin{cases} \mathcal{E}(u_n) \to c \\ \mathcal{E}'(u_n) \to 0 \end{cases} \quad \text{as } n \to \infty.$$

We claim that $\{u_n\}$ admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.

Let us first notice that $\mathcal{E}$ is coercive. Indeed, let $k$ be a positive constant such that $k > S_N^{-\frac{p^*}{p}}$ and $M(t) \geq kt^{\frac{p^*}{p}-1}$ for every $t \geq 0$. Then, $\widehat{M}(t) \geq \frac{p}{p^*}kt^{\frac{p^*}{p}}$ for every $t \geq 0$ and

$$\mathcal{E}(u) \geq \frac{1}{p^*}\left(k - S_N^{-\frac{p^*}{p}}\right)\|u\|^{p^*}, \text{ for every } u \in W_0^{1,p}(\Omega),$$

and coercivity of $\mathcal{E}$ follows at once.

Then, the (PS) sequence $\{u_n\}$ is bounded and there exists $u \in W_0^{1,p}(\Omega)$ such that (up to a subsequence)

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega),\\
&u_n \to u \text{ in } L^q_{\text{loc}}(\Omega),\ q \in [1, p^*),\\
&u_n \to u \text{ a.e. in } \Omega.
\end{aligned}$$

Using the second Concentration Compactness lemma of Lions [15], there exist an at most countable index set $J$, a set of points $\{x_j\}_{j\in J} \subset \overline{\Omega}$ and two families of positive numbers $\{\eta_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ such that

$$\begin{aligned}
|\nabla u_n|^p &\rightharpoonup d\eta \geq |\nabla u|^p + \sum_{j\in J}\eta_j\delta_{x_j},\\
|u_n|^{p^*} &\rightharpoonup d\nu = |u|^{p^*} + \sum_{j\in J}\nu_j\delta_{x_j},
\end{aligned}$$

(weak star convergence in the sense of measures), where $\delta_{x_j}$ is the Dirac mass concentrated at $x_j$ and such that

$$S_N\nu_j^{\frac{p}{p^*}} \leq \eta_j \qquad \text{for every } j \in J.$$

Next, we will prove that the index set $J$ is empty. Arguing by contradiction, we may assume that there exists a $j_0$ such that $\nu_{j_0} \neq 0$. Consider now, for $\varepsilon > 0$ a non negative cut-off function $\phi_\varepsilon$ such that

$$\begin{aligned}
&\phi_\varepsilon = 1 \text{ on } B(x_0, \varepsilon),\\
&\phi_\varepsilon = 0 \text{ on } \Omega \setminus B(x_0, 2\varepsilon),\\
&|\nabla\phi_\varepsilon| \leq \frac{2}{\varepsilon}.
\end{aligned}$$

It is clear that the sequence $\{u_n\phi_\varepsilon\}_n$ is bounded in $W_0^{1,p}(\Omega)$, so that

$$\lim_{n\to\infty}\mathcal{E}'(u_n)(u_n\phi_\varepsilon) = 0.$$

Thus

$$\begin{aligned}
o(1) &= M(\|u_n\|^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n\phi_\varepsilon)\,\mathrm{d}x - \int_\Omega |u_n|^{p^*}\phi_\varepsilon\,\mathrm{d}x\\
&= M(\|u_n\|^p)\left(\int_\Omega |\nabla u_n|^p\phi_\varepsilon\,\mathrm{d}x + \int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\varepsilon\,\mathrm{d}x\right) - \int_\Omega |u_n|^{p^*}\phi_\varepsilon\,\mathrm{d}x. \qquad (3.1)
\end{aligned}$$

Using Hölder inequality one has

$$\begin{aligned}
\left|\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\varepsilon\,\mathrm{d}x\right| &= \left|\int_{B(x_0,2\varepsilon)} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\varepsilon\right|\mathrm{d}x\\
&\leq \left(\int_{B(x_0,2\varepsilon)}|\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B(x_0,2\varepsilon)}|u_n\nabla\phi_\varepsilon|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}\\
&\leq C\left(\int_{B(x_0,2\varepsilon)}|u_n\nabla\phi_\varepsilon|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}},
\end{aligned}$$

where $p'$ is the conjugate of $p$. Since

$$\lim_{n\to\infty}\int_{B(x_0,2\varepsilon)}|u_n\nabla\phi_\varepsilon|^{p'}\,\mathrm{d}x=\int_{B(x_0,2\varepsilon)}|u\nabla\phi_\varepsilon|^{p'}\,\mathrm{d}x,$$

