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Reducing hypergraph coloring to clique search

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ABSTRACT

It is known that the legal coloring of the nodes of a given graph can be reduced to a clique search problem. This paper generalizes this result for hypergraphs. Namely, we will show how legal coloring of the nodes of a hypergraph can be reduced to clique search in a uniform hypergraph. Replacing ordinary graphs by hypergraphs extends the descriptive power of graph models. In addition searching cliques in uniform hypergraphs may improve the efficiency of computations. As an illustration we will apply the reformulation technique to a hypergraph coloring problem due to Voloshin.

1. Introduction

The current paper is motivated by the well-known fact that NP-complete problems can be reduced into each other in polynomial time [7]. So the fact that for example graph coloring with given number of colors can be reduced to a k-clique search – two examples from the original Karp's list of NP-complete problems – should not be surprising. There are different well-known reductions (see [4,5]), but the authors choose a simpler one which could be generalized for a list of different problems [11], using the k-clique problem as a modeling language. The aim of this paper is to continue our previous work and generalize further the proposed concept, this time for solving hypergraph coloring problems. The proposed reformulation should enable solving hypergraph coloring problems using standard heuristic and exact maximum clique solvers (see [2].) We also propose similar method for reducing coloring problems into a k-hyperclique problem that is more memory space efficient. (More on hypergraphs see [1,3].)

Let $H = (V, E)$ be a finite simple hypergraph. The hypergraph has finitely many nodes and finitely many edges. Further it does not have any loop (hyperedge containing only one element) and it does not have double hyperedges.

We color the nodes of the hypergraph H in the following way.

1. Each node receives exactly one color.
2. All the nodes of a hyperedge cannot receive the same color.

This type of coloring of the nodes of the hypergraph is called a legal coloring of the nodes of H. A coloring of the nodes of the hypergraph $H = (V, E)$ can be conveniently given by a map $f : V \to \{1, \ldots, k\}$. Here the numbers $1, \ldots, k$ represent the colors and $f(v)$ is the color of the node $v \in V$. The ith level set of the function f is commonly referred to as the ith colors class. The ith color class C_i is equal to $\{v : v \in V, f(v) = i\}$. A coloring of the nodes can also be given by the colors classes $C_1, \ldots, C_k$.

The next problem is known as the k-colorability problem.

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Table 1
The incidence matrix of the hypergraph H in Example 1.

	1	2	3	4	5	6	7	8
e_1	•		•			•		
e_2	•		•					•
e_3	•				•			•
e_4		•		•	•		•	

Table 2
The tiles assigned to the hyperedges in Example 1. The hyperedges are cut into two tiles.

Hyperedge	Tiles
$\{1, 3, 6\}$	$\{1, 3\}, \{6\}$
$\{1, 3, 8\}$	$\{1, 3\}, \{8\}$
$\{1, 5, 8\}$	$\{1, 5\}, \{8\}$
$\{2, 4, 5, 7\}$	$\{2, 4\}, \{5, 7\}$

Problem 1. Given a finite simple hypergraph $H = (V, E)$ and given a positive integer k. Let us decide if the nodes of H can be legally colored using k colors.

For each finite simple hypergraph H there is a well defined positive integer k such that the nodes of H can be legally colored using k colors but the nodes of H cannot be legally colored using $k - 1$ colors. This k is called the chromatic number of H and is denoted by $\chi(H)$.

By the complexity theory of algorithms, Problem 1 belongs to the NP-complete complexity class even in the $k = 2$ special case (see [6,8].) One may interpret this fact by saying that deciding if the nodes of a given hypergraph can be legally colored using two colors is a computationally demanding problem. Consequently determining the chromatic number of a given hypergraph is a computationally hard problem as well.

There are other types of hypergraph colorings, namely rainbow coloring, mixed coloring, etc. We will describe some in the text, but actually, although they are truly different constructions, from the point of view of our paper there is little difference between them.

