CONNECTED FLOER HOMOLOGY OF COVERING INVOLUTIONS

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ABSTRACT. Using the covering involution on the double branched cover of S^3 branched along a knot, and adapting ideas of Hendricks-Manolescu and Hendricks-Hom-Lidman, we define new knot invariants and apply them to deduce novel linear independence results in the smooth concordance group of knots.

1. INTRODUCTION

Concordance questions of knots has been effectively studied by 4-dimensional topological methods. Indeed, for a knot K in the three-sphere S^3 consider the double branched cover $\Sigma(K)$ of S^3 branched along K. If K is a slice knot (i.e. bounds a smoothly embedded disk in the 4-disk D^4) then $\Sigma(K)$ bounds a four-manifold X having the same rational homology as D^4: this X can be chosen to be the double branched cover of D^4 along the slice disk. The existence of such four-manifold then can be obstructed by various methods, leading to sliceness obstructions of knots. For example, Donaldson's diagonalizability theorem applies in case $\Sigma(K)$ is known to bound a negative definite four-manifold with intersection form which does not embed into the same rank diagonal lattice. This line of reasoning was used by Lisca in his work about sliceness properties of 2-bridge knots, see [6, 15, 19]. A numerical invariant (in the same spirit) was introduced by Manolescu-Owens [20] utilizing the Ozsváth-Szabó correction term of the unique spin structure of $\Sigma(K)$.

Different knots might admit diffeomorphic double branched covers, though; for example, if K and K' differ by a Conway mutation, then $\Sigma(K)$ and $\Sigma(K')$ are diffeomorphic. This implies that if K is slice, all slice obstructions coming from the above strategy must vanish for K' as well. A long-standing problem of this type was whether the Conway knot is slice; it admits a mutant (the Kinoshita-Terasaka knot) which is slice, hence merely considering the double branched cover will not provide sliceness obstruction. (The fact that the Conway knot is not slice has been recently proved by Piccirillo [34], relying on four-dimensional topological methods and results from Khovanov homology.)

The information we neglect in the above approach is that the three-manifold $\Sigma(K)$ (viewed as the double branched cover of S^3 along K) comes with a self-diffeomorphism τ, where pairs of points in $\Sigma(K)$ mapping to the same point of S^3 are interchanged by τ. In this paper we introduce modifications of the usual Heegaard Floer homology groups of $\Sigma(K)$ which take this $\mathbb{Z}/2\mathbb{Z}$-action into account, leading to new knot invariants.

Heegaard Floer homology associates to a closed, oriented, smooth three-manifold a finitely generated $\mathbb{F}[U]$-module $\mathrm{HF}^-(Y)$ (where $\mathbb{F}[U]$ is the polynomial ring over the field $\mathbb{F}$ of two elements): it is the homology of a chain complex $(\mathrm{CF}^-(Y), \partial)$ (defined up to chain homotopy equivalence) and the homology naturally splits according to the spin^c structures of Y as

$$\mathrm{HF}^-(Y) = \oplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \mathrm{HF}^-(Y, \mathfrak{s}).$$

If Y is a rational homology sphere (i.e., $b_1(Y) = 0$) then $\mathrm{HF}^-(Y, \mathfrak{s})$ admits a natural $\mathbb{Q}$-grading, and the graded $\mathbb{F}[U]$-module $\mathrm{HF}^-(Y, \mathfrak{s})$ is a diffeomorphism invariant of the spin^c three-manifold $(Y, \mathfrak{s})$, while the local equivalence class of $(\mathrm{CF}^-(Y, \mathfrak{s}), \partial)$ (for the definition of its notion see Definition 3.2) provides an invariant of the rational spin^c homology cobordism class of $(Y, \mathfrak{s})$. In this case the local equivalence class of $\mathrm{CF}^-(Y, \mathfrak{s})$ can be characterised by a single rational number $d(Y, \mathfrak{s})$, the *Ozsváth-Szabó correction term* of the spin^c three-manifold $(Y, \mathfrak{s})$.

More recently, exploiting a symmetry built in the theory, Hendricks and Manolescu introduced involutive Heegaard Floer homology [11]. The main idea of their construction was that the chain complex $\mathrm{CF}^-(Y)$ admits a map $\iota \colon \mathrm{CF}^-(Y) \to \mathrm{CF}^-(Y)$ which is (up to homotopy) an involution, and the mapping cone of the map $\iota + \mathrm{id}$ provided $\mathrm{HFI}(Y)$, a module over the ring $\mathbb{F}[U, Q]/(Q^2)$. This group is interesting only for those spin^c structures which originate from a spin structure, and provides a new and rather sensitive diffeomorphism invariant of the underlying spin three-manifold. A further application of the above involution appeared in the work of Hendricks, Hom and Lidman [9], where connected Heegaard Floer homology $\mathrm{HF}_{\mathrm{conn}}^-(Y)$, a submodule of $\mathrm{HF}^-(Y)$ was defined. This submodule turned out to be a homology cobordism invariant.

Similar constructions apply for any chain complex equipped with a (homotopy) involution. In this paper we will define the *branched knot Floer homology* of K as $\mathrm{HFB}^-(K) = H_*(\mathrm{Cone}(\tau_\# + \mathrm{id}))$, where $\tau \colon \Sigma(K) \to \Sigma(K)$ is the covering involution, $\tau_\#$ is the map induced by τ on the Heegaard Floer chain complex $\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$, with $\mathfrak{s}_0$ the unique spin^c structure on $\Sigma(K)$, and Cone is the mapping cone of the map $\tau_\# + \mathrm{id}$. (Related constructions have been examined in [10].)

Theorem 1.1. *The group* $\mathrm{HFB}^-(K)$*, as a graded* $\mathbb{F}[U, Q]/(Q^2)$*-module, is an isotopy invariant of the knot* $K \subset S^3$*.*

A simple argument shows that, as an $\mathbb{F}[U]$-module, $\mathrm{HFB}^-(K)$ is isomorphic to $\mathbb{F}[U]_{(\overline{\delta}(K))} \oplus \mathbb{F}[U]_{(\underline{\delta}(K))} \oplus A$, where $\overline{\delta}(K), \underline{\delta}(K) \in \mathbb{Q}$ and A is a finitely generated, graded U-torsion module over $\mathbb{F}[U]$.

Theorem 1.2. *The rational numbers* $\overline{\delta}(K)$ *and* $\underline{\delta}(K)$ *are knot concordance invariants.*

Adapting the method of [9] for defining new homology cobordism invariants of rational homology spheres, we define the *connected branched Floer homology* $\mathrm{HFB}_{\mathrm{conn}}^-(K)$ of a knot $K \subset S^3$ as follows. Consider a sel-local equivalence $f_{\max} \colon \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \to \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$ which commutes (up to homotopy) with $\tau_\#$ and has maximal kernel among such endomorphisms. Then take $\mathrm{HFB}_{\mathrm{conn}}^-(K) = H_*(\mathrm{Im} f_{\max})$.

Theorem 1.3. *The module* $\mathrm{HFB}_{\mathrm{conn}}^-(K)$ *(up to isomorphism) is independent of the choice of the map* $f_{\max}$ *with maximal kernel, and the isomorphism class of the graded module* $\mathrm{HFB}_{\mathrm{conn}}^-(K)$ *is a concordance invariant of the knot* K*.*

It follows from the construction that $\mathrm{HFB}_{\mathrm{conn}}^-(K)$ is an $\mathbb{F}[U]$-submodule of $\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0)$. As a finitely generated $\mathbb{F}[U]$-module, $\mathrm{HFB}_{\mathrm{conn}}^-(K)$ is the sum of cyclic modules, and since it is of rank one, it can be written as

$$\mathrm{HFB}_{\mathrm{conn}}^-(K) = \mathbb{F}[U] \oplus \mathrm{HFB}_{\mathrm{red\text{-}conn}}^-(K),$$

where the second summand (the U-torsion submodule) is the *reduced* connected homology of K.

It is not hard to see that if $\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0) = \mathbb{F}[U]$ holds — for example if $\Sigma(K)$ is an L-space, which is the case if K is a quasi-alternating knot —, then $\tau_\#$ is chain homotopic to id implying

Theorem 1.4. *If K is concordant to an alternating (or more generally to a quasi-alternating) knot, then $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(K) = 0$.*

Somewhat more surprisingly, the same vanishing holds for torus knots (a phenomenon reminiscent to the behaviour of the extension of the Upsilon-invariant of [25] to the Khovanov setting given by Lewark-Lobb in [17]):

Theorem 1.5. *For the torus knot $T_{p,q}$ we have that $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(T_{p,q}) = 0$.*

Lattice homology of Némethi [21] (through results and computations of Dai-Manolescu [2] and Hom-Karakurt-Lidman [13]) provides a computational scheme of the above invariants for Montesinos (and in particular, for pretzel) knots. The combination of some such sample calculations with the two vanishing results above allow us to show that certain families of pretzel knots are linearly independent from alternating and torus knots in the smooth concordance group. To state the results, let us introduce the following notations: let $\mathcal{C}$ denote the (smooth) concordance group of knots in S^3 and $\mathcal{QA}$ (respectively $\mathcal{T}$) those subgroups of $\mathcal{C}$ which are generated by all quasi-alternating (respectively torus) knots. In addition, $\mathcal{QA} + \mathcal{T}$ is the subgroup generated by alternating knots and torus knots. The following theorem extends results from [1, 37].

Theorem 1.6. *Let K be the connected sum of pretzel knots of the form $P(-2, 3, q)$, with $q \geq 7$ odd. Then K is not concordant to any linear combination of alternating or torus knots, i.e. $[K]$ is a nonzero element of the quotient $\mathcal{C}/(\mathcal{T} + \mathcal{QA})$.*

Furthermore, relying on computations from [2, 13], we prove the following.

Theorem 1.7. *The pretzel knots $\{P(-p, 2p - 1, 2p + 1) \mid p \text{ odd}\}$ are linearly independent in the quotient group $\mathcal{C}/(\mathcal{T} + \mathcal{QA})$. In paricular, $\mathbb{Z}^\infty \subset \mathcal{C}/(\mathcal{T} + \mathcal{QA})$.*

The paper is organized as follows. In Section 2 we introduce branched knot Floer homology, and in Section 3 we discuss the details of the definition of the connected Floer homology group $\mathrm{HFB}^-_{\mathrm{conn}}(K)$ of a knot $K \subset S^3$. Section 4 is devoted to the proof of the vanishing results above, while in Section 5 we give a way to compute the invariants for Montesinos knots; finally in Section 6 we derive some independence results in the smooth concordance group.

