Counting copies of a fixed subgraph in F-free graphs

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Abstract

Fix graphs F and H and let $\operatorname{ex}(n,H,F)$ denote the maximum possible number of copies of the graph H in an n-vertex F-free graph. The systematic study of this function was initiated by Alon and Shikhelman $[J.\ Comb.\ Theory,\ B.\ 121\ (2016)]$. In this paper, we give new general bounds concerning this generalized Turán function. We also determine $\operatorname{ex}(n,P_k,K_{2,t})$ (where P_k is a path on k vertices) and $\operatorname{ex}(n,C_k,K_{2,t})$ asymptotically for every k and t. For example, it is shown that for $t\geq 2$ and $k\geq 5$ we have $\operatorname{ex}(n,C_k,K_{2,t})=\left(\frac{1}{2k}+o(1)\right)(t-1)^{k/2}n^{k/2}$. We also characterize the graphs F that cause the function $\operatorname{ex}(n,C_k,F)$ to be linear in n. In the final section we discuss a connection between the function $\operatorname{ex}(n,H,F)$ and Berge hypergraph problems.

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1 Introduction

Let G and F be graphs. We say that a graph G is F-free if it contains no copy of F as a subgraph. Following Alon and Shikhelman [1], let us denote the maximum number of copies of the graph H in an n-vertex F-free graph by

$$ex(n, H, F)$$
.

The case when H is a single edge is the classical Turán problem of extremal graph theory. In particular, the $Turán\ number$ of a graph F is the maximum number of edges possible in an n-vertex F-free graph G. This parameter is denoted ex(n, F) and thus $ex(n, K_2, F) = ex(n, F)$. For more on the ordinary Turán number see, for example, the survey [13]. Recall that the

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Turán graph $T_{k-1}(n)$ is the complete (k-1)-partite graph with n vertices such that the vertex classes are of size as close to each other as possible.

In 1949, Zykov [39] proved that the Turán graph $T_{k-1}(n)$ is the unique graph containing the maximum possible number of copies of K_t in an n-vertex K_k -free graph (when t < k). By counting copies of K_t in $T_{k-1}(n)$ we get the following corollary for $ex(n, K_t, K_k)$. Let $\mathcal{N}(H, G)$ denote the number of copies of the subgraph H in the graph G.

Corollary 1 (Zykov, [39]). If t < k, then

$$ex(n, K_t, K_k) = \mathcal{N}(K_t, T_{k-1}(n)) = {k-1 \choose t} \left(\frac{n}{k-1}\right)^t + o(n^t).$$

This result was rediscovered by Erdős [7] and also follows from a theorem of Bollobás [2] and the case t = 3 and k = 4 was known to Moon and Moser [37]. Another proof appears in Alon and Shikhelman [1] modifying a proof of Turán's theorem.

When H is a pentagon, C_5 , and F is a triangle, K_3 , the determination of $\operatorname{ex}(n, C_5, K_3)$ is a well-known conjecture of Erdős [9]. An upper bound of $1.03(\frac{n}{5})^5$ was proved by Győri [23]. The blow-up of a C_5 gives a lower-bound of $(\frac{n}{5})^5$ when n is divisible by 5. Hatami, Hladký, Král', Norine and Razborov [28] and independently Grzesik [21] proved

$$\operatorname{ex}(n, C_5, K_3) \le \left(\frac{n}{5}\right)^5. \tag{1}$$

Swapping the role of C_5 and K_3 , we count the number of triangles in a pentagon-free graph. Bollobás and Győri [4] determined

$$(1+o(1))\frac{1}{3\sqrt{3}}n^{3/2} \le \operatorname{ex}(n, K_3, C_5) \le (1+o(1))\frac{5}{4}n^{3/2}.$$

The constant in the upper bound was improved to $\frac{\sqrt{3}}{3}$ by Alon and Shikhelman [1] and by Ergemlidze, Győri, Methuku and Salia [11]. Győri and Li [27] give bounds on $ex(n, K_3, C_{2k+1})$. A particularly interesting case to determine the value of $ex(n, K_3, K_{r,r,r})$ was posed by Erdős [8] (second part of Problem 17) and remains open in general.

The systematic study of the function ex(n, H, F) was initiated by Alon and Shikhelman [1] who proved a number of different bounds. Two examples from their paper are as follows. An analogue of the Kővári-Sós-Turán theorem

$$ex(n, K_3, K_{s,t}) = O(n^{3-3/s})$$

which is shown to be sharp in the order of magnitude when t > (s-1)! (see also [30]).

Another example is an Erdős-Stone-Simonovits-type result that for fixed integers t < k and a k-chromatic graph F that

$$\operatorname{ex}(n, K_t, F) = \binom{k-1}{t} \left(\frac{n}{k-1}\right)^t + o(n^t). \tag{2}$$

Gishboliner and Shapira [20] determined the order of magnitude of $ex(n, C_k, C_\ell)$ for every ℓ and $k \geq 3$. Moreover, they determined $ex(n, C_k, C_4)$ asymptotically. We give a theorem (proved independently of the previous authors) that extends this result to $ex(n, C_k, K_{2,t})$. Other results on the function ex(n, H, F) appear in [17, 14, 36, 26, 34, 22, 5].

The goal of this paper is to determine new bounds ex(n, H, F) and investigate its behavior as a function. In Section 2 we give general bounds using standard extremal graph theory techniques. In particular, we give the following extension of (2) using a modification of its proof in [1].

Theorem 2. Let H be a graph and F be a graph with chromatic number k, then

$$ex(n, H, F) \le ex(n, H, K_k) + o(n^{|H|}).$$

Theorem 2 only gives a useful upper-bound if $\operatorname{ex}(n, H, K_k) = \Omega(n^{|H|})$, which happens if and only if K_k is not a subgraph of H. For example, (2) follows by applying Corollary 1 in the case when $H = K_t$. Applying (1) to the case when $H = C_5$ gives

$$\operatorname{ex}(n, C_5, F) \le \left(\frac{n}{5}\right)^5 + o(n^5)$$

for every graph F with chromatic number 3. When F contains a triangle, the construction giving the lower bound in (1) can be used to give an asymptotically equal lower bound on $ex(n, C_5, F)$.

In Section 3 we determine $ex(n, P_k, K_{2,t})$ and $ex(n, C_k, K_{2,t})$ asymptotically.

It is natural to investigate which graphs H and F cause $\operatorname{ex}(n, H, F)$ to be linear in n. When H is K_3 , Alon and Shikhelman [1] characterized the graphs F with $\operatorname{ex}(n, K_3, F) = O(n)$. In Section 4 we determine which graphs F give $\operatorname{ex}(n, C_k, F) = O(n)$. Finally, in Section 5 we establish connections between this counting subgraph problem and Berge hypergraph problems. For notation not defined in this paper, see Bollobás [3].

2 General bounds on ex(n, H, F)

In this section we prove several general bounds on $\operatorname{ex}(n,H,F)$ using standard extremal graph theory techniques. We begin with a proof of Theorem 2. The proof mimics the proof of the Erdős-Stone-Simonovits by the regularity lemma. We use the following version of the regularity lemma and an embedding lemma found in [3]. Recall that the *density* of a pair A,B of vertex sets is d(A,B) = e(A,B)/|A||B|, where e(A,B) is the number of edges between A and B. Furthermore, the pair A,B is ϵ -regular if for any subsets $A' \subset A$, $B' \subset B$ with $|A'| \geq \epsilon |A|$ and $|B'| \geq \epsilon |B|$, we have $|d(A,B) - d(A',B')| < \epsilon$.