A subset I of the nodes of the hypergraph H is called an independent set if a subset of I is never a hyperedge of H. An independent set I is maximal in H if it cannot be extended to a larger independent set by augmenting it by a node of H. An independent set I with cardinality k is a maximum independent set in H if H does not contain any independent set with cardinality $k + 1$.

Legally coloring the nodes of an ordinary graph or a hypergraph has many important applications in various fields besides its theoretical significance. Since finding the optimal number of colors of a legal coloring can easily exceed the available computational resources in many practical situation we settle for approximate greedy coloring procedures. In this work we will reduce hypergraph coloring problems to hyperclique search problems in r-uniform hypergraph. We intend to exploit the many possible greedy clique locating procedures to construct approximate legal coloring of the nodes of a given hypergraph.

Let $H = (V, E)$ be a finite simple r-uniform hypergraph. It means that H is a finite simple hypergraph such that each edge contains exactly r nodes. Let C be a subset of V. We say that C is a clique in H if each r pair-wise distinct nodes in C are the nodes of a hyperedge of H. The size of the clique is the cardinality $|C|$ of C. If $|C| = k$ we speak of a k-clique.

The next problem is the so-called k-clique problem for hypergraphs.

Problem 2. Given a finite simple r-uniform hypergraph H and given a positive integer k. Decide if H contains a k-clique.

For a given finite simple r-uniform hypergraph H there is a well defined positive integer k such that H has a hyperclique of size k and H does not have any hyperclique of size $k + 1$. This k is called the clique number of H and is denoted by $\omega(H)$. It is a well-known result from complexity theory that the k-clique problem is in the NP-complete complexity class even in the $r = 2$ particular case (see [6,8].) As the k-clique problem is computationally challenging it must hold for the problem of determining the clique number too.

To an r-uniform hypergraph H it is customary to assign an r-uniform hypergraph H' such that the nodes of H' are the same as the nodes of H and an r element subset e of the nodes is a hyperedge of H' when e is not a hyperedge of H. The graph H' is called the complement of H. Note that the nodes of hyperclique in H form an independent set in H' and the elements of an independent set in H are the nodes of a hyperclique in H'. We can speak of maximal and maximum cliques in the same way as we spoke about maximal and maximum independent sets.

2. Reducing hypergraph problems to ordinary graph problems

Let $H = (V, E)$ be a finite simple hypergraph. We assign colors to the nodes of H such that the following two conditions are met.

Table 3
The colored tiles assigned to the hyperedges in Example 1. The first rows of the matrices contain the tiles and the second rows contain the colors.

1:	$\begin{bmatrix} 6 \\ 1 \end{bmatrix}$	2:	$\begin{bmatrix} 6 \\ 2 \end{bmatrix}$				
3:	$\begin{bmatrix} 8 \\ 1 \end{bmatrix}$	4:	$\begin{bmatrix} 8 \\ 2 \end{bmatrix}$				
5:	$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$	6:	$\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$	7:	$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$	8:	$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$
9:	$\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$	10:	$\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$	11:	$\begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix}$	12:	$\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix}$
13:	$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$	14:	$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$	15:	$\begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix}$	16:	$\begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix}$
17:	$\begin{bmatrix} 5 & 7 \\ 1 & 1 \end{bmatrix}$	18:	$\begin{bmatrix} 5 & 7 \\ 1 & 2 \end{bmatrix}$	19:	$\begin{bmatrix} 5 & 7 \\ 2 & 1 \end{bmatrix}$	20:	$\begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$

1. Each node receives exactly one color.
2. Two distinct nodes of a hyperedge never receive the same color.

This type of coloring of the nodes of the hypergraph is called a rainbow coloring of the nodes of H. The following problem can be called as the k-rainbow colorability problem.

Problem 3. Given a finite simple hypergraph $H = (V, E)$ and given a positive integer k. Let us decide if the nodes of H can be rainbow colored using k colors.