2. Definition of branched knot Floer homology

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram which represents a rational homology sphere Y, and let J_s be a generic path of almost-complex structures on the g-fold symmetric product $\mathrm{Sym}^g(\Sigma)$ (compatible with a symplectic structure constructed in [33]). Heegaard Floer homology [29] assigns to the pair $(\mathcal{H}, J_s)$ a finitely generated, $\mathbb{Q}$-graded chain complex $\mathrm{CF}^-(\mathcal{H}, J_s)$ over the polynomial ring $\mathbb{F}[U]$, graded so that $\deg U = -2$. This chain complex is defined as the free $\mathbb{F}[U]$-module generated by the intersection point of the Lagrangian tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ in $\mathrm{Sym}^g(\Sigma)$, and is equipped with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1\}} \# \left(\mathcal{M}(\phi)/\mathbb{R} \right) U^{n_z(\phi)} \cdot \mathbf{y} \qquad (1)$$

where $\#(\mathcal{M}(\phi)/\mathbb{R})$ is the (mod 2) number of points in the unparametrized moduli space $\mathcal{M}(\phi)/\mathbb{R}$ of J_s-holomorphic strips with index $\mu(\phi) = 1$ representing the homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and $n_z(\phi)$ is the intersection number of ϕ with the divisor $V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$. For more details about Heegaard Floer homology see [28, 29, 30, 31].

For a knot $K \subset S^3$, let $\Sigma(K)$ denote the double branched cover of S^3 branched along K. The three-manifold $\Sigma(K)$ comes with a natural map $\tau \colon \Sigma(K) \to \Sigma(K)$ (called the *covering involution*) which interchanges points with equal image under the branched covering map $\pi \colon \Sigma(K) \to \Sigma(K)/\tau \simeq S^3$. The fixed point set $\mathrm{Fix}(\tau) = \widetilde{K}$ maps homeomorphically to K under π. As the notation suggests (since $H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$), the branched cover $\Sigma(K)$ in this case is determined by the branch locus $K \subset S^3$.

Pulling back the Heegaard surface, as well as the $\boldsymbol{\alpha}$- and the $\boldsymbol{\beta}$-curves of a doubly-pointed Heegaard diagram $D = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w_1, w_2)$ representing $K \subset S^3$ we get a pointed Heegaard diagram

$$\mathcal{H}_D = \left(\widetilde{\Sigma} = \pi^{-1}(\Sigma), \widetilde{\boldsymbol{\alpha}} = \pi^{-1}(\boldsymbol{\alpha}), \widetilde{\boldsymbol{\beta}} = \pi^{-1}(\boldsymbol{\beta}), z = \pi^{-1}(w_1) \right)$$

of the double branched cover $\Sigma(K)$. The covering projection $\pi \colon \Sigma(K) \to S^3$ restricts to a double branched cover of Riemann surfaces $\pi|_{\widetilde{\Sigma}} \colon \widetilde{\Sigma} \to \Sigma$ with branch set $\{w_1, w_2\}$. The restriction $\tau|_{\widetilde{\Sigma}} \colon \widetilde{\Sigma} \to \widetilde{\Sigma}$ of the covering involution τ represents the covering involution of $\pi|_{\widetilde{\Sigma}} \colon \widetilde{\Sigma} \to \Sigma$, and $\tau|_{\widetilde{\Sigma}} \colon \widetilde{\Sigma} \to \widetilde{\Sigma}$ induces a self-diffeomorphism of the symmetric product

$$\sigma \colon \mathrm{Sym}^g(\widetilde{\Sigma}) \to \mathrm{Sym}^g(\widetilde{\Sigma}) \ , \tag{2}$$

leaving $\mathbb{T}_{\widetilde{\alpha}}$ and $\mathbb{T}_{\widetilde{\beta}}$, as well as the divisor $V_z = \{z\} \times \mathrm{Sym}^{g-1}(\widetilde{\Sigma})$ invariant.

Pick a generic path of almost-complex structures $J_s \in \mathcal{J}(\mathrm{Sym}^g(\widetilde{\Sigma}))$ (satisfying the usual compatibility conditions with the chosen symplectic form on the symmetric product) and consider the Heegaard Floer chain complex $\mathrm{CF}^-(\mathcal{H}_D, J_s)$ associated to $(\mathcal{H}_D, J_s)$. Recall that there is a direct sum decomposition of $\mathrm{CF}^-(\mathcal{H}_D, J_s)$ indexed by spin^c structures:

$$\mathrm{CF}^-(\mathcal{H}_D, J_s) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(\Sigma(K))} \mathrm{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}) \ .$$

The first (singular) homology group of the double branched cover $\Sigma(K)$ can be presented by $\theta + \theta^t$, where θ is a Seifert matrix for K. Thus, $|H_1(\Sigma(K), \mathbb{Z})| = \det(\theta + \theta') = \det(K)$, which is an odd number. In particular, $\Sigma(K)$ has a unique spin structure $\mathfrak{s}_0$. We will focus on $\mathrm{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}_0)$, the summand of $\mathrm{CF}^-(\mathcal{H}_D, J_s)$ associated to $\mathfrak{s}_0$.

Note that if the path of almost-complex structures $J_s \in \mathcal{J}(\mathrm{Sym}^g(\widetilde{\Sigma}))$ is chosen generically, transversality is achieved for both J_s and the push-forward $\sigma_* J_s$ (where σ is given in Equation (2)). For such a choice of almost-complex structures we have well-defined Heegaard Floer chain complexes $\mathrm{CF}^-(\mathcal{H}_D, J_s)$ and $\mathrm{CF}^-(\mathcal{H}_D, \sigma_* J_s)$, and we can consider the map

$$\eta \colon \mathrm{CF}^-(\mathcal{H}_D, J_s) \to \mathrm{CF}^-(\mathcal{H}_D, \sigma_* J_s) \tag{3}$$

sending a generator $\mathbf{x} = x_1 + \cdots + x_g \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}} \subset \mathrm{Sym}^g(\widetilde{\Sigma})$ to $\sigma(\mathbf{x}) = \tau(x_1) + \cdots + \tau(x_g)$.

Lemma 2.1. *The map η is an isomorphism of chain complexes. Furthermore, η maps the summand $\mathrm{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}_0)$ of the spin structure $\mathfrak{s}_0$ into itself.*

Proof. It is obviously an isomorphisms of free $\mathbb{F}[U]$-modules; indeed, $\eta^2 = \mathrm{id}$ since τ is an involution. To see that η commutes with the differential, notice that $u \mapsto \tau \circ u$

provides a diffeomorphism between the moduli space of J_s-holomorphic representatives of a homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and the moduli space of $\sigma_* J_s$-holomorphic representatives of $\tau \circ \phi \in \pi_2(\tau(\mathbf{x}), \tau(\mathbf{y}))$).

To show that η preserves the spin structure we argue as follows. According to [29, Section 2.6] the choice of a basepoint z of $\mathcal{H}_D$ determines a map $\mathfrak{s}_z \colon \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}} \to \text{Spin}^c(\Sigma(K))$ and

$$\text{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}) = \bigoplus_{\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}} \mathbb{F}[U] \cdot \mathbf{x} \ .$$

It follows from the definition of $\mathfrak{s}_z$ that $\tau_*(\mathfrak{s}_z(\mathbf{x})) = \mathfrak{s}_z(\tau(\mathbf{x}))$. Thus if $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_0$, we have that

$$\overline{\mathfrak{s}_z(\tau(\mathbf{x}))} = \overline{\tau_*(\mathfrak{s}_z(\mathbf{x}))} = \overline{\tau_*(\mathfrak{s}_0)} = \tau_*(\overline{\mathfrak{s}_0}) = \tau_*(\mathfrak{s}_0) = \mathfrak{s}_z(\tau(\mathbf{x}))$$

proving that $\mathfrak{s}_z(\tau(\mathbf{x}))$ is a self-conjugate spin^c structure, i.e. spin. The claim now follows from the fact that $\Sigma(K)$ has a unqiue spin structure. $\qquad\square$

We define $\tau_\# \colon \text{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}_0) \to \text{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}_0)$ as the map $\eta \colon \text{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}_0) \to \text{CF}^-(\mathcal{H}_D, \tau_* J_s; \mathfrak{s}_0)$ followed by the continuation map

$$\Phi^-_{J_{s,t}} \colon \text{CF}^-(\mathcal{H}_D, \tau_* J_s; \mathfrak{s}_0) \to \text{CF}^-(\mathcal{H}_D, J_s; \mathfrak{s}_0)$$

from [29, Section 6], induced by a generic two-parameter family $J_{s,t}$ of almost-complex structures interpolating between J_s and $\tau_* J_s$:

$$\tau_\# = \Phi_{J_{s,t}} \circ \eta.$$

Lemma 2.2. $\tau_\#^2 \simeq \text{id}$, where $\simeq$ denotes chain homotopy equivalence.

Proof. Consider

$$\Phi^-_{J_{s,t}}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \ \mid \ \mu(\phi) = 0\}} \# \left(\mathcal{M}_{J_{s,t}}(\phi) \right) \ U^{n_z(\phi)} \cdot \mathbf{y}$$

where $\mathcal{M}_{J_{s,t}}(\phi)$ denotes the moduli spaces of $J_{s,t}$-holomorphic strips.

Given $\mathbf{x} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}$ one computes

$$\eta \circ \Phi^-_{J_{s,t}}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \ \mid \ \mu(\phi) = 0\}} \# \left(\mathcal{M}_{J_{s,t}}(\phi) \right) \ U^{n_z(\phi)} \cdot \tau(\mathbf{y})$$

$$= \sum_{\mathbf{y} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \tau(\mathbf{y})) \ \mid \ \mu(\phi) = 0\}} \# \left(\mathcal{M}_{J_{s,t}}(\phi) \right) \ U^{n_z(\phi)} \cdot \mathbf{y}$$

$$= \sum_{\mathbf{y} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}} \sum_{\{\phi \in \pi_2(\tau(\mathbf{x}), \mathbf{y}) \ \mid \ \mu(\phi) = 0\}} \# \left(\mathcal{M}_{J_{s,t}}(\tau \circ \phi) \right) \ U^{n_z(\tau \circ \phi)} \cdot \mathbf{y}$$

$$= \sum_{\mathbf{y} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}} \sum_{\{\phi \in \pi_2(\tau(\mathbf{x}), \mathbf{y}) \ \mid \ \mu(\phi) = 0\}} \# \left(\mathcal{M}_{J_{s,t}}(\phi) \right) \ U^{n_z(\phi)} \cdot \mathbf{y}$$

$$= \Phi^-_{\tau_* J_{s,t}}(\tau(\mathbf{x})) \ ,$$

hence the identity $\eta \circ \Phi^-_H = \Phi^-_{\tau_* H} \circ \eta$ follows. Thus,

$$\tau_\#^2 = \Phi^-_{J_{s,t}} \circ \eta \circ \Phi^-_{J_{s,t}} \circ \eta = \Phi^-_{J_{s,t}} \circ \Phi^-_{\tau_* J_{s,t}} \circ \eta^2 = \Phi^-_{J_{s,t}} \circ \Phi^-_{\tau_* J_{s,t}} \ ,$$

where the last identity holds because $\eta^2 = \text{id}$ (a consequence of the fact that $\tau \colon \Sigma(K) \to \Sigma(K)$ is an involution).