Lemma 3 (Regularity Lemma). For an integer m and $0 < \epsilon < 1/2$ there exists an integer $M = M(\epsilon, m)$ such that every graph on $n \ge m$ vertices has a partition V_0, V_1, \ldots, V_r with $m \le r \le M$ where $|V_0| < \epsilon n$, $|V_1| = |V_2| = \cdots = |V_r|$ and all but at most ϵr^2 of the pairs $V_i, V_i, 1 \le i < j \le r$ are ϵ -regular.

Lemma 4 (Embedding lemma). Let F be a k-chromatic graph with $f \geq 2$ vertices. Fix $0 < \delta < \frac{1}{k}$, let G be a graph and let V_1, \ldots, V_k be disjoint sets of vertices of G. If each V_i has $|V_i| \geq \delta^{-f}$ and each pair of partition classes is δ^f -regular with density $\geq \delta + \delta^f$, then G contains F as a subgraph.

Proof of Theorem 2. Fix $\delta > 0$ and an integer $m \geq k$ such that the following inequality holds

$$\left(\frac{1}{2m} + 2\delta^f + \frac{\delta + \delta^f}{2}\right) \mathcal{N}(H, K_{|H|}) < \alpha. \tag{3}$$

Let us apply the regularity lemma with $\epsilon = \delta^f$ and m to get $M = M(\epsilon, m)$. Let G be a graph on $n > M\delta^{-f}$ vertices and more than

$$ex(n, H, K_k) + \alpha n^{|H|}$$

copies of H. We will show that G contains F as a subgraph.

Let V_0, V_1, \ldots, V_r be the partition of G given by the regularity lemma. We will remove the following edges.

- 1. Remove the edges inside of each V_i . There are at most $r\binom{n/r}{2} \leq \frac{n^2}{2r} \leq \frac{1}{2m}n^2$ such edges.
- 2. Remove the edges between all pairs V_i, V_j that are not ϵ -regular. There are at most ϵr^2 such pairs and each has at most $(\frac{n}{r})^2$ edges. So we remove at most ϵn^2 such edges.
- 3. Remove the edges between all pairs V_i, V_j if the density $d(V_i, V_j) < \delta + \delta^f$. There are less than $\binom{r}{2}(\delta + \delta^f)(\frac{n}{r})^2 < \frac{\delta + \delta^f}{2}n^2$ such edges.
- 4. Remove all edges incident to V_0 . There are at most ϵn^2 such edges.

In total we have removed at most

$$\left(\frac{1}{2m} + 2\delta^f + \frac{\delta + \delta^f}{2}\right)n^2$$

edges. There are at most $\mathcal{N}(H, K_{|H|})n^{|H|-2}$ copies of H containing a fixed edge. Therefore, by (3) we have removed less than $\alpha n^{|H|}$ copies of H. Thus, the resulting graph still has more than $\operatorname{ex}(n, H, K_k)$ copies of H so it contains K_k as a subgraph.

The k classes of the resulting graph that correspond to the vertices of K_k satisfy the conditions of the embedding lemma so G contains F.

Using a standard first-moment argument of Erdős-Rényi [10] we can get a lower-bound on the number of copies of H in an F-free graph.

Proposition 5. Let F and H be graphs such that e(F) > e(H). Then

$$\operatorname{ex}(n, H, F) = \Omega\left(n^{|H| - \frac{e(H)(|F| - 2)}{e(F) - e(H)}}\right).$$

Proof. Let G be an n-vertex random graph with edge probability

$$p = cn^{-\frac{|F|-2}{e(F)-e(H)}}$$

where $c = |H|^{|H|(e(F) - e(H))} + 1$.

Among |F| vertices in G there are at most |F|! copies of the graph F. Therefore, the expected number of copies of F is at most

$$|F|! \binom{n}{|F|} p^{e(F)} \le n^{|F|} p^{e(F)}.$$

Fix |H| vertices in G. The probability of a particular copy of H appearing among those vertices is $p^{e(H)}$. Thus, the probability of at least one copy of H appearing among those |H| vertices is at least $p^{e(H)}$. Therefore, the expected number of copies of H is at least

$$\binom{n}{|H|} p^{e(H)} \ge \left(\frac{n}{|H|}\right)^{|H|} p^{e(H)}.$$

We remove an edge from each copy of F in G and count the remaining copies of H. There are at most $n^{|H|-2}$ copies of H destroyed for each edge removed from G.

Let X be the random variable defined by the difference between the number of copies of H and the number of copies of H destroyed by the removal of edges. The expectation of X is

$$E[X] \ge \left(\frac{n}{|H|}\right)^{|H|} p^{e(H)} - n^{|H|-2} n^{|F|} p^{e(F)}.$$

Which simplifies to

$$E[X] = \Omega\left(n^{|H| - \frac{e(H)(|F| - 2)}{e(F) - e(H)}}\right).$$

This implies that there exists a graph such that after removing an edge from each copy of F we are left with at least E[X] copies of H.

We conclude this section with two simple bounds on ex(n, H, F). Neither result is likely to give a sharp bound, but may be useful as simple tools.

Proposition 6.
$$ex(n, H, F) \ge ex(n, F) - ex(n, H)$$
.

Proof. Consider an edge-maximal n-vertex F-free graph G. Remove an edge from each copy of the subgraph H in G. The resulting graph does not contain H and therefore has at most $\operatorname{ex}(n,H)$ edges. This means we have removed at least $\operatorname{ex}(n,F)-\operatorname{ex}(n,H)$ edges from G, thus G (which is an F-free graph on n vertices) contained at least $\operatorname{ex}(n,F)-\operatorname{ex}(n,H)$ copies of H.

The other simple observation is a consequence of the Kruskal-Katona theorem [32, 29]. A hypergraph \mathcal{H} is k-uniform if all hyperedges have size k. For a k-uniform hypergraph \mathcal{H} , the i-shadow is the i-uniform hypergraph $\Delta_i \mathcal{H}$ whose hyperedges are the collection of all subsets of size i of the hyperedges of \mathcal{H} . We denote the collection hyperedges of a hypergraph \mathcal{H} by $E(\mathcal{H})$. Here we use a version of the Kruskal-Katona theorem due to Lovász [35].

Theorem 7 (Lovász, [35]). If \mathcal{H} is a k-uniform hypergraph and

$$|E(\mathcal{H})| = {x \choose k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

for some real number $x \geq k$, then

$$|E(\Delta_i \mathcal{H})| \ge {x \choose i}.$$

This gives the following easy corollary,

Corollary 8.

$$ex(n, K_t, F) \le ex(n, F)^{t/2}.$$

Proof. Suppose G is F-free and has the maximum number of copies of K_t . Let us consider the hypergraph \mathcal{H} whose hyperedges are the vertex sets of each copy of K_t in G. Pick x such that the number of hyperedges in \mathcal{H} is

$$|E(\mathcal{H})| = \binom{x}{t}.\tag{4}$$

Applying Theorem 7 we get that the 2-uniform hypergraph (i.e., graph) $\Delta_2 \mathcal{H}$ has size at least $\binom{x}{2}$.

On the other hand, the family $\Delta_2 \mathcal{H}$ is a subgraph of G. Therefore,

$$\binom{x}{2} \le e(G) \le \exp(n, F). \tag{5}$$

Combining (4) and (5) gives the corollary.