and

$$\begin{aligned}\left(\int_{B(x_0,2\varepsilon)}|u\nabla\phi_\varepsilon|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}} &\leq \left(\int_{B(x_0,2\varepsilon)}|u|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}\left(\int_{B(x_0,2\varepsilon)}|\nabla\phi_\varepsilon|^{\frac{pN}{p(N+1)-2N}}\,\mathrm{d}x\right)^{\frac{p(N+1)-2N}{pN}}\\ &\leq C\left(\int_{B(x_0,2\varepsilon)}|u|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}\longrightarrow 0,\ \text{as } \varepsilon\to 0\end{aligned}$$

by the boundedness of the sequence $\{M(\|u_n\|^p)\}_n$ we get

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}M(\|u_n\|^p)\left|\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\varepsilon\,\mathrm{d}x\right|=0.$$

Moreover, as $0\leq\phi_\varepsilon\leq 1$,

$$\begin{aligned}\lim_{n\to\infty}M(\|u_n\|^p)\int_\Omega|\nabla u_n|^p\phi_\varepsilon\,\mathrm{d}x &\geq k\lim_{n\to\infty}\left(\int_{B(x_0,2\varepsilon)}|\nabla u_n|^p\phi_\varepsilon\,\mathrm{d}x\right)^{\frac{p^*}{p}}\\ &\geq k\left(\int_{B(x_0,2\varepsilon)}|\nabla u|^p\phi_\varepsilon\,\mathrm{d}x+\eta_{j_0}\right)^{\frac{p^*}{p}}.\end{aligned}$$

Also, $\displaystyle\int_{B(x_0,2\varepsilon)}|\nabla u|^p\phi_\varepsilon\,\mathrm{d}x\to 0$ as $\varepsilon\to 0$, so

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}M(\|u_n\|^p)\int_\Omega|\nabla u_n|^p\phi_\varepsilon\,\mathrm{d}x\geq k\eta_{j_0}^{\frac{p^*}{p}}.$$

Finally,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|u_n|^{p^*}\phi_\varepsilon\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_{B(x_0,2\varepsilon)}|u|^{p^*}\phi_\varepsilon\,\mathrm{d}x+\nu_{j_0}=\nu_{j_0}.$$

Summing up the above outcomes, from (3.1) one obtains

$$0\geq k\eta_{j_0}^{\frac{p^*}{p}}-\nu_{j_0}=\left(k-S_N^{-\frac{p^*}{p}}\right)\eta_{j_0}^{\frac{p^*}{p}}\geq 0.$$

Therefore $\eta_{j_0}=0$, which is a contradiction. Such conclusion implies that $J$ is empty, that is

$$\lim_{n\to\infty}\int_\Omega|u_n|^{p^*}\,\mathrm{d}x=\int_\Omega|u|^{p^*}\,\mathrm{d}x$$

and the uniform convexity of $L^{p^*}(\Omega)$ implies that

$$u_n\to u \text{ in } L^{p^*}(\Omega).$$

Since $\{u_n-u\}$ is bounded in $W_0^{1,p}(\Omega)$,

$$\lim_{n\to\infty}\mathcal{E}'(u_n)(u_n-u)=\lim_{n\to\infty}\left[M(\|u_n\|^p)\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,\mathrm{d}x-\int_\Omega|u_n|^{p^*-2}u_n(u_n-u)\,\mathrm{d}x\right]=0.$$

From Hölder inequality,

$$\left|\int_\Omega|u_n|^{p^*-2}u_n(u_n-u)\,\mathrm{d}x\right|\leq\left(\int_\Omega|u_n|^{p^*}\,\mathrm{d}x\right)^{\frac{pN-N+p}{pN}}\left(\int_\Omega|u_n-u|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}},$$

so we deduce that

$$\lim_{n\to\infty}M(\|u_n\|^p)\left|\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,\mathrm{d}x\right|=0.$$

We claim that

$$\lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,\mathrm{d}x=0. \tag{3.2}$$

If $\limsup_{n\to\infty} M(\|u_n\|^p) > 0$, then, (3.2) follows at once. If $\lim_{n\to\infty} M(\|u_n\|^p) = 0$, then, by *iii*), we obtain that $u_n \to 0$ strongly in $W_0^{1,p}(\Omega)$ and (3.2) holds true also in this case.

