ON ALTERNATING POWER SUMS OF ARITHMETIC PROGRESSIONS

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Abstract. Depending on the parity of the positive integer n, the alternating power sum

$T^k_{a,b}(n) = b^k - (a+b)^k + (2a+b)^k - \ldots + (-1)^{n-1}(a(n-1)+b)^k$.

can be extended to a polynomial in two ways, say as $\mathtt{T}^{k+}_{a,b}(x)$ and $\mathtt{T}^{k-}_{a,b}(x)$. In this note we classify all the possible decompositions of these polynomials.

1. Introduction

We denote by $\mathbb{C}[x]$ the ring of polynomials in the variable x with complex coefficients. A *decomposition* of a polynomial $F(x) \in \mathbb{C}[x]$ is an equality of the following form

$$F(x) = G_1(G_2(x)) \quad (G_1(x), G_2(x) \in \mathbb{C}[x]),$$

which is nontrivial if

$$\deg G_1(x) > 1 \quad \text{and} \quad \deg G_2(x) > 1.$$

Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are said to be *equivalent* if there exists a linear polynomial $\ell(x) \in \mathbb{C}[x]$ such that $G_1(x) = H_1(\ell(x))$ and $H_2(x) = \ell(G_2(x))$. The polynomial $F(x)$ is called *decomposable* if it has at least one nontrivial decomposition; otherwise it is said to be *indecomposable*.

It is well known (see e.g [1]) that the alternating power sum

$$T_k(n) := -1^k + 2^k - \ldots + (-1)^{n-1}(n-1)^k$$

can be expressed by means of the classical Euler polynomials $E_k(x)$ via the identity:

$$T_k(n) = \frac{E_k(0) + (-1)^{n-1}E_k(n)}{2}, \tag{1}$$

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where the classical Euler polynomials $E_k(x)$ are usually defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi).$$

The motivation for studying the decomposition of alternating power sums is the lack of results in this direction as well as the long history of the investigation of the decomposition of the related power sum

$$S_k(n) := 1^k + 2^k + \ldots + (n-1)^k.$$

It started in the 16th century when Johann Faulhaber [6] discovered that for odd values of k, $S_k(n)$ can be written as a polynomial of the simple sum N given by

$$N = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Analogously to (1), there exists a relation between $S_k(n)$ and the classical Bernoulli polynomials $B_k(x)$ (those defined by their generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi)),$$

namely, we have

$$S_k(n) = \frac{1}{k+1}\left(B_{k+1}(n) - B_{k+1}\right) \tag{2}$$

with of course the classical Bernoulli numbers B_n given by $B_n = B_n(0)$. Using this connection (2), one can extend $S_k(n)$ appropriately to a polynomial

$$\mathtt{S}_k(x) = \frac{1}{k+1}\left(B_{k+1}(x) - B_{k+1}\right) \quad (x \in \mathbb{R})$$

In [8], Rakaczki proved that the polynomial $\mathtt{S}_k(x)$ is indecomposable for even values of k. Further, he observed that, for $k = 2v-1$, all the decompositions of $\mathtt{S}_k(x)$ are equivalent to the following decomposition:

$$\mathtt{S}_k(x) = \tilde{S}_v\left(\left(x - \frac{1}{2}\right)^2\right),$$

where $\tilde{S}_v(x)$ is a rational polynomial of degree v. His result is a consequence of a theorem of Bilu *et al.* [5], which states that the Bernoulli polynomial $B_k(x)$ is indecomposable for odd k, while if $k = 2m$ is even, then any nontrivial decomposition of $B_k(x)$ is equivalent to

$$B_k(x) = \tilde{B}_m\left(\left(x - \frac{1}{2}\right)^2\right),$$

with a rational polynomial $\tilde{B}_m(x)$ of degree m.