3 Counting paths and cycles in $K_{2,t}$ -free graphs

The maximum number of edges in a $K_{2,t}$ -free graph is

$$ex(n, K_{2,t}) = \left(\frac{1}{2} + o(1)\right)\sqrt{t - 1}n^{3/2}.$$
(6)

The upper bound above is given by Kővári, Sós and Turán [31] and the lower bound is given by an algebraic construction of Füredi [12]. We will refer to this construction as the Füredi graph $F_{q,t}$. We recall some well-known properties of $F_{q,t}$ without giving a full description of its construction. For fixed t and q a prime power such that t-1 divides q-1, the graph $F_{q,t}$ has $n = (q^2 - 1)/(t-1)$ vertices. All but at most 2q vertices have degree q and the others have degree q-1, thus the number of edges is $(1/2 + o(1))\sqrt{t-1}n^{3/2}$. Furthermore, every pair of vertices has at most t-1 common neighbors while every pair of non-adjacent vertices has exactly t-1 common neighbors.

Alon and Shikhelman [1] used the Füredi graph to give a lower bound in the following theorem.

Theorem 9 (Alon, Shikhelman, [1]).

$$\operatorname{ex}(n, K_3, K_{2,t}) = \left(\frac{1}{6} + o(1)\right) (t-1)^{3/2} n^{3/2}.$$

We generalize this theorem to cycles of arbitrary length and paths. We use the notation $v_1v_2\cdots v_k$ for the path P_k with vertices v_1,\ldots,v_k and edges v_iv_{i+1} (for $1\leq i\leq k-1$). The cycle C_k that includes this path and the edge $v_k v_1$ is denoted $v_1 v_2 \cdots v_k v_1$.

Proposition 10. For $t \geq 3$,

$$\operatorname{ex}(n, C_4, K_{2,t}) = \left(\frac{1}{4} + o(1)\right) {t-1 \choose 2} n^2.$$

Proof. We begin with the upper bound. Consider an n-vertex graph G that is $K_{2,t}$ -free. Fix two vertices u and v. As G is $K_{2,t}$ -free, u and v have at most t-1 common neighbors. Therefore the number of C_4 s with u and v as non-adjacent vertices is at most $\binom{t-1}{2}$. Therefore, the number of C_4 s in G is at most

$$\frac{1}{2} \binom{n}{2} \binom{t-1}{2} \le \frac{1}{4} \binom{t-1}{2} n^2$$

as each cycle is counted twice.

The lower bound is given by the Füredi graph $F_{q,t}$. Every pair of non-adjacent vertices has t-1 common neighbors, so there are $\binom{t-1}{2}$ copies of C_4 containing them. There are $(1/2 + o(1))n^2$ pairs of non-adjacent vertices in $F_{q,t}$. Each C_4 is counted twice in this way, so the number of C_4 s in $F_{q,t}$ is at least

$$\frac{1}{2} \left(\frac{1}{2} + o(1) \right) n^2 \binom{t-1}{2}$$

A slightly more sophisticated argument than the proof of Proposition 10 is needed to count longer cycles and paths.

Theorem 11. Fix $t \geq 2$. For $k \geq 5$,

$$ex(n, C_k, K_{2,t}) = \left(\frac{1}{2k} + o(1)\right) (t-1)^{k/2} n^{k/2}$$

and for $k \geq 2$,

$$\operatorname{ex}(n, P_k, K_{2,t}) = \left(\frac{1}{2} + o(1)\right) (t-1)^{(k-1)/2} n^{(k+1)/2}.$$

Proof. We begin with the upper bound for $ex(n, C_k, K_{2,t})$. Let G be a $K_{2,t}$ -free graph. We distinguish two cases based on the parity of k.

Case 1: k is even. Fix a (k/2)-tuple $(x_1, x_2, \ldots, x_{k/2})$ of distinct vertices of G. This can be done in at most $n^{k/2}$ ways. We count the number of cycles $v_1v_2\cdots v_kv_1$ such that $x_i=v_{2i}$ for $1 \le i \le k/2$. As G is $K_{2,t}$ -free, there are at most t-1 choices for each vertex v_{2i+1} on the cycle (for $0 \le i \le (k-2)/2$) as v_{2i+1} must be joined to both v_{2i+2} and v_{2i} (where the indicies are modulo k). Each cycle $v_1v_2\cdots v_kv_1$ is counted by 2k different (k/2)-tuples, so the number of copies of C_k is at most

$$\left(\frac{1}{2k}\right)(t-1)^{k/2}n^{k/2}.$$

Case 2: k is odd. Fix a ((k+1)/2)-tuple $(x_1, x_2, \ldots, x_{(k-3)/2}, y, z)$ of distinct vertices such that yz is an edge. This can be done in at most

$$2e(G)n^{(k-3)/2} \le (1+o(1))(t-1)^{1/2}n^{3/2}n^{(k-3)/2} = (1+o(1))(t-1)^{1/2}n^{k/2}$$

ways by (6). We count the number of cycles $v_1v_2\cdots v_kv_1$ such that $x_i=v_{2i}$ for $1\leq i\leq (k-3)/2$, $y=v_{k-1}$, and $z=v_k$. Similar to Case 1, as G is $K_{2,t}$ -free, there are at most t-1 choices for each of the (k-1)/2 remaining vertices v_{2i+1} of the cycle. Each cycle $v_1v_2\cdots v_kv_1$ is counted by 2k different ((k+1)/2)-tuples, so the number of copies of C_k is at most

$$\frac{1}{2k}(t-1)^{(k-1)/2}(1+o(1))(t-1)^{1/2}n^{k/2} = \left(\frac{1}{2k} + o(1)\right)(t-1)^{k/2}n^{k/2}.$$

For the upper bound on $\operatorname{ex}(n, P_k, K_{2,t})$ we fix a tuple of distinct vertices of G as above. We sketch the proof and leave the remaining details to the reader. If k is odd we fix a ((k+1)/2)-tuple $(x_1, x_2, \ldots, x_{(k+1)/2})$ and if k is even we fix a ((k+2)/2)-tuple $(x_1, x_2, \ldots, x_{(k-2)/2}, y, z)$ such that yz is an edge. In both cases we count the paths $v_1v_2\cdots v_k$ such that $x_i=v_{2i-1}$ and with the additional conditions that $y=v_{k-1}$, and $z=v_k$ in the case k even. Similar to the case for cycles there are at most t-1 choices for each of the remaining vertices of the path. Each path is counted exactly two times in this way.

Both lower bounds are given by the Füredi graph $F_{q,t}$ for q large enough compared to t and k. We begin by counting copies of the path $P_k = v_1 v_2 \cdots v_k$ greedily. The vertex v_1 can be chosen in n ways. As the Füredi graph $F_{q,t}$ has minimum degree q-1, we can pick vertex v_i (for i>1) in at least q-i+1 ways. Each path is counted twice in this way, therefore, we have at least

$$\frac{1}{2}n(q-k+1)^{k-1} = \left(\frac{1}{2} + o(1)\right)(t-1)^{(k-1)/2}n^{(k+1)/2}$$

paths of length k in the Füredi graph $F_{q,t}$.

For counting copies of the cycle $C_k = v_1 v_2 \cdots v_k v_1$ we proceed as above with the addition that v_k should be adjacent to v_1 . In order to do this, we pick v_1 arbitrarily and v_2, \ldots, v_{k-3} greedily as in the case of paths. As $k \geq 5$ the vertex v_{k-3} is distinct from v_1 . From the

neighbors of v_{k-3} we pick v_{k-2} that is not adjacent to v_1 . The number of choices for v_{k-2} is at least q-k+3-(t-1) as v_{k-3} and v_1 have at most t-1 common neighbors. From the neighbors of v_{k-2} we pick v_{k-1} that is not adjacent to any of the vertices v_1, \ldots, v_{k-3} . Each v_i has at most t-1 common neighbors with v_{k-2} which forbids at most (k-3)(t-1) vertices as a choice for v_{k-1} . Therefore, we have at least q-k-2-(k-3)(t-1) choices for v_{k-1} .