By concatenating $J_{s,t}$ and $\tau_* J_{s,t}$ we obtain a one-parameter family of paths of almost-complex structures describing a closed loop based at the path J_s. Since the space of

almost complex structures (compatible with the fixed symplectic structure) is contractible, we can find a three-parameter family of almost complex structures $J_{s,t,x}$ interpolating between the juxtaposition of $J_{s,t}$ and $\tau_* J_{s,t}$, and $J_{s,t,1} \equiv J_s$. As pointed out in [29, Section 6], a generic choice of $J_{s,t,x}$ produces smooth moduli spaces

$$\mathcal{M}_{J_{s,t,x}}(\phi) = \bigcup_{c \in [0,1]} \mathcal{M}_{J_{s,t,c}}(\phi) \qquad \phi \in \pi_2(\mathbf{x}, \mathbf{y})$$

of dimension $\mu(\phi) + 1$. These can be used to produce a chain homotopy equivalence

$$H^-_{J_{s,t,x}}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\tilde{\alpha}} \cap \mathbb{T}_{\tilde{\beta}}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = -1\}} \# \left(\mathcal{M}_{J_{s,t,x}}(\phi) \right) \ U^{n_z(\phi)} \cdot \mathbf{y}$$

between $\Phi^-_{J_{s,t}} \circ \Phi^-_{\tau_* J_{s,t}}$ and id, concluding the argument. $\qquad\square$

In summary, for a knot $K \subset S^3$ there is a homotopy involution $\tau_\# \colon \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \to \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$ associated to the covering involution $\tau \colon \Sigma(K) \to \Sigma(K)$. In order to derive knot invariants from the pair $(\mathrm{CF}^-(\mathcal{H}, \mathfrak{s}_0), \tau_\#)$, we follow ideas from [11] and form the mapping cone of $\tau_\# + \mathrm{id} \colon \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \to \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$, written equivalently as

$$\mathrm{CFB}^-(K) = \left(\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)[-1] \otimes \mathbb{F}[Q]/(Q^2), \ \partial_{\mathrm{cone}} = \partial + Q \cdot (\tau_\# + \mathrm{id}) \right),$$

where $\deg Q = -1$. Recall that $\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$ admits an absolute $\mathbb{Q}$-grading, and $\tau_\#$ preserves this grading, hence $\mathrm{CFB}^-(K)$ also admits an absolute $\mathbb{Q}$-grading. Taking homology we get the group $\mathrm{HFB}^-(K) = H_*(\mathrm{CFB}^-(K))$, which is now a module over the ring $\mathbb{F}[U, Q]/(Q^2)$. We call $\mathrm{HFB}^-(K)$ the **branched Heegaard Floer homology** of the knot $K \subset S^3$. Now we are ready to turn to the proof of the first statement announced in Section 1:

Proof of Theorem 1.1. The proof is similar to the one of [11, Proposition 2.8]. Independence from the chosen path of almost-complex structures is standard Floer theory. For independence from the chosen doubly pointed Heegaard diagram of K, we argue as follows: A doubly pointed Heegaard diagram $D = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w_1, w_2)$ representing the knot $K \subset S^3$ can be connected to any other doubly pointed diagram $D' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w_1', w_2')$ of K by a sequence of isotopies and handleslides of the $\boldsymbol{\alpha}$-curves (or $\boldsymbol{\beta}$-curves) supported in the complement of the two basepoints, and by stabilizations (i.e., forming the connected sum of Σ with a torus T^2 equipped with a new pair of curves α_{g+1} and β_{g+1} which meet transversally in a single point). A sequence of these moves lifts to a sequence of pointed Heegaard moves of the pull-back diagrams $\mathcal{H}_D$ and $\mathcal{H}_{D'}$ with underlying three-manifold the double branched cover $\Sigma(K)$. According to [14] the choice of such a sequence of Heegaard moves yields a natural chain homotopy equivalence $\psi \colon \mathrm{CF}^-(\mathcal{H}_D) \to \mathrm{CF}^-(\mathcal{H}_{D'})$ which fits into the diagram

$$
\begin{array}{ccc}
\mathrm{CF}^-(\mathcal{H}_D) & \xrightarrow{\ \tau_\#\ } & \mathrm{CF}^-(\mathcal{H}_D) \\
\ \downarrow{\scriptstyle \psi} & & \ \downarrow{\scriptstyle \psi} \\
\mathrm{CF}^-(\mathcal{H}_{D'}) & \xrightarrow{\ \tau_\#'\ } & \mathrm{CF}^-(\mathcal{H}_{D'})
\end{array}
\qquad (4)
$$

that commutes up to chain homotopy. Let $\Gamma \colon \mathrm{CF}^-(\mathcal{H}_D) \to \mathrm{CF}^-(\mathcal{H}_{D'})$ be a map realizing the chain homotopy equivalence. Then the map $f \colon \mathrm{Cone}(\tau_\# + \mathrm{id}) \to \mathrm{Cone}(\tau_\#' + \mathrm{id})$ defined by $f = \psi + Q \cdot (\psi + \Gamma)$ is a quasi-isomorphism. Indeed, f is a filtered map

with respect to the two step filtration of the mapping cones, and since it induces an isomorphism on the associated graded objects, it is a quasi-isomorphism. $\qquad\square$

Notice that the mapping cone exact sequence associated to $\mathrm{CFB}^-(K) = \mathrm{Cone}(\tau_\# + \mathrm{id})$ reads as an exact triangle

$$\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0) \xrightarrow{\ \tau_* + \mathrm{id}\ } \mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0) \tag{5}$$

$$\underset{p_*}{\nwarrow} \qquad \underset{j_*}{\swarrow}$$

$$\mathrm{HFB}^-(K)$$

in which j_* preserves the grading, and p_* drops it by one. In particular, if $\tau_* = \mathrm{id}$, the horizontal map in the above triangle is zero, and in that case $\mathrm{HFB}^-(K)$ is the sum of two copies of $\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0)$ (with the grading on one copy shifted by one).

A close inspection of the exact triangle above reveals that, as $\mathbb{F}[U]$-modules, we have

$$\mathrm{HFB}^-(K) = \mathbb{F}[U]_{(\overline{\delta})} \oplus \mathbb{F}[U]_{(\underline{\delta}+1)} \oplus (\mathbb{F}[U]-\text{torsion}) \ .$$

We set $\underline{\delta}(K) = \underline{\delta}$ and $\overline{\delta}(K) = \overline{\delta}$, which (by the above discussion) are knot invariants. Notice that $\underline{\delta}(K), \overline{\delta}(K) \in \mathbb{Q}$, $\underline{\delta}(K) \equiv \delta(K) \equiv \overline{\delta}(K) \bmod 2$, and $\underline{\delta}(K) \le \delta(K) \le \overline{\delta}(K)$, where $\delta(K)$ is the Ozsváth-Szabó correction term of $(\Sigma(K), \mathfrak{s}_0)$, hence $\delta(K)$ is half the Manolescu-Owens invariant of K introduced in [20].

3. Concordance invariants from $(\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0), \tau_\#)$

Adapting ideas from [9], the chain complex $\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$, equipped with $\tau_\#$, provides concordance invariants of the knot K as follows. Recall [9, Definition 2.5] regarding ι-complexes:

Definition 3.1. *An ι-**complex** (C, ι) is a finitely generated, free, $\mathbb{Q}$-graded chain complex C over $\mathbb{F}[U]$ together with a chain map $\iota\colon C \to C$ where C is supported in degree $\tau + \mathbb{Z}$ for some $\tau \in \mathbb{Q}$ (multiplication by U drops the $\mathbb{Q}$-grading by two), the homology of the localization $U^{-1}H_*(C) = H_*(C \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}])$ is isomorphic to $\mathbb{F}[U, U^{-1}]$ via an isomorphism preserving the relative $\mathbb{Z}$-grading, and ι is a grading preserving, U-equivariant chain map which is a homotopy involution.*

We will consider ι-complexes up to *local equivalence* (see [9, Definitions 2.6 and 2.7]).

Definition 3.2. *A **local equivalence** $f\colon C \to C'$ of two ι-complexes (C, ι) and (C', ι') is a grading preserving, U-equivariant chain map $f\colon C \to C'$ such that*

- *$\iota' \circ f \simeq f \circ \iota$, i.e. the two compositions are chain homotopy equivalent,*
- *f induces an isomorphisms f_{loc} on the localization $U^{-1}H_*(C)$.*

Definition 3.3. *Two ι-complexes (C, ι) and (C', ι') are **locally equivalent** if there exist local equivalences $f\colon C \to C'$ and $g\colon C' \to C$. If in addition we have $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$, then (C, ι) and (C', ι') are **chain homotopy equivalent** ι-complexes.*

Given an ι-complex (C, ι) we can look at the set $\mathrm{End}_{\mathrm{loc}}(C, \iota)$ of its self-local equivalences $f\colon C \to C$. This can be partially ordered by defining $f \preceq g$ if and only if $\mathrm{Ker}\, f \subset \mathrm{Ker}\, g$. We say that $f \in \mathrm{End}_{\mathrm{loc}}(C)$ is a **maximal self-local equivalence** if it is maximal with respect to this ordering. By Zorn's lemma maximal self-local equivalences always exist. The following lemma summarises the results of [9, Section 3].