In a recent paper, Bazsó *et al.* [3] considered the more general power sum

$$S_{a,b}^k(n) := b^k + (a+b)^k + (2a+b)^k + \ldots + (a(n-1)+b)^k$$

for positive integers $n > 1, a \neq 0, b$ with $\gcd(a,b) = 1$. In particular, $S_{1,0}^k(n) = S_k(n)$. Again, by (2), the above definition of $S_{a,b}^k(n)$ can be extended to hold true for every real value of x as

$$\mathtt{S}_{a,b}^k(x) = \frac{a^k}{k+1}\left(B_{k+1}\left(x + \frac{b}{a}\right) - B_{k+1}\left(\frac{b}{a}\right)\right).$$

Bazsó *et al.* [3] observed that the polynomial $\mathtt{S}_{a,b}^k(x)$ is indecomposable for even k. If $k = 2v - 1$ is odd, then any nontrivial decomposition of $\mathtt{S}_{a,b}^k(x)$ is equivalent to the following decomposition

$$\mathtt{S}_{a,b}^k(x) = \widehat{S}_{a,b}^v\left(\left(x + \frac{b}{a} - \frac{1}{2}\right)^2\right),$$

where $\widehat{S}_{a,b}^v$ is a rational polynomial of degree v.

Recently, Rakaczki and Kreso [9] proved the following result, which will be used in the proof of our main result below, about the decomposition of Euler polynomials:

Proposition 1. *Euler polynomials $E_k(x)$ are indecomposable over $\mathbb{C}$ for all odd k. If $k = 2m$ is even, then every nontrivial decomposition of $E_k(x)$ over $\mathbb{C}$ is equivalent to*

$$E_k(x) = \tilde{E}_m\left(\left(x - \frac{1}{2}\right)^2\right), \text{ where } \tilde{E}_m(x) = \sum_{n=0}^{m} \binom{2m}{2n} \frac{E_{2n}}{2^{2n}} x^{m-n},$$

and $E_j = 2^j E_j(1/2)$. In particular, the polynomial $\tilde{E}_m(x)$ is indecomposable over $\mathbb{C}$ for any $m \in \mathbb{N}$.

Proof. This is Theorem 1 in [9]. □

For a positive integer $n > 1$, and for $a \neq 0, b$ coprime integers, let

$$T_{a,b}^k(n) := b^k - (a+b)^k + (2a+b)^k - \ldots + (-1)^{n-1}(a(n-1)+b)^k.$$

Clearly, $T_{1,0}^{k}(n) = T_k(n)$. Using generating functions, Howard [7] showed that $T_{a,b}^{k}(n)$ can be written as follows by means of Euler polynomials:

$$T_{a,b}^{k}(n) = \frac{a^k}{2}\left(E_k\left(\frac{b}{a}\right) + (-1)^{n-1}E_k\left(n+\frac{b}{a}\right)\right). \tag{3}$$

Thus, depending on the power of -1 in (3), we can extend $T_{a,b}^{k}(n)$ to a polynomial in the following two ways:

$$\mathtt{T}_{a,b}^{k+}(x) := \frac{a^k}{2}\left(E_k\left(\frac{b}{a}\right) + E_k\left(x+\frac{b}{a}\right)\right),$$

$$\mathtt{T}_{a,b}^{k-}(x) := \frac{a^k}{2}\left(E_k\left(\frac{b}{a}\right) - E_k\left(x+\frac{b}{a}\right)\right).$$

In this work, our goal is to determine all the possible decompositions of the polynomials $\mathtt{T}_{a,b}^{k+}(x)$ and $\mathtt{T}_{a,b}^{k-}(x)$ defined above.

We note that the decomposition properties of a polynomial with rational coefficients play an important role in the theory of separable Diophantine equations of the form $f(x) = g(y)$ (see [4]). For related results on such equations involving the above mentioned power sums we refer to [5], [8], [2] and [9].

2. The main result

In this section we apply Proposition 1 in order to derive a refinement of Faulhaber's theorem [6] and an analogue of the results of Bazsó *et al.* [3] in the case of alternating sums of powers of arithmetic progressions.