Since v_{k-1} is not joined to v_1 by an edge they have t-1 common neighbors and none of these neighbors are among $v_1, v_2, \ldots, v_{k-1}$. Hence we can pick any of the common neighbors as v_k . Every copy of C_k is counted 2k times, thus altogether we have at least

$$\frac{1}{2k}n(q-t(k-3))^{k-2}(t-1) = \left(\frac{1}{2k} + o(1)\right)(t-1)^{k/2}n^{k/2}$$

copies of C_k .

4 Linearity of the function $ex(n, C_k, F)$

Recall that Alon and Shikhelman [1] characterized the graphs F with $\operatorname{ex}(n, K_3, F) = O(n)$. For trees they also essentially answer the question by determining the order of magnitude of $\operatorname{ex}(n, T, F)$ where both T and F are trees. One can easily see that their proof extends to the case when F is a forest. On the other hand, if F contains a cycle and T is a tree, then $\operatorname{ex}(n, F)$ is superlinear and $\operatorname{ex}(n, T)$ is linear. Thus by Proposition 6 we have that $\operatorname{ex}(n, T, F)$ is superlinear.

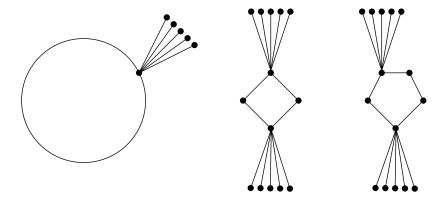


Figure 1: The graphs C_k^{*r} , C_4^{**r} and, C_5^{**r}

Now we turn our attention to the case when H is a cycle. We begin by introducing some notation. Let C_k^{*r} be a cycle C_k with r additional vertices adjacent to a vertex x of the C_k . For k=4, let C_4^{**r} be a cycle $v_1v_2v_3v_4$ with 2r additional vertices; r are adjacent to v_1 and r are adjacent to v_3 . Similarly, let C_5^{**r} be a cycle $v_1v_2v_3v_4v_5$ with 2r additional vertices; r are adjacent to v_1 and r are adjacent to v_3 . See Figure 1 for examples of these graphs.

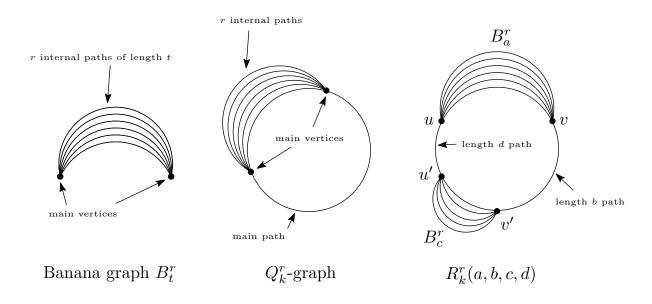


Figure 2: A banana graph B_t^r , a Q_k^r -graph, and $R_k^r(a, b, c, d)$

A banana graph B_t^r is the union of r internally-disjoint u-v paths of length t. We call the vertices u, v the main vertices of B_t^r and the u-v paths of B_t^r are its internal paths.

For t < k, let $Q_k^r(t)$ be the graph consisting of a banana graph B_t^r with main vertices u, v and a u-v path of length k-t that is otherwise disjoint from B_t^r . Alternatively, $Q_k^r(t)$ -graph is a C_k with r-1 additional paths of length t between two vertices that are joined by a path of length t in the C_k . For simplicity, we call any graph $Q_k^r(t)$ a Q_k^r -graph. The internal paths and main vertices of a Q_k^r -graph are simply the internal paths and main vertices of the associated banana graph B_t^r . The main path of a Q_k^r -graph is the associated u-v path of length k-t.

For $a, c \geq 2$ and $b, d \geq 0$ such that a+b+c+d=k, let $R_k^r(a,b,c,d)$ be the graph formed by a copy of B_a^r with main vertices u, v and a copy of B_c^r with main vertices u', v' together with a v-v' path of length b and a u-u' path of length d=k-(a+b+c). When b=0 we identify the vertices v and v' and when d=0 we identify the vertices v and v'. Note that the last parameter v is redundant, but we include it for ease of visualizing individual instances of this graph. For simplicity, we call any graph v0 and v1 and v2 and v3 are v4 and v5 are v6 and v5 are v6 and v6 are v7 and v8 are v8 are v9 are v9 and v9 are v9 are v9. Finally, call a graph v9 and v9 are v9 are v9 are v9 are v9 are v9. Finally, call a graph v9 are v9. Finally, call a graph v9 are v9 and v9 are v9 and v9 are v9 are

We now characterize those graphs F for which the function $ex(n, C_k, F)$ is linear.

Theorem 12. For k = 4 and k = 5, if F is a subgraph of C_k^{**r} (for some r large enough), then $ex(n, C_k, F) = O(n)$. For k > 5, if F is a subgraph of C_k^{**r} or an F_k^r -graph (for some r large enough), then $ex(n, C_k, F) = O(n)$. On the other hand, for every k > 3 and every other F we have $ex(n, C_k, F) = \Omega(n^2)$.

It is difficult to give a simple characterization of F_k^r -graphs. However, the following lemma gives some basic properties of these forests. For simplicity, the term *high degree* refers to a

vertex of degree greater than 2. A *star* is a single high degree vertex joined to vertices of degree 1. A *broom* is a path (possibly of a single vertex) with additional leaves attached to one of its end-vertices. Finally, let c(F) be the sum of the number of vertices in the longest path in each component of F (excluding the isolated vertex components).

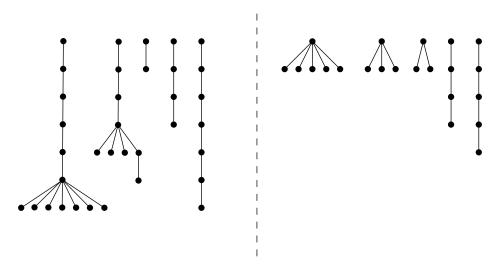


Figure 3: An F_k^r -graph with a non-broom component and an F_k^r -graph F with c(F) = k + 4.

Proposition 13. Let F be an F_k^r -graph, i.e., F is a subforest of every R_k^r -graph. Then the following properties hold when k > 5:

- 1. F has at most two vertices of high degree. This implies that all but at most two components of F are paths.
- 2. Each component of F has at most one vertex of high degree.
- 3. Each vertex of high degree in F is adjacent to at most two vertices of degree 2.
- 4. If F has two high degree vertices, then at least one of them is contained in a component that is a broom.
- 5. The number of vertices in the longest path in F is at most k.
- 6. $c(F) \le k + 4$.
- 7. If c(F) = k + 4, then F contains three components that are stars on at least 3 vertices. Furthermore, each component of F with a high degree vertex is a star.

Proof. The first property follows as F is a subgraph of the graph $R_k^r(2,0,k-2,0)$ which has exactly two high degree vertices.

For property two, consider the graphs $R_k^r(2,0,k-2,0)$ and $R_k^r(3,0,k-3,0)$. Each graph has two high degree vertices and they are at distance 2 and 3, respectively. If F had a component with two high degree vertices, then these vertices would be at distance 2 and 3 simultaneously; a contradiction. Note that we use k > 5 here.