Lemma 3.4. *Let (C, ι) be an ι-complex. Then*

(1) *if $f \in \mathrm{End}_{\mathrm{loc}}(C, \iota)$ is a maximal self-local equivalence, then ι restricts to a homotopy involution $\iota^{\mathrm{Im}\, f}$ of $\mathrm{Im}\, f$. Furthermore, $(\mathrm{Im}\, f, \iota^{\mathrm{Im}\, f})$ is locally equivalent to (C, ι);*

(2) *if $f, h \in \mathrm{End}_{\mathrm{loc}}(C, \iota)$ are two maximal self-local equivalences, then there is a chain homotopy equivalence $(\mathrm{Im}\, f, \iota^{\mathrm{Im}\, f}) \simeq (\mathrm{Im}\, h, \iota^{\mathrm{Im}\, h})$ of ι-complexes;*

(3) *if (C', ι') is an ι-complex locally equivalent to (C, ι) and $f \in \mathrm{End}_{\mathrm{loc}}(C, \iota)$, and $h \in \mathrm{End}_{\mathrm{loc}}(C', \iota')$ are self-local equivalences then there is a chain homotopy equivalence $(\mathrm{Im}\, f, \iota^{\mathrm{Im}\, f}) \simeq (\mathrm{Im}\, h, \iota^{\mathrm{Im}\, h})$ of ι-complexes.* $\square$

Since

$$H_*(\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]) = \mathrm{HF}^\infty(\Sigma(K), \mathfrak{s}_0) = \mathbb{F}[U, U^{-1}] \ ,$$

the pair $(\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0), \tau_\#)$ of the Heegaard Floer chain complex of the double branched cover of a knot $K \subset S^3$ (equipped with the homotopy involution $\tau_\#$ induced by the covering involution) is an ι-complex associated to K. Given a maximal self-local equivalence $f_{\max} \colon \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \to \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$ we define $\mathrm{HFB}^-_{\mathrm{conn}}(K)$, the ***connected Floer homology*** of the knot $K \subset S^3$ as $H_*(\mathrm{Im}\, f_{\max})$. As an application of Lemma 3.4, it is then easy to see that the resulting group is a knot invariant:

Theorem 3.5. *The chain homotopy type of $\mathrm{Im}\, f_{\max}$ is independent of the choice of the maximal self-local equivalence $f_{\max} \in \mathrm{End}_{\mathrm{loc}}(\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0), \tau_\#)$.* $\square$

We now turn to the proof of concordance invariance of the groups $\mathrm{HFB}^-_{\mathrm{conn}}(K)$. The following naturality statement will be needed in the proof.

Lemma 3.6. *(Ozsváth & Szabó, Zemke, [32, 38]) Let Y and Y' be two three-manifolds equipped with self-diffeomorphisms $\tau \colon Y \to Y$ and $\tau' \colon Y' \to Y'$. Suppose that $W \colon Y \to Y'$ is a cobordism and that there exists a self-diffeomorphism $T \colon W \to W$ restricting to τ and τ' on the two ends of W. Then*

$$\tau'_\# \circ F^-_{W, \mathfrak{t}} = F^-_{W, T_* \mathfrak{t}} \circ \tau_\#$$

for every $\mathfrak{t} \in \mathrm{Spin}^c(W)$. $\square$

Proof of Theorem 1.3. Suppose that $K' \subset S^3$ is concordant to K, i.e. there exists a smoothly embedded annulus $C \subset S^3 \times [0,1]$ with $\partial C = C \cap S^3 \times [0,1] = K \times \{1\} \cup K' \times \{0\}$. By taking the double branched cover $\Sigma(C)$ of $S^3 \times [0,1]$ branched along C we get a smooth rational homology cobordism from $\Sigma(K)$ to $\Sigma(K')$. By adapting [8, Lemma 2.1] for $n = 2$, we get that the four-manifold $\Sigma(C)$ comes with a distinguished spin structure $\mathfrak{t}$ restricting to the canonical spin structure on the two ends. In addition, this spin structure is invariant under the covering involution of the double branched cover $\Sigma(C)$. Let $F^-_C \colon \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \to \mathrm{CF}^-(\Sigma(K'), \mathfrak{s}_0)$ denote the cobordism map induced by $(\Sigma(C), \mathfrak{t})$.

Since $\Sigma(C)$ is a rational homology cobordism, it follows that

$$F^-_C \colon \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \to \mathrm{CF}^-(\Sigma(K'), \mathfrak{s}_0)$$

and

$$F^-_{-C} \colon \mathrm{CF}^-(\Sigma(K'), \mathfrak{s}_0) \to \mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0) \ ,$$

are local equivalences. (Recall that according to [32] a rational homology cobordism induces an isomorphism on $\mathrm{HF}^\infty = U^{-1}\mathrm{HF}^-$.) The fact that F^-_C and F^-_{-C} homotopy commute with the homotopy $\mathbb{Z}/2\mathbb{Z}$-actions follows from Lemma 3.6 and the fact that (in the notations of that lemma) we have $T_* \mathfrak{t} = \mathfrak{t}$. Then Lemma 3.4 concludes the argument. $\square$

Proof of Theorem 1.2. Let $f \in \text{End}_{\text{loc}}(\text{CF}^-(\Sigma(K), \mathfrak{s}_0), \tau_\#)$ be a maximal self-local equivalence. As a consequence of Lemma 3.4, the chain homotopy type of the mapping cone of the restriction $\tau_\#^{\text{Im} f} + \text{id}: \text{Im} f \to \text{Im} f$ is a concordance invariant of K. On the other hand,

$$H_*(\text{Cone}(\tau_\#^{\text{Im} f} + \text{id})) = \mathbb{F}[U]_{\underline{\delta}(K)} \oplus \mathbb{F}[U]_{\overline{\delta}(K)+1} \oplus (\mathbb{F}[U]-\text{torsion}),$$

implying the claim. $\qquad\square$

Since $\text{HF}^-(\Sigma(K), \mathfrak{s}_0)$ is of rank one (as an $\mathbb{F}[U]$-module), and $f_{\max}$ is a local equivalence, it follows that $\text{HFB}^-_{\text{conn}}(K) \subset \text{HF}^-(\Sigma(K), \mathfrak{s}_0)$ is also of rank one. The U-torsion submodule of $\text{HFB}^-_{\text{conn}}(K)$ is the **reduced connected Floer homology** of K, and it will be denoted by $\text{HFB}^-_{\text{red-conn}}(K)$.

The following simple adaptation of [9, Proposition 4.6] allows us to prove triviality of $\text{HFB}^-_{\text{red-conn}}(K)$.

Proposition 3.7. $\text{HFB}^-_{\text{red-conn}}(K) = 0$ *if and only if* $\overline{\delta}(K) = \underline{\delta}(K) = \delta(K)$. $\qquad\square$

Given a knot $K \subset S^3$ we denote by $-K$ its mirror image.

Lemma 3.8. *For a knot $K \subset S^3$ we have that* $\underline{\delta}(K) = -\overline{\delta}(-K)$

Proof. The double branched cover of $-K$ is $-\Sigma(K)$. The argument of [11, Proposition 5.2] provides the claimed identity. $\qquad\square$

Lemma 3.9. *If $K = K_1 \# K_2$ for two knots K_1 and $K_2 \subset S^3$ then*

$$\underline{\delta}(K_1) + \underline{\delta}(K_2) \leq \underline{\delta}(K) \leq \overline{\delta}(K) \leq \overline{\delta}(K_1) + \overline{\delta}(K_2). \tag{6}$$

Proof. Suppose that D_i is a doubly pointed Heegaard diagram for $K_i \subset S^3$ ($i = 1, 2$). Then a doubly pointed Heegaard diagram D can be constructed for K by taking the connected sums of D_i (along w_2^1 in D_1 and w_2^2 in D_2). This construction shows that $\text{CF}^-(\mathcal{H}_D)$ is the tensor product of $\text{CF}^-(\mathcal{H}_{D_1})$ and $\text{CF}^-(\mathcal{H}_{D_2})$. It then obviously follows that the map η_D of Equation (3) for $\mathcal{H}_D$ is the tensor product of the similar maps η_{D_1} and η_{D_2} for $\mathcal{H}_{D_1}$ and $\mathcal{H}_{D_2}$. This implies that τ_D and $\tau_{D_1} \otimes \tau_{D_2}$ are chain homotopic maps, from which [12, Proposition 1.3] implies the result. $\qquad\square$

4. Vanishing results

In some cases $\text{HFB}^-(K)$ and $\text{HFB}^-_{\text{conn}}(K)$ can be easily determined. As customary in Heegaard Floer theory, these invariants do not capture any new information for alternating (or, more generally quasi-alternating) knots. It is a more surprising (and as we will see, very convenient) feature of $\text{HFB}^-_{\text{conn}}$ that it is rather trivial for torus knots as well. In this section we show some vanishing results about the group $\text{HFB}^-_{\text{conn}}(K)$, while the next section provides methods to determine our invariants for Montesinos (and more generally for arborescent) knots. We start with a simple motivating example.

4.1. **An example.** Consider the Brieskorn sphere $\Sigma(2, 3, 7)$; it can be given as (-1)-surgery on the right-handed trefoil knot $T_{2,3}$. It is an integral homology sphere with Heegaard Floer homology $\widehat{\text{HF}}(\Sigma(2, 3, 7)) = \mathbb{F}^2_{(0)} \oplus \mathbb{F}_{(-1)}$ and $\text{HF}^-(\Sigma(2, 3, 7)) = \mathbb{F}[U]_{(-2)} \oplus \mathbb{F}_{(-2)}$ in its unique spin^c (hence spin) structure, see [26, Equation (25)].

This three-manifold can be presented as the double branched cover of S^3 either along the torus knot $T_{3,7}$, or along the pretzel knot $P(2, -3, -7)$. The two presentations provide two involutions on $\Sigma(2, 3, 7)$, which potentially provide two different

maps on the Heegaard Floer chain complex. Indeed, let ϕ_1 denote the involution $\Sigma(2,3,7)$ admits as double branched cover along $T_{3,7}$ and let ϕ_2 denote the involution it gets as double branched cover along $P(2,-3,-7)$. Through direct calculation, the actions of these maps on Heegaard Floer homology has been identified in [10, Propositions 6.26 and 6.27].