Putting together (3.2) with the limit

$$\lim_{n\to\infty} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_n - u)\,\mathrm{d}x = 0,$$

we obtain

$$\lim_{n\to\infty} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u)\,\mathrm{d}x = 0,$$

which implies at once that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. $\square$

**Remark 3.1.** The above result extends the result of [5, Lemma 2.2] where the simple case $M(t) = a+bt$ when $p = 2$ is considered. The Palais Smale property for $\mathcal{E}$ in such case is proved when (see also the proof of [5, Lemma 2.2])

- $N = 4$, $a \geq 0, b > 0$ with $b > C_2(N)$,
- $N > 4$, $a > 0, b > 0$ with $a^{\frac{N-4}{2}} b > C_2(N)$,

where

$$C_2(N) = \begin{cases} \dfrac{2(N-4)^{\frac{N-4}{2}}}{(N-2)^{\frac{N-2}{2}} S_N^{\frac{N}{2}}} & N > 4 \\ S_4^{-2} & N = 4. \end{cases}.$$

Notice that when $p = 2$, assumption *iii*) reads as

$$\inf_{t>0} \frac{M(t)}{t^{\frac{2^*}{2}-1}} > S_N^{-\frac{2^*}{2}}.$$

and it is fulfilled by the particular case investigated in [5]. It is worth mentioning that Palais–Smale property for $\mathcal{E}$ when $M(t) = a + bt$ and $p = 2$, was first proved by Hebey on compact Riemannian manifolds ( [9]).

## 4. An application

In this section, we provide an application of Theorem 1.1 to the following Kirchhoff problem on an exterior domain:

$$\begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)\Delta_p u = \lambda(u^{p^*-1} + u^{q-1}), & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{P}_\lambda)$$

where $\Omega = \mathbb{R}^N \setminus B(0,R)$ for some positive $R$, $2 \leq p < q < p^*$, and $\lambda > 0$.

As far a we know there are no contributions about subcritical and critical Kirchhoff equations on exterior unbounded domains beside [7] where an existence result is proved via a careful analysis of Palais Smale sequences and an application of the mountain pass theorem.

We will prove a multiplicity result for problem $(\mathcal{P}_\lambda)$ by employing an abstract well–posedness result for a class of constrained minimization problem which is derived by a general minimax theory of Ricceri ( [20]). Let us first recall the following definition: if $X$ is a topological space, $C \subset X$, $f : X \to \mathbb{R}$, we say that the problem of minimizing $f$ over $C$ is well-posed if the following two conditions hold:

- the restriction of $f$ to $C$ has a unique global minimum, say $x_0$;
- every sequence $\{x_n\}$ in $C$ such that $\lim_{n\to\infty} f(x_n) = \min_C f$, converges to $x_0$.

**Theorem A** ( [20, Theorem 1]). *Let $X$ be a Hausdorff topological space, $a > 0$, $\Phi, \Psi : X \to \mathbb{R}$, two functions such that, for each $\mu > a$, $\mu\Psi + \Phi$ has sequentially compact sub-level sets and admits a unique global minimum in $X$. Denote by $\mathscr{M}_a$ the set of global minima of $a\Psi + \Phi$ and assume that*

$$\inf_X \Psi < \inf_{\mathscr{M}_a} \Psi$$

*(if $\mathscr{M}_a = \emptyset$, put $\inf_{\mathscr{M}_a} \Psi = +\infty$). Then, for each $r \in ]\inf_X \Psi, \inf_{\mathscr{M}_a} \Psi[$ the problem of minimizing $\Phi$ on $\Psi^{-1}(r)$ is well posed.*

Our multiplicity result reads as follows:

**Theorem 4.1.** *Let $\Omega = \mathbb{R}^N \setminus B(0,R)$ for some $R > 0$, $2 \le p < q < p^*$, $M : [0,+\infty[ \to [0,+\infty[$ a continuous function such that if $\widehat{M}$ denotes its primitive, the following conditions hold:*

*i)* $\widehat{M}(t+s) \ge \widehat{M}(t) + \widehat{M}(s)$, *for every* $t, s \in [0,+\infty[$;

*ii)* $\inf_{t>0} \dfrac{\widehat{M}(t)}{t^{\frac{p^*}{p}}} > c_p$;

*iv)* $\lim_{t\to 0} \dfrac{\widehat{M}(t)}{t^{\frac{q}{p}}} = 0$,

*where $c_p$ is from (1.1).*

*Then, there exists $\lambda^* \in ]0,1[$ such that the problem $(\mathcal{P}_{\lambda^*})$ has two nontrivial solutions.*