For property three, consider the graph $R_k^r(2,0,2,k-4)$. This graph contains three high degree vertices x, y, z such that every vertex adjacent to y is adjacent to either x or z. If F has a component with a high degree vertex adjacent to more than two vertices of degree 2, then that component contains a cycle; a contradiction.

For property four, again consider the graph $R_k^r(2,0,2,k-4)$ and denote the three high degree vertices x,y,z as before. If F has two components each with a high degree vertex, then without loss of generality one of these high degree vertices is x. If x is adjacent to two vertices of degree 2 in F, then one of these vertices is y. Therefore, the other high degree vertex in F is z. The component containing z cannot contain y, so z is adjacent to at most one vertex of degree 2, i.e., the component containing z is a broom.

For property five, observe that the number of vertices in a longest path in $R_k^r(2,0,2,k-4)$ is k.

For property six and seven we can assume that all the components of F are paths (by deleting unnecessary leaves) and that each component contains at least two vertices.

Consider again the graph $R_k^r(2,0,2,k-4)$ with high degree vertices x,y,z as above. Note that this graph contains an x-z path on k-3 vertices. The components of F containing x or z have at most 2 additional vertices not on this path. Moreover, the component of F containing y has at most 3 vertices not on this path (this includes y itself). Therefore, $c(F) \leq k-3+2+2+3=k+4$. This proves property six. In order to achieve equality c(F)=k+4 there must be three distinct components containing x,y and z and each of these components has 3 vertices in their longest path, i.e., each such component is a star. This proves property seven.

The next lemma establishes another class of graphs that contains each F_k^r -graph as a subgraph.

Lemma 14. Let k > 5 and H be a graph formed by two Q_k^r -graphs Q_1, Q_2 such that Q_1 and Q_2 share at most one vertex and such a vertex is on the main path of both Q_1 and Q_2 . Then each F_k^r -graph is a subgraph of H.

Proof. Let F be an F_k^r -graph. We will show that F can be embedded in H. Suppose Q_1, Q_2 share a vertex x on their main paths as this is the more difficult case.

Let F' be a graph formed by components of F such that $c(F') \leq k$ and there is at most one vertex of high degree in F'. We claim that F' can be embedded into Q_2 . Indeed, first we embed the component of F' containing the high degree vertex using a main vertex of Q_2 . The remaining (path) components of F' can be embedded into the remaining vertices of the C_k in Q_2 greedily. Now, if we can embed components of F into Q_1 without using the vertex x such that the remaining components satisfy the conditions of F' above, then we are done.

First suppose that c(F) = k + 4. By property seven of Proposition 13 let T and T' be distinct star components of F such that T has exactly 3 vertices. It is easy to see that T

and T' can both be embedded to Q_1 without using the vertex x. Therefore, the remaining components of F can be embedded into Q_2 .

We may now assume $c(F) \leq k + 3$. If F contains a single component, then it can be embedded into Q_1 by property five of Proposition 13. If F has no high degree vertex, then every component is a path. In this case it is easy to embed F into Q_1 and Q_2 . So let us assume that F contains at least two components and at least one high degree vertex.

The graph Q_1 has two high degree vertices. Therefore, one of them is connected to x by a path P_{ℓ} with $\ell > (k+2)/2$.

Suppose F contains two high degree vertices, then let T be a component containing a high degree vertex. We may assume that the number of vertices on the longest path in T is at most (k+3)/2 (as there are two components with a high degree vertex). Therefore, we may embed T into Q_1 without using the vertex x. The remaining components of F can be embedded into Q_2 .

Now suppose F contains exactly one high degree vertex. If F contains a component with longest path on k vertices, then it can be embedded into Q_2 and the remaining component of F can be embedded into Q_1 without using vertex x. So we may assume all components in F have longest paths with less than k vertices. If there is a (path) component on at least three vertices, then it can be embedded into Q_1 without using the vertex x and the remaining components of F can be embedded into Q_2 . If there is no such path, then all path components are single edges. Two such edges can be embedded into Q_1 without using the vertex x and the remaining components can be embedded into Q_2 as before.

A version of the next lemma has already appeared in a slightly different form in [15].



Figure 4: The graph B from Lemma 15

Lemma 15. Fix integers $s \geq 2$ and $i \geq 2$. Let G be a graph containing a family \mathcal{P} of $(si)^{2i-2}$ u–v paths of length i. Then G contains a subgraph B consisting of a banana graph B_t^s (for some $t \leq i$) with main vertices u', v' together with a u–u' path and and v'–v path that are disjoint from each other and otherwise disjoint from B (we allow that the additional paths be of length 0, i.e., u = u' and v = v'), such that each u–v path in B is a sub-path of some member of \mathcal{P} . Moreover, if each member of \mathcal{P} is a sub-path of some copy of C_k in G, then G contains a $Q_k^{s'}$ -graph where s' = s - k.

Proof. We prove the first part of the lemma by induction on i. The statement clearly holds for i = 2, as such a collection of paths is a banana graph. Let i > 2 and suppose the lemma holds for smaller values of i. If there are s disjoint paths of length i between u and v then

we are done. So we may assume that there are at most s-1 disjoint paths of length i from u to v. The union of a set of disjoint paths of length i from u to v has at most si vertices. Furthermore, every other u-v path of length i must intersect this set of vertices. Therefore, there is a vertex w that is contained in at least $(si)^{2i-3}$ of these paths. This w can be in different positions in those paths, but there are at least $(si)^{2i-4}$ paths where w is the (p+1)st vertex (counting from u) with $1 \le p < i$. Then there are either at least $(si)^{2p-2} > (sp)^{2p-2}$ sub-paths of length p from u to w or at least $(si)^{2(i-p)-2} > (s(i-p))^{2(i-p)-2}$ sub-paths of length i-p from i to i0. Without loss of generality, suppose there are at least $(si)^{2p-2} > (sp)^{2p-2}$ sub-paths of length i1 from i2 to i3. Then, by induction on this collection of paths of length i3 we find a banana graph i4 with main vertices i5. As there is a path from i6 to i7 we have the desired subgraph i8.

Now it remains to show that if each member of \mathcal{P} is a sub-path of some copy of C_k in G, then G contains a $Q_k^{s'}$ -graph. Suppose we have a graph B from the first part of the lemma. Let C be a cycle of length k that contains any u-v path of length i in B. Note that C also contains a u-v path P of length k-i. The internal vertices of P intersect at most k of the internal paths of the banana graph B_t^s in B. Remove these internal paths from B_t^s and let B' be the resulting subgraph of B. Now B' together with P forms a $Q_k^{s'}$ -graph. \square

Proof of Theorem 12. First let us suppose that F is a graph such that $\operatorname{ex}(n, C_k, F) = o(n^2)$. Therefore, F must be a subgraph of every graph with $\Omega(n^2)$ copies of C_k . It is easy to see that each R_k^r -graph contains $\Omega(n^2)$ copies of C_k . Thus, F is a subgraph of every R_k^r -graph.

The Füredi graph $F_{q,2}$ does not contain a copy of C_4 and contains $\Omega(n^2)$ copies of C_k for $k \geq 5$. Furthermore, $F_{q,3}$ contains $\Omega(n^2)$ copies of C_4 . This follows from the proof of the lower bound in Theorem 11. Therefore, when $k \geq 5$ and r and q are large enough, F is a subgraph of $F_{q,2}$. When k = 4, and r and q are large enough, F is a subgraph of $F_{q,3}$.

Claim 16. The graph F contains at most one cycle and it is of length k.

Proof. As F is a subgraph of $R_k^r(2,0,2,k-4)$, every cycle in F is of length k or 4. If k > 4, as F is a subgraph of $F_{q,2}$, it does not contain cycles of length 4. Therefore, all cycles in F are of length k.