Theorem 4.1 ([10]). *The map $(\phi_1)_*$ induces the identity map on $\widehat{\mathrm{HF}}(\Sigma(2,3,7))$ (and hence on $\mathrm{HF}^-(\Sigma(2,3,7))$), while the map $(\phi_2)_*$ is different from the identity on $\widehat{\mathrm{HF}}(\Sigma(2,3,7))$.* $\square$

This allows us to compute the invariants HFB^- and $\mathrm{HFB}^-_{\mathrm{conn}}$ for $T_{3,7}$ and $P(2,-3,-7)$, showing that

- $\mathrm{HFB}^-(T_{3,7}) = \mathbb{F}[U]_{(-2)} \oplus \mathbb{F}[U]_{(-3)} \oplus \mathbb{F}_{(-2)} \oplus \mathbb{F}_{(-3)}$, or, as an $\mathbb{F}[U,Q]/(Q^2)$-module (and ignoring gradings) $\mathrm{HFB}^-(T_{3,7}) = (\mathbb{F}[U,Q]/(Q^2)) \oplus \mathbb{F}^2$,
- $\mathrm{HFB}^-_{\mathrm{conn}}(T_{3,7}) = \mathbb{F}[U]_{(-2)}$, hence $\mathrm{HFB}^-_{\mathrm{red-conn}}(T_{3,7}) = 0$; and
- $\mathrm{HFB}^-(P(2,-3,-7)) = (\mathbb{F}[U,Q]/(Q^2)) \oplus \mathbb{F}$,
- $\mathrm{HFB}^-_{\mathrm{conn}}(P(2,-3,-7)) = \mathbb{F}[U]_{(-2)} \oplus \mathbb{F}_{(-2)}$, hence $\mathrm{HFB}^-_{\mathrm{red-conn}}(P(2,-3,-7)) = \mathbb{F}_{(-2)} \neq 0$.

These calculations generalize to show that any torus knot has trivial reduced connected Floer homology $\mathrm{HFB}^-_{\mathrm{red-conn}}$, while for pretzel knots there is a combinatorial method to determine this quantity. In particular, the above results will be reproved in Subsection 4.3 and in Section 6.

4.2. Quasi-alternating knots.

Proof of Theoem 1.4. If K is an alternating (or more generally, quasi-alternating) knot, then the double branched cover $\Sigma(K)$ is an L-space, and hence $\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0) = \mathbb{F}[U]$; in particular the homology is only in even degrees. Results of [2] imply that $\tau_\#$ is determined (up to homotopy) by its action on homology, which (for a grading preserving map) for $\mathbb{F}[U]$ must be equal to the identity. Therefore $\tau_\#$ is homotopic to the identity, and so $\tau_\# + \mathrm{id} = 0$, hence the exact triangle of Equation (5) determines $\mathrm{HFB}^-(K)$ as the sum of two copies of $\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0)$ (one with shifted grading). Furthermore, since the homotopy commuting assumption of a self-local equivalence in this case is vacuous, we get that $\mathrm{HFB}^-_{\mathrm{conn}}(K) = \mathbb{F}[U](= \mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0))$, $\mathrm{HFB}^-_{\mathrm{red-conn}}(K) = 0$, and the only invariant we get from this picture is the d-invariant of $(\Sigma(K), \mathfrak{s}_0)$, which is (half of) the Manolescu-Owens invariant of the knot K from [20]. $\square$

4.3. Torus knots.
Next we turn to the discussion of Theorem 1.5. The proof of this result is significantly easier when pq is odd.

Proposition 4.2. *Suppose that pq is odd. Then the covering involution τ on the double branched cover $\Sigma(T_{p,q})$ is isotopic to* id.

Proof. The double branched cover of the torus knot $T_{p,q}$ is diffeomorphic to the link of the complex surface singularity given by the equation $z^2 = x^p + y^q$, which is the Brieskorn sphere $\Sigma(2,p,q)$. The covering involution $\tau \colon \Sigma(2,p,q) \to \Sigma(2,p,q)$ of $T_{p,q}$ acts as $(z,x,y) \mapsto (-z,x,y)$.

Fix $t \in S^1$ and consider the diffeomorphism

$$(z,x,y) \mapsto (t^{pq}z, t^{2q}x, t^{2p}y).$$

Clearly we get an S^1-family of diffeomorphisms, where $t = 1$ gives id, while $t = -1$ (under the condition pq odd) gives τ, concluding the proof of the proposition. $\square$
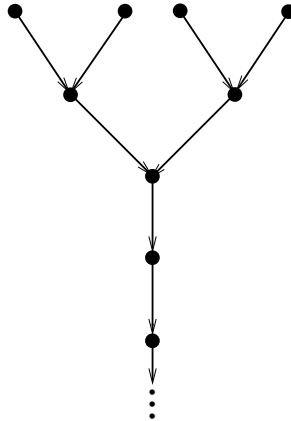
FIGURE 1. A graded root

Next we turn to the case when exactly one of p and q is even. The proof in this case requires some preparation from lattice homology. (For a more thorough introduction to this subject see [21, 23].)

4.4. **Lattice homology.** In determining the ι-complex associated to the double branched cover of a torus knot $T_{p,q}$, the concept of lattice homology will be extremally useful. This theory was motivated by computational results of Ozsváth and Szabó (for manifolds given by negative definite plumbing trees of at most one 'bad' vertex) in [27] and extended by Némethi [21] to any negative definite plumbing trees. The isomorphism of lattice homology with Heegaard Floer homology was established for almost-rational graphs by Némethi in [21], which was extended in [24] to graphs with at most two 'bad'vertices. Below we recall the basic concepts and results of this theory.

A **graded root** is a pair (R, w) where

- R is a directed, infinite tree with a finite number of leaves and a unique end modelled on the infinite stem $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots$, subject to the condition that every vertex has one and only one successor (see Figure 1 for an example),
- $w \colon R \to \mathbb{Q}$ is a function associating to each vertex x of R a rational number $w(x) \in \mathbb{Q}$ such that $w(x_2) = w(x_1) - 2$ if (x_1, x_2) is an edge of R.

A graded root (R, w) specifies a graded $\mathbb{F}[U]$-module $\mathbb{H}^-(R, w)$ as follows: As a graded vector space, $\mathbb{H}^-(R, w)$ is generated over $\mathbb{F}$ by the vertices of R, graded so that $\mathrm{gr}(x) = w(x)$ for all $x \in R$. Multiplication by U is defined on the set of generators by saying that $U \cdot x = y$ if for the vertex x of R the vertex y is its successor.

Manolescu and Dai showed in [2] that the lattice homology $\mathbb{H}^-(R, w_k)$ corresponding to the graded root (R, w_k) can be represented as the homology of a *model complex* $C(R)$, which is defined as follows. Let $C(R)$ be generated (as an $\mathbb{F}[U]$-module), by

- the leaves $\{v_l\}$ of R (with grading $w(v_l)$), called the *even generators*, and
- by the *odd generators* defined as follows: for a vertex a of R with valency greater than 2, let $V = \{v_1, \cdots, v_n\}$ be the set of all vertices of R satisfying $Uv = a$. Then we take the formal sums $v_1 - v_2, \cdots, v_{n-1} - v_n$ of vertices of R as generators of $C(R)$ at degree $w(a) + 1$.

The U-action in this module lowers degree by 2. The differential ∂ on $C(R)$ vanishes on all even generator v, while for an odd generator a the definition is slightly more complicated. Let v and w be two even generators such that $a = U^m v - U^n w$ as

formal sums of vertices of R, for some nonnegative integers m and n. Then set $\partial(a) = U^{m+1}v - U^{n+1}w$. In this case we say that the odd generator a is an *angle* between the even generators v and w. (For pictorial descriptions of $C(R)$, see [2].)

A negative definite plumbing tree Γ (with associated plumbed four-manifold X_Γ) and a characteristic vector k of $H^2(X_\Gamma, \mathbb{Z})$ determines a graded root (R_Γ, w_k) as follows. Let L be the non-compact 1-dimensional CW-complex having the points of $H_2(X_\Gamma, \mathbb{Z}) = \mathbb{Z}^{|\Gamma|}$ as 0-cells and a 1-cell connecting two vertices $\ell, \ell' \in H_2(X_\Gamma; \mathbb{Z})$ if $\ell' = \ell + v$ for some $v \in \Gamma$. The characteristic vector $k \in H^2(X_\Gamma, \mathbb{Z}) = \mathrm{Hom}(H_2(X_\Gamma), \mathbb{Z})$ determines a quadratic function through the formula

$$\chi_k(\ell) = -\frac{1}{2}(k(\ell) + \ell^2). \tag{7}$$

For each $n \in \mathbb{Z}$ let S_n be the set of connected components of the subcomplex of L spanned by the vertices satisfying $\chi_k \le n$. We define R_Γ to be the graph with vertex set $\bigcup_{n \ge 0} S_n$, in which two vertices x_1 and x_2 are connected by a directed edge if and only if the elements $x_1 \in S_n$ and $x_2 \in S_{n+1}$ (corresponding to components of the sublevel sets $\chi_k \le n$ and $\chi_k \le n + 1$, respectively) satisfy $x_1 \subset x_2$. We define $w_k \colon R_\Gamma \to \mathbb{Q}$ for $x \in S_n$ by the formula

$$w_k(x) = \frac{k^2 + |\Gamma|}{4} - 2n.$$

A negative definite plumbing graph Γ is **rational** if it is the resolution graph of a rational singularity, i.e. a singularity with geometric genus $p_g = 0$. Note that according to [22, Theorem 1.3] a negative definite graph is rational if and only if the boundary Y_Γ of the associated four-dimensional plumbing X_Γ is a Heegaard Floer L-space. We say that Γ is **almost-rational** if there is a vertex v of Γ on which we can change the weight in such a way that the result is rational.

Theorem 4.3. *([27]) Let Γ be an almost-rational graph, $\mathfrak{s} \in \mathrm{Spin}^c(Y_\Gamma)$ a spin^c structure on Y_Γ, and k a characteristic vector of the intersection lattice of X_Γ representing a spin^c structure that restricts to $\mathfrak{s}$ on the boundary $\partial X_\Gamma = Y_\Gamma$. Then there is an isomorphism $\mathbb{H}^-(R_\Gamma, w_k) \simeq \mathrm{HF}^-(Y_\Gamma, \mathfrak{s})$ of graded $\mathbb{F}[U]$-modules.* $\square$

4.5. **Torus knots again.** With this preparation in place, we now return to the computation of invariants of torus knots. We start by describing a plumbing presentation of $\Sigma(T_{p,q})$, where the covering involution τ is also visible.