*Proof.* Denote by $X = W^{1,p}_{0,\mathrm{rad}}(\Omega)$, the subspace of $W^{1,p}_0(\Omega)$ consisting of radial functions. It is well known that such space is embedded into $L^r(\Omega)$ continuously for $r \in [p,p^*]$, and compactly for $r \in ]p,p^*[$ ( [14]). We apply Theorem **A**, to the space $X$ endowed with the weak topology, choosing $a = 1$ and functions

$$\Psi(u) = \frac{1}{p}\widehat{M}(\|u\|^p) \quad \text{and} \quad \Phi(u) = -\frac{1}{p^*}\|u^+\|^{p^*}_{p^*} - \frac{1}{q}\|u^+\|^q_q.$$

For $\mu \ge 1$, define in $X$ the functional

$$\mathcal{F}_\mu(u) = \mu\Psi(u) + \Phi(u) = \frac{\mu}{p}\widehat{M}(\|u\|^p) - \frac{1}{p^*}\|u^+\|^{p^*}_{p^*} - \frac{1}{q}\|u^+\|^q_q, \quad \text{for every } u \in X.$$

Thus, if $\mathcal{E}_\mu(u) = \frac{\mu}{p}\widehat{M}(\|u\|^p) - \frac{1}{p^*}\|u\|^{p^*}_{p^*}$, then

$$\mathcal{F}_\mu(u) = \mathcal{E}_\mu(u) - \frac{1}{p^*}\|u^-\|^{p^*}_{p^*} - \frac{1}{q}\|u^+\|^q_q. \tag{4.1}$$

From $ii)$, $\mathcal{F}_\mu$ is coercive. The sequential weak lower semicontinuity of $\mathcal{F}_\mu$ follows from Theorem 1.1. Indeed, if $\{u_n\} \subset X$ is weakly convergent to $u \in X$, then $u_n^- \rightharpoonup u^-$ in $X$, and in particular in $L^{p^*}(\Omega)$, and $u_n^+ \to u^+$ (strongly) in $L^q(\Omega)$, therefore from Remark 2.1 and from the lower semicontinuity of the norm we have that

$$\begin{aligned} &\mathcal{E}_\mu(u) \le \liminf_n \mathcal{E}_\mu(u_n), \\ &\|u^-\|^{p^*}_{p^*} \le \liminf_n \|u_n^-\|^{p^*}_{p^*}, \\ &\|u^+\|^q_q = \lim_n \|u_n^+\|^q_q. \end{aligned} \tag{4.2}$$

Therefore, on account of (4.1) and (4.2) we have that

$$\liminf_{n\to\infty}(\mathcal{F}_\mu(u_n) - \mathcal{F}_\mu(u)) \ge 0,$$

hence, $\mu\Psi + \Phi$ has sequentially weakly compact sub-level sets.

Fix now a positive function $\bar{u} \in X$ and $\varepsilon \in \left]0, \dfrac{p\|\bar{u}\|^q_q}{q\|\bar{u}\|^q}\right[$. From assumption $iv)$ it follows that there exists $\delta > 0$ such that $\widehat{M}(t) < \varepsilon t^{\frac{q}{p}}$ for $0 < t \le \delta$. If $\rho \in \left]0, \dfrac{\delta}{\|\bar{u}\|}\right[$ then,

$$\mathcal{F}_1(\rho\bar{u}) < \rho^q\left(\frac{1}{p}\varepsilon\|\bar{u}\|^q - \frac{1}{q}\|\bar{u}\|^q_q\right) - \frac{1}{p^*}\rho^{p^*}\|\bar{u}\|^{p^*}_{p^*} < 0.$$

Thus, if we denote by $\mathscr{M}_1$ the set of global minima of $\mathcal{F}_1 = \Phi + \Psi$ (which is non empty), $0 \notin \mathscr{M}_1$.

We claim that $\inf_{\mathscr{M}_1} \Psi > 0$. Indeed, if $\inf_{\mathscr{M}_1} \Psi = 0$, there would exist a sequence $\{u_n\}$ in $\mathscr{M}_1$ such that $\Psi(u_n) \to 0$, that is $u_n \to 0$, but then, by continuity of $\Psi + \Phi$,

$$(\Psi + \Phi)(0) = \min_X(\Psi + \Phi),$$

which is in contradiction with $0 \notin \mathscr{M}_1$. Thus,

$$\inf_X \Psi = 0 < \inf_{\mathscr{M}_1} \Psi.$$

Let $r \in ]0, \inf_{\mathscr{M}_1} \Psi[$. Assume by contradiction that $\Phi$ has a global minimum $u_0$ on

$$\Psi^{-1}(r) = \{u \in X : \|u\|^p = (\widehat{M})^{-1}(pr)\}.$$

Thus, by the Lagrange multiplier rule, there exists $\sigma_0 \leq 0$ such that

$$\Phi'(u_0) = \sigma_0 \Psi'(u_0),$$

or

$$\sigma_0 \int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \nabla v \, \mathrm{d}x = -\int_\Omega ({u_0}_+^{p^*-1} + {u_0}_+^{q-1}) v \, \mathrm{d}x, \quad \text{for every } v \in X$$