Suppose there is more than one copy of C_k in F. For k=4, as F is a subgraph of $R_4^r(2,0,2,0)$ it is easy to see that any two copies of C_4 in F form a $K_{2,3}$ or $K_{2,4}$. This contradicts the fact that F is also a subgraph of $F_{q,3}$. For k=5, as F is a subgraph of $R_5^r(2,0,3,0)$ it is easy to see that any two copies of C_5 in F form a C_4 or C_6 . This contradicts the fact that all cycles are of length k. For k>5, as F is a subgraph of $R_k^r(2,0,2,k-4)$, every pair of C_k s in F share k-3 or k-1 vertices. On the other hand, as F is a subgraph of $R_k^r(3,0,3,k-6)$, every pair of C_k s in F share k-4 or k-2 vertices; a contradiction. \square

We distinguish three cases based on the value of k.

Case 1: k = 4. The graph F is a subgraph of $R_4^r(2,0,2,0)$. By Claim 16, F has at most one cycle. The subgraphs of $R_4^r(2,0,2,0)$ with at most one cycle are clearly subgraphs of C_4^{**2r} .

Case 2: k = 5. The graph F is a subgraph of $R_5^r(2,0,3,0)$ and therefore has at most 2 vertices of degree greater than 2 and they are non-adjacent. Furthermore, F is a subgraph of $R_5^r(2,0,2,1)$. By Claim 16, F has at most one cycle. The subgraphs of $R_5^r(2,0,2,1)$ with at most one cycle that are simultaneously subgraphs of $R_5^r(2,0,3,0)$ are subgraphs of C_5^{**2r} .

Case 3: k > 5. First assume that F is a forest. As every R_k^r -graph contains $\Omega(n^2)$ copies of C_k , each must contain F as a subgraph. Therefore, F is an F_k^r -graph by definition.

Now consider the remaining case when F contains a cycle C. As F is a subgraph of $R_k^r(2,0,2,k-2)$, every edge of F is incident to C. If F has at least two vertices of degree greater than 2 on C, then as F is a subgraph of both $R_k^r(2,0,k-2,0)$ and $R_k^r(3,0,k-3,0)$, we have that these two vertices should be at distance 2 and 3 from each other in F; a contradiction. Thus, there is only one vertex of degree greater than 2 on C. Therefore, F is a subgraph of C_k^{*r} .

This completes the first part of the proof that if F is a graph such that $ex(n, C_k, F) = O(n)$, then F is as characterized in the theorem.

Now it remains to show that if F is as characterized in the theorem, then $ex(n, C_k, F) < cn$ for some constant c. The constants k and r are given by the statement of the theorem. Fix constants r'', r', γ, c', c in the given order such that each is large enough compared to k, r and the previously fixed constants.

Let G be a vertex-minimal counterexample, i.e, G is an n-vertex graph with at least cn copies of C_k and no copy of F such that n is minimal. We may assume every vertex in G is contained in at least c copies of C_k , otherwise we can delete such a vertex (destroying fewer than c copies of C_k) to obtain a smaller counterexample.

Case 1: F contains a cycle. Thus, for k = 4, 5 we have that F is a subgraph of C_k^{**r} and for k > 5 we have that F is a subgraph of C_k^{*r} . If every vertex of G has degree at least 2r + k, then on any C_k in G we can build a copy of C_k^{*r} or C_k^{**r} greedily. These graphs contain F; a contradiction.

Now let x be a vertex of degree less than 2r + k. This implies that there is an edge xy contained in at least c/(2r + k) copies of C_k . Therefore, there are at least c/(2r + k) x-y paths of length k-1. As c is large enough compared to k and r, we may apply Lemma 15 to this collection of paths of length k-1 (each a subgraph of a C_k) to get a Q_k^r -graph Q. The graph Q contains C_k^{*r} when k > 5 and C_k^{**r} when k = 4, 5. This implies that G contains F; a contradiction.

Case 2: F is a forest. Note that if k = 4, 5, then F is a subgraph of C_k^{**r} , and we are done by the same argument as in Case 1. Thus, we may assume k > 5.

Claim 17. Suppose G contains a collection C of at least c'n copies of C_k . Then there is an integer $\ell < k$ such that G contains a $Q_k^{r'}$ -graph Q with main vertices x, y and internal paths of length ℓ such that less than c'n members of C contain x, y at distance ℓ .

Proof. We distinguish two cases.

Case 1: There exists two vertices u, v of G in at least c'n members of C. Then there are at least (c'/k)n members of C that contain a u-v sub-path of length i < k. Let us suppose that u and v are chosen such that i is minimal. Among these u-v paths of length

i we can find a collection \mathcal{P} of $(c'/k)n/(ni) \geq c'/k^2$ of them that contain some fixed vertex w (different from u and v) such that w is at distance j < i from u in each such u-v path. Applying Lemma 15 this collection \mathcal{P} of u-w paths of length j gives a $Q_k^{r'}$ -graph Q. Let x, y be the main vertices of Q and let $\ell \leq j < i$ be the length of the main paths in Q. By the minimality of i, there are less than c'n members of \mathcal{C} that contain x, y at distance ℓ .

Case 2: The graph G does not contain two vertices in c'n members of C. As G is F-free and F is a forest there are at most 2|V(F)|n edges in G. Thus, there is an edge uv contained in at least c'/(2|V(F)|) members of C. Let P be a collection of c'/(2|V(F)|) u-v paths of length k-1 defined by these members of C. Applying Lemma 15 to P gives a $Q_k^{r'}$ -graph Q. Let x,y be the main vertices of Q. By the assumption in Case 2, the vertices x,y are contained in less than c'n total copies of C_k in G.

Now let us apply Claim 17 repeatedly in the following way. Let C_0 be the collection of all cn copies of C_k in G. We may apply Claim 17 to C_0 to find a $Q_k^{r'}$ -graph Q_1 with main vertices x_1, y_1 at distance ℓ_1 in Q_1 . Now remove from C_0 the copies of C_k that contain x, y at distance ℓ_1 and let C_1 be the remaining copies of C_k in C_0 . Note that $|C_1| \geq (c - c')n$ and that none of the copies of C_k in Q_1 are present in C_1 . Repeating the argument above on C_1 in place of C_0 gives another $Q_k^{r'}$ -graph Q_2 with main vertices x_2, y_2 . We can continue this argument until we have $k\gamma$ different $Q_k^{r'}$ -graphs (as c is large enough compared to c').

A pair of vertices x, y can appear as main vertices in at most k of the graphs $Q_1, Q_2, \ldots, Q_{k\gamma}$ Indeed, as once they appear as main vertices at distance $\ell \leq k$ in some $Q_k^{r'}$ -graph we remove all copies of C_k that have x, y at distance ℓ . Therefore, there is a collection of $\gamma = k\gamma/k$ different $Q_k^{r'}$ -graphs such that no two of the $Q_k^{r'}$ -graphs have the same two main vertices. Let $Q_1', Q_2', \ldots, Q_{\gamma}'$ be this collection of $Q_k^{r'}$ -graphs.

The internal paths of any Q_i' may share vertices with Q_j' (for $j \neq i$). However, for r' large enough compared to r'', we may remove internal paths from each of the Q_i' s to construct a collection of $Q_k^{r''}$ -graphs $Q_1'', Q_2'', \ldots, Q_{\gamma}''$ such that each pair Q_i'', Q_j'' only share vertices on their respective main paths (for $i \neq j$).