Lemma 4.4. *Suppose that $(p, q) = 1$ and pq even. The plumbing graph $\Gamma = \Gamma_{p,q}$ presenting the double branched cover $\Sigma(T_{p,q})$ of S^3 branched along $T_{p,q}$ can be assumed to have the following properties:*

- *Γ is star-shaped with three legs L_{fix}, L_1, L_2 going out from the central vertex c.*
- *The coefficients on L_1 and L_2 are the same.*
- *The covering involution on $\Sigma(T_{p,q})$ can be modelled by a map on Γ which fixes L_{fix} and c, and flips L_1 and L_2.*
- *The covering involution on $\Sigma(K)$ extends smoothly to the plumbed 4-manifold X_Γ, which is compatible with the above involution on Γ.*

Proof. Recall that the double branched cover $\Sigma(T_{p,q})$ is equal to the link of the hypersurface singularity $z^2 = x^p + y^q$, hence $\Sigma(T_{p,q})$ can be presented as a plumbed manifold along the resolution dual graph of the above singularity. This graph can be easily determined by computing first the embedded resolution of the curve singularity $x^p + y^q = 0$, and then using a simple algorithm (described, for example in [5, Section 7.2]) for computing the resolution graph of the singularity. The embedded

resolution of $T_{p,q}$ gives a linear graph, where the single (-1)-curve is intersected by the proper transform of the knot, and the multiplicities at the two ends are p and q, respectively (see [3]). Now [5, Lemma 7.2.8] shows that if say p is even, then the curves between the leaf with multiplicity p and the (-1)-curve intersecting the proper transform have all even multiplicities. The algorithm described in [5, Section 7.2] then provides the resolution graph, together with the information about the covering transformation, satisfying the properties listed in the lemma. $\qquad\square$

Lemma 4.5. *Suppose that Γ is a plumbing graph as in Lemma 4.4 and the action of τ exchanges the two legs. Then the associated reduced connected homology vanishes.*

Proof. This information will be sufficient to identify the $\mathbb{Z}/2\mathbb{Z}$-action on the associated graded root R. Recall that in computing a graded root, we have to choose a characteristic vector $k \in H^2(X_\Gamma; \mathbb{Z})$, so that we can work with the induced weight function χ_k. Here, we choose k to be the canonical characteristic vector, which is given by $k(v) = -2 - v^2$, where v^2 denotes the self-intersection of the vertex v in the plumbing graph. Then k is clearly $\mathbb{Z}/2\mathbb{Z}$-invariant, so the induced spin^c structure $[k]|_{\partial W_\Gamma} \in \mathrm{Spin}^c(\Sigma(K))$ on the boundary is also $\mathbb{Z}/2\mathbb{Z}$-invariant. Recall that the $\mathbb{Z}/2\mathbb{Z}$-action on the set of spin^c structures on $\Sigma(K)$ leaves the unique spin structure $\mathfrak{s}_0$ invariant. Actually, even more is true: $\mathfrak{s}_0$ is the unique fixed point of the action, as shown in [7, Page 1378] and [16, Remark 3.4]. Hence the lattice homology of Γ with respect to k computes $\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0)$. Since we are using a $\mathbb{Z}/2\mathbb{Z}$-invariant characteristic vector, the associated graded root R_k, computed using the weight function χ_k, admits a $\mathbb{Z}/2\mathbb{Z}$-action which acts by permuting its vertices. In conclusion, R_k is a symmetric graded root (in the terminology of [2, Definition 2.11]).

Next, we claim that there exists a $\mathbb{Z}/2\mathbb{Z}$-invariant vertex $v \in V(R_k)$ with minimal χ_k-value. To prove this, we have to find a lattice point $x = \sum x_v \cdot v \in \mathbb{Z}^{|\Gamma|}$ which satisfies the following properties:

- x is $\mathbb{Z}/2\mathbb{Z}$-invariant, i.e. the coefficients of x on the arm L_1 and the coefficients of x on L_2 are the same.
- $\chi_k(x) \le \chi_k(y)$ for any other lattice point $y = \sum_{v \in V(\Gamma)} y_v \cdot v$.

Once we have found such a lattice points, the claim about the vertex v can be proved using the following argument. Recall that vertices of R_k are components of sublevel sets of χ_k. The $\mathbb{Z}/2\mathbb{Z}$-action on R_k permutes the components of sublevel sets, hence if we denote the component of the minimally-weighted sublevel set which contains the invariant lattice point x by C, then C is fixed by the $\mathbb{Z}/2\mathbb{Z}$-action.

To see the existence of x with the above properties, we first choose any lattice point $x_0 = \sum_{v \in V(\Gamma)} (x_0)_v \cdot v$ such that $\chi_k(x_0)$ is minimal. Then we can write $\chi_k(x_0)$ as

$$\chi_k(x_0) = \chi_k(x_0^{fix}) + Q_1(x_0) + Q_2(x_0),$$

where $x_0^{fix} = \sum_{v \in L_{fix} \cup c} x_v \cdot v$ is the "fixed part" x_0 and Q_1, Q_2 are functions defined on $\mathbb{Z}^{|\Gamma|}$ using the formula

$$-2Q_i(x) = \sum_{v \in L_i} (x_v^2 \cdot e_v + x_v k_v) + \sum_{\text{edge } (v_1 v_2) \text{ in } L_i} x_{v_1} x_{v_2} + x_{v_c^i} x_c$$

for $i = 1, 2$. (Here, we have denoted the vertex in L_i which is connected to the central vertex c as v_c^i.)

Assume, without loss of generality, that $Q_1(x_0) \leq Q_2(x_0)$, and consider the lattice point x' defined as

$$x' = x_0^{fix} + \sum_{v \in V(L_1)} x_v \cdot (v + \sigma(v)),$$

where σ denotes the $\mathbb{Z}/2\mathbb{Z}$-action on Γ. Then we have

$$\chi_k(x') = \chi_k(x_0^{fix}) + 2Q_1(x_0) \leq \chi_k(x_0) + Q_1(x_0) + Q_2(x_0) = \chi_k(x_0).$$

Since we assumed that $\chi_k(x_0)$ is minimal among all lattice points, we get $\chi_k(x') = \chi_k(x_0)$, implying x' satisfies the desired properties.

Now we claim that our symmetric graded root $R = R_k$ is locally equivalent to another symmetric graded root R', where the $\mathbb{Z}/2\mathbb{Z}$-action on R' is trivial. This can be verified by induction on the number n_R of non-$\mathbb{Z}/2\mathbb{Z}$-invariant leaves of R. In this part of the proof, in the notation we will confuse graded roots with their associated model complexes. For example, when we write that two given symmetric graded roots are locally equvialent, we will actually mean that their associated model chain complexes are locally equivalent.

The base case is simple: if $n_R = 0$, then the $\mathbb{Z}/2\mathbb{Z}$-action on R is already trivial, so we are done. In the general case, choose a non-$\mathbb{Z}/2\mathbb{Z}$-invariant leaf v of R. Since R carries a $\mathbb{Z}/2\mathbb{Z}$-invariant leaf in its top-degree level, we can always find a $\mathbb{Z}/2\mathbb{Z}$-invariant vertex x of R which lies in the same grading as v does. Denote the angle between the infinite monotone path starting at v and at x by α. Then we define R_1 as the graded root associated to the model complex we get from the model complex of R by deleting $v, \sigma(v)$ and the angles $\alpha, \sigma(\alpha)$. Define a map F from the associated model complex of R to that of R_1 as follows.

- $F(v) = F(\sigma(v)) = x$ and $F(w) = w$ for any leaf $w \neq v$.
- $F(\alpha) = F(\sigma(\alpha)) = 0$ and $F(\beta) = \beta$ for any angle $\beta \neq \alpha$.
- Extend this map $\mathbb{F}[U]$-linearly to the model chain complex of R.

This map F is obviously $\mathbb{F}[U]$-linear and $\mathbb{Z}/2\mathbb{Z}$-equivariant, so if we only prove that F is a chain map, it would automatically be a local equivalence to its image. To check that F is a chain map, it suffices to check that $\partial(F(v)) = F(\partial v)$ and $\partial(F(\alpha)) = F(\partial \alpha)$ by linearity and equivariance. Indeed,

- $\partial(F(v)) = \partial(\text{even generator}) = 0$ and $F(\partial v) = F(0) = 0$,
- $\partial(F(\alpha)) = \partial(0) = 0$ and $F(\partial \alpha) = F(v + x) = F(v) + F(x) = x + x = 0$.

Consequetly F is a local equivalence. This implies that R is locally equivalent to R_1, and the number of non-$\mathbb{Z}/2/\mathbb{Z}$-invariant leaves of R_1 is strictly smaller than the number of non-$\mathbb{Z}/2\mathbb{Z}$-invariant leaves of R. Thus, by induction, we deduce that R is locally equivalent to a symmetric graded root R' whose $\mathbb{Z}/2\mathbb{Z}$-action is trivial. This gives us the equality $\overline{\delta}(K) = \underline{\delta}(K)$, which by Proposition 3.7 then implies the claim. $\qquad \square$

Proof of Theorem 1.5. Proposition 4.2 in the case pq odd, and the combination of Lemmas 4.4 and 4.5 show that the covering transformation on $\Sigma(T_{p,q})$ is homotopic to id. The rest of the statement then follows as for the case of alternating knots. $\qquad \square$

5. ARBORESCENT AND MONTESINOS KNOTS

A plumbing tree Γ is a tree whose vertices are labelled by integers (see Γ on Figure 2 below). Above we associated a four-manifold X_Γ (and its three-dimensional boundary Y_Γ) to a plumbing tree Γ. A variant of this construction associates a surface $F_\Gamma \subset S^3$ to Γ: for every vertex we consider an annulus or a Möbius band, given by introducing
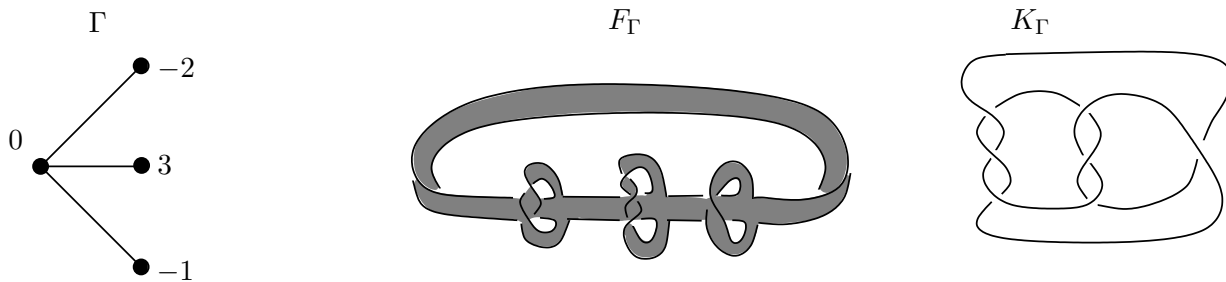
FIGURE 2. The plumbing graph Γ on the left determines a surface F_Γ (shown in the middle) with boundary K_Γ.

half-twists dictated by the label of the vertex, and plumb (or Murasugi sum) these annuli and Möbius bands together according to Γ. (See Figure 2 for a simple example.) The boundary of F_Γ specifies a link $K_\Gamma = \partial F_\Gamma$, called an ***arborescent link*** associated to Γ.