Theorem 1. *The polynomials $\mathtt{T}_{a,b}^{k+}(x)$ and $\mathtt{T}_{a,b}^{k-}(x)$ are both indecomposable for any odd k. If $k = 2m$ is even, then any nontrivial decomposition of $\mathtt{T}_{a,b}^{k+}(x)$ or $\mathtt{T}_{a,b}^{k-}(x)$ is equivalent to*

$$\mathtt{T}_{a,b}^{k+}(x) = \widehat{T}_{a,b}^{m+}\left(\left(x+\frac{b}{a}-\frac{1}{2}\right)^2\right) \quad \text{or} \quad \mathtt{T}_{a,b}^{k-}(x) = \widehat{T}_{a,b}^{m-}\left(\left(x+\frac{b}{a}-\frac{1}{2}\right)^2\right),$$

respectively, where

$$\widehat{T}_{a,b}^{m+}(x) = \frac{a^{2m}}{2}\left(E_{2m}\left(\frac{b}{a}\right) + \tilde{E}_m(x)\right),$$

$$\widehat{T}_{a,b}^{m-}(x) = \frac{a^{2m}}{2}\left(E_{2m}\left(\frac{b}{a}\right) - \tilde{E}_m(x)\right)$$

with $\tilde{E}_m(x)$ specified in Proposition 1.

Proof. We detail the proof for $\mathtt{T}^{k+}_{a,b}(x)$. For the polynomial $\mathtt{T}^{k-}_{a,b}(x)$, the proof is essentially the same.

Let k be an odd positive integer and put

$$t := x + \frac{b}{a}.$$

Suppose that there exists polynomials $f_1(t), f_2(t) \in \mathbb{C}[t]$ such that

$$\deg f_1(t) > 1 \text{ and } \deg f_2(t) > 1,$$

and

$$\mathtt{T}^{k+}_{a,b}(x) = \frac{a^k}{2}\left(E_k\left(\frac{b}{a}\right) + E_k\left(t\right)\right) = f_1(f_2(t)). \tag{4}$$

From the second equality in (4) we obtain

$$E_k(t) = \frac{2}{a^k} f_1(f_2(t)) - E_k\left(\frac{b}{a}\right). \tag{5}$$

Now putting $f(t) := \frac{2}{a^k} f_1(t) - E_k\left(\frac{b}{a}\right)$, (5) implies that

$$E_k(t) = f(f_2(t)). \tag{6}$$

Thus we obtained a nontrivial decomposition of the kth Euler polynomial, which contradicts Proposition 1.

If $k \in \mathbb{N}$ is even, then similarly we have relation (6), whence by Proposition 1 it follows that

$$f_2(t) = \ell\left(\left(t - \frac{1}{2}\right)^2\right) = \ell\left(\left(x + \frac{b}{a} - \frac{1}{2}\right)^2\right),$$

where $\ell(x)$ is a linear polynomial. This completes the proof of Theorem 1.

□

As an example, we consider the alternating sum of the squares of the arithmetic progression $b, a+b, 2a+b, \dots, a(n-1)+b$:

$$T^2_{a,b}(n) = (-1)^{n-1}\left(\frac{a^2}{2}n^2 + \frac{a(2b-a)}{2}n + \frac{b(b-a)}{2}\right) + \frac{b(b-a)}{2}.$$

One can easily obtain the following decompositions:

$$T^2_{a,b}(n) = \begin{cases} \frac{a^2}{2}\left(n + \frac{b}{a} - \frac{1}{2}\right)^2 + \frac{-a^2+4b^2-4ab}{8} & \text{if } n \text{ is odd,} \\ -\frac{a^2}{2}\left(n + \frac{b}{a} - \frac{1}{2}\right)^2 + \frac{(a-2b)^2}{8} & \text{if } n \text{ is even.} \end{cases}$$

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