Now let $M_1, M_2, \ldots, M_{\gamma}$ be the collection of main paths of the $Q_k^{r''}$ -graphs $Q_1'', Q_2'', \ldots, Q_{\gamma}''$. If there are two paths M_i and M_j that share at most one vertex, then we may apply Lemma 14 to Q_i'' and Q_j'' to find a copy of F in G; a contradiction.

So we may assume that each M_i shares at least two vertices with each other M_j . Recall that Q_i'' and Q_j'' share at most one of their main vertices. Therefore, there is a vertex $u \in M_1$ that is contained in at least γ/k of the paths $M_2, M_3, \ldots, M_{\gamma}$. Moreover, u is the ith vertex in at least γ/k^2 of those paths. Each of these paths contains another vertex from M_1 . At least γ/k^3 of them contain the same vertex v, and it is the jth vertex in at least γ/k^4 of them. Thus, there are at least γ/k^4 u-v paths of length |j-i|. As γ/k^4 is large enough we may apply Lemma 15 to this collection of u-v paths of length |j-i| to get a subgraph B consisting of a banana graph B_t^r (for some t < k) with main vertices u', v' together with a u-u' path and a v-v' path. Each u-v path of B is a sub-path of some Q_i'' . Pick any such Q_i'' and take its union with B. The vertices of B intersect at most kr internal paths of Q_i'' . As r'' is large enough compared to r, we may remove internal paths of Q_i'' that intersect the vertices of B to get a graph containing an R_k^r -graph. As F is a subgraph of every R_k^r -graph,

5 Connection to Berge-hypergraphs

The problem of counting copies of a graph H in an n-vertex F-free graph is closely related to the study of Berge hypergraphs. Generalizing the notion of hypergraph cycles due to Berge, the authors introduced [18] the notion of Berge copies of any graph. Let F be a graph. We say that a hypergraph \mathcal{H} is a Berge-F if there is a bijection $f: E(F) \to E(\mathcal{H})$ such that $e \subseteq f(e)$ for every $e \in E(F)$. Note that Berge-F actually denotes a class of hypergraphs. The maximum number of hyperedges in an n-vertex hypergraph with no sub-hypergraph isomorphic to any Berge-F is denoted $\operatorname{ex}(n,\operatorname{Berge-}F)$. When we restrict ourselves to r-uniform hypergraphs, this maximum is denoted $\operatorname{ex}_r(n,\operatorname{Berge-}F)$.

Results of Győri, Katona and Lemons [24] together with Davoodi, Győri, Methuku and Tompkins [6] give tight bounds on $\exp(n, \text{Berge-}P_\ell)$. Upper-bounds on $\exp(n, \text{Berge-}C_\ell)$ are given by Győri and Lemons [25] when $r \geq 3$. A brief survey of Turán-type results for Berge-hypergraphs can be found in Subsection 5.2.2 in [19].

An early link between counting subgraphs and Berge-hypergraph problems was established by Bollobás and Győri [4] who investigated both $ex_3(n, Berge-C_5)$ and $ex(n, K_3, C_5)$. The connection between these two parameters is also examined in two recent manuscripts [16, 38]. In this section we prove two new relationships between these problems.

Proposition 18. Let F be a graph. Then

$$\operatorname{ex}(n, K_r, F) \le \operatorname{ex}_r(n, \operatorname{Berge-}F) \le \operatorname{ex}(n, K_r, F) + \operatorname{ex}(n, F).$$

and

$$\operatorname{ex}(n, \operatorname{Berge-}F) = \max_{G} \left\{ \sum_{i=0}^{n} \mathcal{N}(K_i, G) \right\} \leq \sum_{i=0}^{n} \operatorname{ex}(n, K_i, F)$$

where the maximum is over all n-vertex F-free graphs G.

Proof. Given an F-free graph G, let us construct a hypergraph \mathcal{H} on the vertex set of G by replacing each clique of G by a hyperedge containing exactly the vertices of that clique. The hypergraph \mathcal{H} contains no copy of a Berge-F. This gives $\operatorname{ex}(n, K_r, F) \leq \operatorname{ex}_r(n, \operatorname{Berge-}F)$ and

$$\max_{G} \left\{ \sum_{i=0}^{n} \mathcal{N}(K_i, G) \right\} \le \exp(n, \text{Berge-}F)$$

where the maximum is over all n-vertex F-free graphs G.

Given an n-vertex hypergraph \mathcal{H} with no Berge-F subhypergraph, we construct a graph G on the vertex set of \mathcal{H} as follows. Consider an order h_1, \ldots, h_k of the hyperedges of \mathcal{H} such that the hyperedges of size two appear first. We proceed through the hyperedges in order and at each step try to choose a pair of vertices in h_i to be an edge in G. If no such pair is available, then each pair of vertices in h_i is already adjacent in G. In this case, we

add no edge to G. A copy of F in G would correspond exactly to a Berge-F in \mathcal{H} , so G is F-free.

For each hyperedge h_i where we did not add an edge to G, there is a clique on the vertices of h_i in G. Thus, the number of hyperedges of \mathcal{H} is at most the number of cliques in G. If \mathcal{H} is r-uniform, then each hyperedge h_i of \mathcal{H} corresponds to either an edge in G or a clique K_r on the vertices of h_i (when we could not add an edge to G). Therefore, the number of hyperedges in \mathcal{H} is at most $\operatorname{ex}(n, K_r, F) + \operatorname{ex}(n, F)$.

As in the case of traditional Turán numbers we may forbid multiple hypergraphs. In particular, let $\exp(n, \{\text{Berge-}F_1, \text{Berge-}F_2, \dots, \text{Berge-}F_k\})$ denote the maximum number of hyperedges in an r-uniform n-vertex hypergraph with no subhypergraph isomorphic to any $\operatorname{Berge-}F_i$ for all $1 \leq i \leq k$. Similarly, $\exp(n, H, \{F_1, F_2, \dots, F_k\})$ denotes the maximum number of copies of the graph H in an n-vertex graph that contains no subgraph F_i for all $1 \leq i \leq k$.

Proposition 19. For $k \geq 4$,

$$ex_3(n, \{Berge-C_2, \dots, Berge-C_k\}) = ex(n, K_3, \{C_4, \dots, C_k\}).$$

Proof. Let \mathcal{H} be an n-vertex 3-uniform hypergraph with no Berge- C_i for i = 2, 3, ..., k and the maximum number of hyperedges. Consider the graph G on the vertex set of \mathcal{H} where a pair of vertices are adjacent if and only if they are contained in a hyperedge of \mathcal{H} . As \mathcal{H} is C_2 -free (i.e., each pair of hyperedges share at most one vertex) each edge of G is contained in exactly one hyperedge of \mathcal{H} .

Each hyperedge of \mathcal{H} contributes a triangle to G. We claim that G contains no other cycles of length i for $i=3,4,5,\ldots,k$. That is, G contains no cycle with two edges coming from different hyperedges of \mathcal{H} . Suppose (to the contrary) that G does contain such a cycle C. If two edges of C come from the same hyperedge, then they are incident in C. Therefore, these two edges can be replaced by the edge between their disjoint endpoints (which is contained in the same hyperedge) to get a shorter cycle. We may repeat this process until we are left with a cycle such that each edge comes from a different hyperedge of \mathcal{H} . Then this cycle corresponds exactly to a Berge-cycle of at most k hyperedges in \mathcal{H} ; a contradiction. Thus, $\exp(n, \{\{1, 2, \ldots, n\}\}) \le \exp(n, K_3, \{1, 2, \ldots, n\})$.