Remark 5.1. *Notice that the link is not determined uniquely by the graph, since at vertices of higher valency we need to determine an order for the edges when considering a planar presentation, and the link might depend on this choice. In addition, the location of the plumbing region relative to the twists also might influence the resulting link. With slightly more information (see Gabai's introductory work in [4]) attached to the tree this procedure can be made unique, though.*

The resulting K_Γ is called a ***Montesinos link*** if the tree Γ is star-shaped (i.e., has at most one vertex with valency more than 2); it is a ***pretzel link*** if Γ is star-shaped with all legs of length one, and is 2-bridge if Γ is linear (i.e., all valencies are either 1 or 2).

The construction of the four-manifold and the knot associated to Γ is connected by the fact that (by repeated application of Montesinos' trick) the double branched cover $\Sigma(K_\Gamma)$ of an arborescent link K_Γ is homeomorphic to the boundary Y_Γ of the four-dimensional plumbing X_Γ associated to Γ. Indeed, X_Γ can be presented as the double branched cover of D^4 branched along the surface we get by pushing the interior of the surface F_Γ of the above pluming into D^4.

Next we give an algorithm for computing the connected Floer homology group $\mathrm{HFB}^-_{\mathrm{conn}}(K)$ of an arborescent knot based on lattice homology. Recall that the graded module $\mathbb{H}^-(R, w)$ associated to a graded root (R, w) is the homology of a canonical free, finitely generated chain complex (the model complex) $(C(R), \partial)$ over the polynomial ring $\mathbb{F}[U]$. The following was observed in [2, Remark 4.3].

Lemma 5.2. *If (R, w) is a graded root, then a grading preserving homomorphism $f : \mathbb{H}^-(R, w) \to \mathbb{H}^-(R, w)$ lifts, uniquely up to chain homotopy, to a grading preserving chain map $f_\# : C(R) \to C(R)$. In addition, if C is a free chain complex with $H_*(C) \simeq \mathbb{H}^-(R, w)$ then $C \simeq C(R)$.* $\qquad\square$

If k is a characteristic vector which restricts to a self-conjugate spin^c structure $\mathfrak{s}_0 \in$ Spin$^c(Y_\Gamma)$, then the graded root (R_Γ, w_k) comes with an involution $J : \mathbb{H}^-(R, w) \to \mathbb{H}^-(R, w)$ [2, Section 2.3]. This is the map induced on $\mathbb{H}^-(R, w)$ by

$$\ell \mapsto -\ell - \frac{1}{2} PD(k), \tag{8}$$

for $\ell \in H_2(X_\Gamma, \mathbb{Z})$. We denote its lift to $C(R)$ by $J_\# : C(R_\Gamma) \to C(R_\Gamma)$.

Theorem 5.3. *Let $K = K_\Gamma$ be an arborescent knot associated to an almost-rational plumbing tree Γ. Then there exists a chain homotopy equivalence $(\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0), \tau_\#) \simeq (C(R_\Gamma), J_\#)$ of ι-complexes.*

Proof. Let $k_0 \in H^2(X_\Gamma, \mathbb{Z})$ be a characteristic vector of the intersection lattice of X_Γ which restricts to $\mathfrak{s}_0$ on the boundary. Theorem 4.3 provides an isomorphism

$$H_*(\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)) = \mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0) \simeq \mathbb{H}^-(R_\Gamma, w_{k_0}) = H_*(C(R_\Gamma)) \ .$$

As a consequence of Lemma 5.2, if we prove that the push-forward of $J \colon \mathbb{H}^-(R_\Gamma, w_{k_0}) \to \mathbb{H}^-(R_\Gamma, w_k)$ through this isomorphism agrees with $\tau_* \colon \mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0) \to \mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0)$, we are done.

Denote by W_Γ the cobordism $S^3 \to Y_\Gamma$ obtained from X_Γ by removing the interior of a small ball $D^4 \subset X_\Gamma$. Unravelling the definition of the isomorphism of Theorem 4.3 as it was done in [11, Theorem 3.1], the claim boils down to the identity

$$\tau_\# \circ F^-_{W_\Gamma, k} = F^-_{W_\Gamma, -k} \ ,$$

where k is a characteristic vector which restricts to $\mathfrak{s}_0$ on the boundary. According to [36] the covering involution $\tau \colon Y_\Gamma \to Y_\Gamma$ extends over X_Γ to the complex conjugation $T \colon X_\Gamma \to X_\Gamma$. Since T acts on spin^c structures as spin^c conjugation, Lemma 3.6 implies the claimed identity. $\qquad\square$

By applying similar arguments of [2, Section 5] we get the following results.

Corollary 5.4. *Let $K = K_\Gamma$ be an arborescent knot associated to an almost-rational plumbing tree Γ. Then there is an isomorphism of graded $\mathbb{F}[U]$-modules $\mathrm{HFB}^-(K) \simeq \mathrm{Ker}\,(1 + J)[-1] \oplus \mathrm{CoKer}\,(1 + J)$. Under this isomorphism the action of Q on $\mathrm{HFB}^-(K)$ is given by the projection $\mathrm{Ker}\,(1 + J) \to \mathrm{Ker}\,(1 + J)/\mathrm{Im}(1 + J) \subset \mathrm{CoKer}\,(1 + J)$.* $\quad\square$

Corollary 5.5. *Let $K = K_\Gamma$ be an arborescent knot associated to an almost-rational plumbing tree Γ. Then $\overline{\delta}(K) = \delta(K)$ and $\underline{\delta}(K) = -\sigma(K)/4$, where $\sigma(K)$ denotes the signature of K.*

Proof. Following the argument of [11, Theorem 1.2] we can identify $\overline{\delta}(K)$ with the Ozsváth-Szabó correction term of the double branched cover $d(\Sigma(K), \mathfrak{s}_0) = \delta(K)$. Furthermore, $\underline{\delta}(K) = -2 \cdot \overline{\mu}(\Gamma, \mathfrak{s}_0)$ where $\overline{\mu}(\Gamma, \mathfrak{s}_0)$ denotes the Neumann-Siebenmann $\overline{\mu}$-invariant of the plumbing Γ in the spin structure $\mathfrak{s}_0$. On the other hand, according to [35, Theorem 5] we have $\sigma(K) = 8 \cdot \overline{\mu}(\Gamma, \mathfrak{s}_0)$ thus $\underline{\delta}(K) = -\sigma(K)/4$. $\qquad\square$

Note that since a star-shaped plumbing tree Γ is almost-rational, for Montesinos knots the assumption of the above results modifies to demand that the intersection form of X_Γ (equivalently, the inertia matrix of the plumbing graph Γ) is negative definite. This method of approaching computability questions will be utilized in the next section.

The connected group associated to the ι-complex $(C(R), J_\#)$ of a graded root (R, w) can be easily computed. Given a vertex $v \in R$ denote by $C(v)$ the set of all leaves of R that are connected to v by an oriented path. We construct a subset S of the leaves of R by the following algorithm. Let v_0 denote the J-invariant vertex v_0 of R with highest weight. If $C(v_0)$ consists of only one vertex, we add it to S; otherwise, we can find a pair $\{v, Jv\}$ in $C(v_0)$ and in this case we add both v and Jv to S. Next consider the path γ connecting v_0 to infinity. Take $v_1 \in R$ to be the first vertex along γ for which $C(v_0) \subsetneq C(v_1)$. If $C(v_1)$ contains a pair of leaves $\{v, Jv\}$ with weight larger than the weight of any leaf in S then we choose one such pair with largest possible weight and we add it to S. By keep iterating this procedure until γ merges with the

long stem we end up with a set S of distiguished leaves. We denote by M the smallest graded root $M \subset R$ containing S. According to [9, Proposition 7.5] we have that $\mathrm{HFB}^-_{\mathrm{conn}}(K) = \mathbb{H}^-(M, w|_M)$. The resulting M is the *monotone subroot* of (R, w).

6. Some independence results

The following observation is a simple corollary of the concluding statements of Section 3.

Corollary 6.1. *Suppose that $\mathcal{F} = \{K_i \mid i \in I\}$ is a family of knots with $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(K) = 0$ for all $K \in \mathcal{F}$. If $\langle \mathcal{F} \rangle$ denotes the subgroup of $\mathcal{C}$ generated the equivalences of $K_i \in \mathcal{F}$, then $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(K) = 0$ for all $K \in \langle \mathcal{F} \rangle$.*

Proof. Lemmas 3.8 and 3.9, together with the equivalence provided by Proposition 3.7 imply the result. □

Recall the definition of the subgroups $\mathcal{QA}$ and $\mathcal{T}$ (generated by quasi-alternating and torus knots, respectively) of the smooth concordance group $\mathcal{C}$ from Section 1.

Proposition 6.2. *For $[K] \in \mathcal{QA} + \mathcal{T}$ we have that $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(K) = 0$.*

Proof. In Section 4 we showed that $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(K) = 0$ once K is either a quasi-alternating knot or a torus knot. Application of Corollary 6.1 concludes the argument. □

Based on this proposition, nonvanishing results for $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}$ then immediately imply nonvanishing in the quotient group $\mathcal{C}/(\mathcal{QA} + \mathcal{T})$.

Recall that in [18] Lidman and Moore showed that a pretzel knot $P(a_1, \ldots, a_k)$ is an L-space knot (i.e. it has L-space surgeries) if and only if it is a torus knot of the form $T_{2,2n+1}$, or a pretzel knot of the form $P(-2, 3, q)$ for some $q \geq 7$ odd. In [37] the question of which linear combinations of L-space knots is concordant to a linear combination of algebraic knots was studied. In [1, Theorem 1.1] the first author showed that pretzel knots of the form $P(-2, 3, q)$ are not concordant to positive sums of algebraic knots. Note that for these knots the obstruction found in [37, Corollary 3.5] vanishes. By computing connected Floer homology now we will prove Theorem 1.6.