Thus, $\sigma_0 < 0$, $u_0 \geq 0$ and plugging $v = u_0$ in the above equality we deduce that

$$\int_\Omega |\nabla u_0|^p \, \mathrm{d}x = -\frac{1}{\sigma_0} \int_\Omega ({u_0}^{p^*} + {u_0}^q) \, \mathrm{d}x.$$

On the other hand, by the Pohožaev equality, one has

$$\begin{aligned} \frac{p-1}{p} \int_{\partial\Omega} |\nabla u_0|^p \sigma \cdot \nu \mathrm{d}\sigma &= -\frac{N}{\sigma_0} \int_\Omega \left( \frac{u_0^{p^*}}{p^*} + \frac{u_0^q}{q} \right) \mathrm{d}x - \frac{N-p}{p} \int_\Omega |\nabla u_0|^p \, \mathrm{d}x = \\ &= \frac{N}{\sigma_0} \left( -\frac{1}{q} + \frac{1}{p^*} \right) \int_\Omega u_0^q \, \mathrm{d}x > 0, \end{aligned}$$

against the fact that the left hand side is non positive (due to the shape of the domain $\Omega$). Thus, the problem of minimizing $\Phi$ on $\Psi^{-1}(r)$ is not well posed and by Theorem **A**, we conclude that there exists $\mu^* > 1$ such that $\mu^* \Psi + \Phi$ has two distinct global minima in $X$. By the Palais principle of symmetric criticality [18] these minima are critical points of $\mathcal{F}_{\mu^*}$, i.e. solutions of $(\mathcal{P}_{\lambda^*})$ with $\lambda^* = \frac{1}{\mu^*}$. □

**Remark 4.1.** In order to prove that the problem of minimizing $\Phi$ on $\Psi^{-1}(r)$ is not well posed, one could prove either that $\Phi$ has two distinct global minima on $\Psi^{-1}(r)$ or that $\Phi$ has no global minima on the level set. This is the first application of Theorem **A** where such conclusion is obtained by showing that through the Pohozaev inequality $\Phi$ has no global minima on $\Psi^{-1}(r)$. For contributions where the claim is achieved via the existence of two global minima for the constrained problem we mention for instance [21, 22].

**Remark 4.2.** Theorem 4.1 applies for instance to the simple case $N = 4$, $\Omega = \mathbb{R}^4 \setminus B(0, R)$, $M(t) = bt$ where $b > 2c_2$ and $p = 2 < q < 4$.

## 5. Concluding remarks

In the present paper we presented some energy properties of the energy functional associated to a critical Kirchhoff problem with an application to a nonlocal problem on an exterior domain. We believe that such properties can be used in different settings to establish, by the means of critical point theory, existence and multiplicity results for perturbations of $(\mathcal{P})$. We formulate some open problems which could be object of forthcoming investigations.

(1) Theorem 1.1 allows us to consider critical Kirchhoff equation on unbounded domain while the boundedness of $\Omega$ is needed in the proof of Theorem 1.2. We conjecture that the sequential weak lower semicontinuity property holds true for general domains even when $1 < p < 2$.

(2) In Theorem 4.1 we proved the existence of $\mu^* > 1$, such that the functional $\mathcal{F}_{\mu^*}$ (see (4.1)) has two distinct global minima in $X = W^{1,p}_{0,\mathrm{rad}}(\Omega)$. Can we say that such points are global minima of $\mathcal{F}_{\mu^*}$ in $W^{1,p}_0(\Omega)$?

(3) In Theorem 4.1, under the additional assumption *iii*) which ensures the Palais–Smale condition for $\mathcal{F}_{\mu^*}$, the energy functional admits a third critical point in $X$ as it follows by [19]. Can we exclude, under additional assumptions on $M$, that such third solution for $(\mathcal{P}_{\lambda^*})$ is trivial?

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