On the other hand, let G be an n-vertex graph with no cycle C_4, C_5, \ldots, C_k and the maximum number of triangles. Construct a hypergraph \mathcal{H} on the vertex set of G where the hyperedges of \mathcal{H} are the triangles of G. The graph G is C_4 -free, so each pair of triangles share at most one vertex, i.e., \mathcal{H} contains no Berge- C_2 . If \mathcal{H} contains a Berge- C_3 , then it is easy to see that G contains a C_4 ; a contradiction.

Therefore, if \mathcal{H} contains any Berge- C_i for $i=4,\ldots,k$, then G contains a cycle C_i ; a contradiction. Thus, $\exp(n, \{\text{Berge-}C_2, \ldots, \text{Berge-}C_k\}) \geq \exp(n, K_3, \{C_4, \ldots, C_k\})$.

Alon and Shikhelman [1] showed that for every k > 3, $\operatorname{ex}(n, K_3, \{C_4, \dots, C_k\}) \ge \Omega(n^{1 + \frac{1}{k-1}})$. For k = 4 they showed that $\operatorname{ex}_3(n, K_3, C_4) = (1 + o(1)) \frac{1}{6} n^{3/2}$. Lazebnik and Verstraëte [33] proved $\operatorname{ex}_3(n, \{\operatorname{Berge-}C_2, \operatorname{Berge-}C_3, \operatorname{Berge-}C_4\}) = (1 + o(1)) \frac{1}{6} n^{3/2}$. By Proposition 19 these two statements are equivalent.

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References

- [1] N. Alon, C. Shikhelman, Many T copies in H-free graphs. Journal of Combinatorial Theory, Series B 121 (2016) 146–172.
- [2] B. Bollobás, On complete subgraphs of different orders. *Math. Proc. Cambridge Philos.* Soc. **79** (1976) 19–24.
- [3] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, 184. Springer-Verlag, New York, 1998. xiv+394 pp. ISBN: 0-387-98488-7
- [4] B. Bollobás, E. Győri, Pentagons vs. triangles. *Discrete Mathematics* **308** (2008) 4332-4336.
- [5] J. Cutler, J. Nir, J. Radclifee, Supersaturation for subgraph counts. arXiv:1903.08059 (2019).
- [6] A. Davoodi, E. Győri, A. Methuku, C. Tompkins, An Erdős-Gallai type theorem for uniform hypergraphs. *European Journal of Combinatorics* **69** (2018) 159–162.
- [7] P. Erdős, On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. KutatóInt. Közl.* **7** (1962) 459–464.
- [8] P. Erdős, Problems and results in graph theory and combinatorial analysis, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), Congress. Numer. XV, pp. 169–192, Utilitas Math., Winnipeg, Man., 1976 MR53 #13006; Zentralblatt 335.05002.
- [9] P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory. Graph Theory and Combinatorics, Proc. Conf. Hon. P. Erdős, Cambridge 1983 (1984) 1–17.
- [10] P. Erdős, A. Rényi, On the evolution of random graphs. *Magyar Tud. Akad. Mat. KutatóInt. Közl.* **5** (1960) 17–61.
- [11] B. Ergemlidze, E. Győri, A. Methuku, N. Salia, A note on the maximum number of triangles in a C₅-free graph. Electronic Notes in Discrete Mathematics 61 (2017) 395– 398.

- [12] Z. Füredi. New asymptotics for bipartite Turán numbers. *Journal of Combinatorial Theory*, *Series A*, **75** (1996) 141–144.
- [13] Z. Füredi, M. Simonovits, The history of degenerate (bipartite) extremal graph problems. Erdős Centennial. Springer, Berlin, Heidelberg, (2013) 169–264.
- [14] D. Gerbner, E. Győri, A. Methuku, M. Vizer, Generalized Turán problems for even cycles. arXiv:1712.07079 (2017).
- [15] D. Gerbner, B. Keszegh, C. Palmer, B. Patkós, On the number of cycles in a graph with restricted cycle lengths. SIAM Journal on Discrete Mathematics 32 (2018) 266–279.
- [16] D. Gerbner, A. Methuku, M. Vizer, Asymptotics for the Turán number of Berge- $K_{2,t}$. Journal of Combinatorial theory, Series B 137 (2019) 264–290.
- [17] D. Gerbner, A. Methuku, M. Vizer, Generalized Turán problems for disjoint copies of graphs. arXiv:1712.07072 (2017).
- [18] D. Gerbner, C. Palmer, Extremal Results for Berge Hypergraphs. SIAM Journal on Discrete Mathematics 31 (2017) 2314–2327.
- [19] D. Gerbner, B. Patkós. Extremal Finite Set Theory, 1st Edition, CRC Press, 2018.
- [20] L. Gishboliner, A. Shapira, A Generalized Turán Problem and its Applications. Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing (2018) 760–772.
- [21] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph. *Journal of Combinatorial Theory, Series B*, **102** (2012) 1061–1066.
- [22] A. Grzesik, B. Kielak, On the maximum number of odd cycles in graphs without smaller odd cycles, arXiv:1806.09953 (2018).
- [23] E. Győri, On the number of C_5 's in a triangle-free graph. Combinatorica **9** (1989) 101–102.
- [24] E. Győri, G. Y. Katona, N. Lemons, Hypergraph extensions of the Erdős-Gallai Theorem, European Journal of Combinatorics 58 (2016) 238–246.
- [25] E. Győri, N. Lemons, Hypergraphs with no cycle of a given length. *Combinatorics*, *Probability and Computing* **21** (2012) 193–201.
- [26] E. Győri, J. Pach, M. Simonovits, On the maximal number of certain subgraphs in K_r -free graphs, *Graphs and Combinatorics* **7** (1991) 31–37.
- [27] E. Győri, H. Li, The maximum number of triangles in C_{2k+1} -free graphs. Combinatorics, Probability and Computing 21 (2011) 187–191.

- [28] H. Hatami, J. Hladký, D. Král', S. Norine, A. Razborov. On the number of pentagons in triangle-free graphs. *Journal of Combinatorial Theory, Series A* **120** (2013) 722–732.
- [29] G.O.H. Katona, A theorem for finite sets. Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, (1968) 187–207.
- [30] A. Kostochka, D. Mubayi and J. Verstraëte, Turán problems and shadows III: expansions of graphs, SIAM Journal on Discrete Math. 29 (2015) 868–876.
- [31] T. Kővári, V. Sós, P. Turán, On a problem of K. Zarankiewicz. In Colloquium Mathematicae Vol. 3, No. 1, (1954) 50–57.
- [32] J. Kruskal. The optimal number of simplices in a complex. *Mathematical Optimization Techniques* (1963) 251–268.
- [33] F. Lazebnik, J. Verstraëte, On hypergraphs of girth five. *Electronic Journal of Combinatorics* **10** (2003) #R25.
- [34] S. Letzter, Many H-copies in graphs with a forbidden tree. arXiv:1811.04287 (2018).
- [35] L. Lovász, Combinatorial Problems and Exercises. Akadémiai Kiadó, North Holland, 1979.
- [36] Jie Ma and Yu Qiu, Some sharp results on the generalized Turán numbers. arXiv:1802.01091 (2018).
- [37] I. W. Moon, L. Moser, On a problem of Turán. Magyar Tud. Akad. Mat. KutatóInt. Közl. 7 (1962) 311–314.
- [38] C. Palmer, M. Tait, C. Timmons, A. Z. Wagner, Turán numbers for Berge-hypergraphs and related extremal problems. *Discrete Mathematics* **342** (2019) 1553–1563.
- [39] A. A. Zykov, On some properties of linear complexes. *Matematicheskii sbornik*, **66** (1949) 163–188.