Let $C = (C_0, \partial_0, \iota_0)$ denote the ι-complex where C_0 (as an $\mathbb{F}[U]$-module) is generated by three generators a, b, c, the boundary map ∂_0 is given by $\partial_0 a = \partial_0 b = 0$ and $\partial_0 c = Ua + Ub$ and $\iota_0(a) = b, \iota_0(b) = a, \iota_0(c) = c$. Define the grading by $\mathrm{gr}(a) = \mathrm{gr}(b) = 0$ and $\mathrm{gr}(c) = -1$, and denote by $C[r]$ the same chain complex with grading shifted by $r \in \mathbb{Q}$. (The same chain complex has been denoted by $\mathcal{C}_1$ in [9, Subsection 7.1].)

Lemma 6.3. *Let $q \geq 7$ be any odd integer. Then $(\mathrm{CF}^-(\Sigma(2, -3, -q), \mathfrak{s}_0), \tau_\#)$ is locally equivalent to $C[r]$ for some r. In particular, $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(\Sigma(2, -3, -q)) = \mathbb{F}$.*

Proof. The pretzel knot $-P(-2, 3, q) = P(2, -3, -q)$ is associated to the negative definite star-shaped (hence almost-rational) plumbing tree Γ_q shown in Figure 3(a). Let us label the vertices of Γ_q by c, v_1, v_2, v_3 as shown by Figure 3(a). (Notice that by successively blowing down c, v_1 and v_2 we can see that the three-manifold Y_{Γ_q} defined by the plumbing is the result of $(-q + 6)$-surgery on the right-handed trefoil knot.)

In determining the connected Floer homology of these pretzel knots we will appeal to the computational scheme through lattice homology (as outlined in Section 5 for appropriate arborescent knots). Recall that the canonical characteristic vector $K_q \in H^2(X_{\Gamma_q}; \mathbb{Z})$ is the class satisfying $K_q(v) = -2 - v^2$ for each vertex (where a vertex v is viewed as an element of $H_2(X_{\Gamma_q}; \mathbb{Z})$ and v^2 denotes the weight on the vertex v, which is
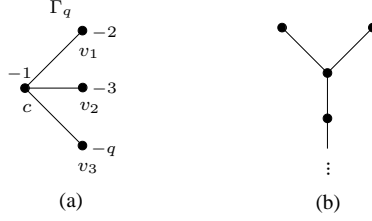
FIGURE 3. On the left we give the plumbing graph Γ_q giving the pretzel knot $P(2, -3, -q)$. On the right the monotone subroot of the associated graded root is shown.

equal to self-intersection of the corresponding homology class). In particular, $K_q(c) = -1, K_q(v_1) = 0, K_q(v_2) = 1$ and $K_q(v_3) = q - 2$. Every characteristic cohomology element determines a spin^c structure on the boundary three-manifold; by following the blow-down sequence described above with K_q we see that it induces the unqiue spin structure $\mathfrak{s}_0$ on the boundary. For an element $x = \alpha c + \beta v_1 + \gamma v_2 + \delta v_3 \in H_2(X_{\Gamma_q}; \mathbb{Z})$ the quadratic function χ_{K_q} of Equation (7) is given by

$$2\chi_{K_q}(\alpha c + \beta v_1 + \gamma v_2 + \delta v_3) = \alpha^2 + 2\beta^2 + 3\gamma^2 + q\delta^2 - 2\alpha(\beta + \gamma + \delta) + \alpha - \gamma - (q-2)\delta.$$

It is not hard to see that $\chi_{K_q}(x) \geq 0$ for any $x \in H_2(X_{\Gamma_q}; \mathbb{Z})$. In order to determine the monotone subroot M_q of the graded root (R_{K_q}, w_{K_q}) it will be sufficient to understand S_0 and S_1 (in the terminology of Subsection 4.4, i.e., $S_n = \{\chi_{K_q} \leq n\}$). For $q = 7$ this calculation has been performed in [27, Subsection 3.2], where it has been shown that $\{\chi_{K_7} \leq 0\}$ has two components, while $\{\chi_{K_7} \leq 1\}$ has a single component. This shows that M_7 is of the form given by Figure 3(b).

Notice, however, that for $x = \alpha c + \beta v_1 + \gamma v_2 + \delta v_3$ we have $2\chi_{K_q}(x) = 2\chi_{K_7}(x) + (q-7)(\delta^2 - \delta)$. Since $\chi_{K_q}(x) \geq 0$ for all x, we have that S_0 and S_1 have the same cardinality for all $q \geq 7$, hence the monotone subroot will be the same for all q.

The explicit description of the two vectors representing S_0 (as given in [27]) and the description of the involution of Equation (8) now identifies the action of $\tau_\#$, verifying the claim about the action of $\tau_\#$. Then the definition of $\mathrm{HFB}^-_{\mathrm{conn}}(K)$ shows that in this case it is isomorphic to $\mathrm{HF}^-(\Sigma(K), \mathfrak{s}_0)$, and hence $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(K) = \mathbb{F}$, completing the proof. □

Remark 6.4. *Indeed, $\widehat{\mathrm{HF}}(\Sigma(P(2, -3, -q)))$ can be easily computed using the surgery exact triangle (by viewing this three-manifold as surgery along the trefoil knot). This computation shows that $\dim \widehat{\mathrm{HF}}(\Sigma(P(2, -3, -q)), \mathfrak{s}_0) = 3$ (and all other spin^c structures have one-dimensional invariant), determining $\widehat{\mathrm{HF}}(\Sigma(P(2, -3, -q)), \mathfrak{s})$ as well, and showing that the local equivalence claimed in Lemma 6.3 is, indeed, a chain homotopy equivalence. Also, the gradings can be easily determined by computing K_q^2. The detailed calculation in the proof of Lemma 6.3 is crucial in the identification of the involution $\tau_\#$.*

Proof of Theorem 1.6. Suppose that K is the connected sum of pretzel knots $P(2, -3, -q)$ for some q's (all with $q \geq 7$). By [9, Proposition 7.1] the tensor product of these ι-complexes have nonvanishing connected homology, hence the combination of this nonvanishing result with Proposition 6.2 implies that $[K] \in \mathcal{C}/(\mathcal{QA} + \mathcal{T})$ is nontrivial. In the statement of the theorem we considered mirrors of the knots encountered above; since $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(-K)$ and $\mathrm{HFB}^-_{\mathrm{red\text{-}conn}}(K)$, as well as $[K]$ and $[-K]$ in $\mathcal{C}/(\mathcal{QA} + \mathcal{T})$ are trivial at the same time, the proof is complete. □
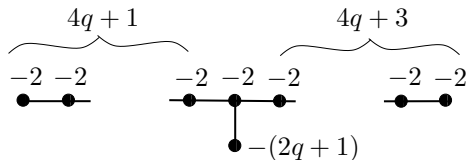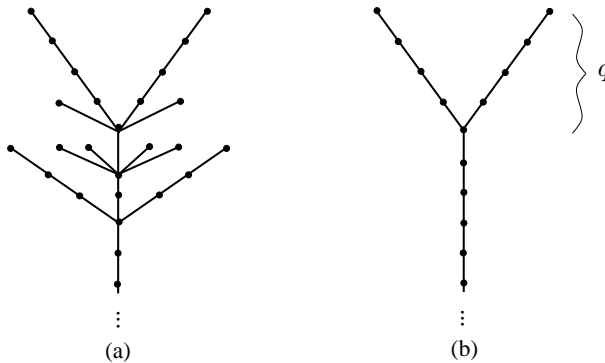
FIGURE 4. The plumbing graph Γ_q.

FIGURE 5. Schematic picture of the graded root associated to Γ_q in (a) and its monotone subroot in (b). Notice that there is no element of top degree fixed by the reflection.

Next we verify Theorem 1.7 from Section 1. For a knot $K \subset S^3$ define $\omega(K) = \min\{n \geq 0 \mid U^n \cdot \mathrm{HFB}^-_{\text{red-conn}}(K) = 0\}$.

Proof of Theorem 1.7. The double branched cover of the pretzel knot $K_q = P(4q + 3, -2q - 1, 4q + 1)$ can be expressed as boundary of the negative definite plumbing Γ_q of Figure 4. The associated graded root R_q was partially computed in [13]. According to [13] the top part of R_q, describing the truncated Heegaard Floer group $\mathrm{HF}^-_{\geq \delta(K_q) - 2q}(\Sigma(K_q))$, looks like the oriented graph of Figure 5.

The monotone sub-root $M_q \subset R_q$ of a graded root of this form has been determined in [2, Theorem 6.1], providing the graded roots of Figure 5(b) (see also [2, Figure 4]), yielding

$$\mathrm{HFB}^-_{\text{conn}}(K_q) = \mathbb{F}[U] \oplus \mathbb{F}[U]/(U^q).$$

The application of [9, Proposition 7.1] calculates $\mathrm{HFB}^-_{\text{conn}}(K)$ for a knot K that is a positive linear combination of the pretzel knots K_q:

$$\mathrm{HFB}^-_{\text{conn}}(K_{q_1}\# \ldots \#K_{q_s}) = \mathbb{F}[U] \oplus \bigoplus_{i=1}^{s} \mathbb{F}[U]/U^{q_i} .$$

In particular, for a knot of this form we have that

$$\omega(K_{q_1}\# \ldots \#K_{q_s}) = \max_i q_i . \tag{9}$$

Suppose that there is a non-trivial linear relation among the classes represented by the pretzel knots K_q in $\mathcal{C}/(\mathcal{T} + \mathcal{QA})$. By grouping the summands of such a linear relation according to their sign we end up with an identity of the form

$$K_{a_1}\# \ldots \#K_{a_s} = K_{b_1}\# \ldots \#K_{b_l}\#P$$

in the knot concordance group $\mathcal{C}$, for some $P \in \mathcal{T} + \mathcal{QA}$. Without loss of generality we can assume that the relation is reduced, i.e. that $a_i \neq b_j$ for each i and j. On the

other hand as a consequence of Equation (9) we have that

$$\max_i a_i = \omega(K_{a_1} \# \ldots \# K_{a_s}) = \omega(K_{b_1} \# \ldots \# K_{b_l} \# P) = \omega(K_{b_1} \# \ldots \# K_{b_l}) = \max_j b_j$$

a contradiction. This shows that the subgroup $\langle [K_q] \mid q \text{ odd and } q \geq 7 \rangle \subset \mathcal{C}/(\mathcal{T} + \mathcal{QA})$ generated by the above pretzel knots is isomorphic to $\mathbb{Z}^\infty$, concluding the proof. $\square$

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