One can observe that Problem 3 is not a genuine hypergraph problem in the sense that it can be reduced to the coloring of the nodes of an ordinary graph. Let us define an ordinary graph G. The nodes of G are identical to the nodes of the hypergraph H. Two distinct nodes u, v of G will be adjacent in G if u, v are elements of a hyperedge of H simultaneously. It is easy to verify that if the nodes of the hypergraph H have a rainbow coloring with k colors then the nodes of the ordinary graph G have a legal coloring using k colors. And conversely, if the nodes of G have a legal coloring with k colors, then the nodes of H have a rainbow coloring with k colors.

One may define cliques in a hypergraph $H = (V, E)$ in the next way. A subset C of V is a clique in H if for each distinct nodes u, v in V there is a hyperedge of H that contains both u and v. We may consider Problem 2 in connection with this clique concept. Locating a k-clique in the hypergraph H can be reduced to find a k-clique in an ordinary graph G. For this purpose it is enough to introduce the ordinary graph G we have described above. In this sense this new clique concept is not a genuine generalization of the ordinary clique concept for hypergraphs.

Let $H = (V, E)$ be a 3-uniform hypergraph and suppose that we are looking for a k-hyperclique in H. This problem can be reduced to a clique search in an ordinary graph G. Let $e_1, \ldots, e_m$ be all the hyperedges of H. These hyperedges will be the nodes of G. Two distinct edges $e_i = \{u_i, v_i, w_i\}$, $e_j = \{u_j, v_j, w_j\}$ are adjacent in G if the unordered triplets

$$\begin{array}{lll} \{u_i, u_j, v_j\}, & \{u_i, u_j, w_j\}, & \{u_i, v_j, w_j\}, \\ \{v_i, u_j, v_j\}, & \{v_i, u_j, w_j\}, & \{v_i, v_j, w_j\}, \\ \{w_i, u_j, v_j\}, & \{w_i, u_j, w_j\}, & \{w_i, v_j, w_j\} \end{array}$$

are all hyperedges of the hypergraph H. Both of the sets $\{u_i, v_i, w_i\}$, $\{u_j, v_j, w_j\}$ have three elements. It can happen that these sets are not disjoint. In this case not all of the listed nine sets have three elements. We should check if the three elements subsets among the listed nine sets are hyperedges of the hypergraph H.

We claim that if the hypergraph H has a hyperclique of size k, then the ordinary auxiliary graph G has a clique of size $\binom{k}{3}$.

In order to prove the claim let $C \subseteq V$ with $|C| = k$ such that for each pair-wise distinct $u, v, w \in C$ the unordered triplet $\{u, v, w\}$ is a hyperedge of H. We can form $\binom{k}{3}$ unordered triples from the elements of C. All these triplets are hyperedges of H. Further these hyperedges are pair-wise adjacent nodes in the graph G. Thus G has a clique of size $\binom{k}{3}$.

Next we claim that if G has a clique of size $\binom{k}{3}$, then H has a hyperclique of size k. Let $m = \binom{k}{3}$ and let $e_1 = \{u_1, v_1, w_1\}, \ldots, e_m = \{u_m, v_m, w_m\}$ be all the nodes of a clique of size m in G. Of course $e_1, \ldots, e_m$ are hyperedges of the hypergraph H. Set $C = \{u_1, v_1, w_1, \ldots, u_m, v_m, w_m\}$. There maybe repetition among the elements $u_1, v_1, w_1, \ldots, u_m, v_m, w_m$. In other words these elements are not necessarily pair-wise distinct. Let us suppose that $|C| = t$. As $e_1, \ldots, e_m$ are pair-wise distinct three element subsets of C, it follows that $m = \binom{k}{3} \leq \binom{t}{3}$ and so $k \leq t$.

Choose $u, v, w \in C$ such that u, v, w are pair-wise distinct. By the definition of C there are hyperedges e_p, e_q, e_r of H for which $u \in e_p$, $v \in e_q$, $w \in e_r$ and $e_p, e_q, e_r \subseteq C$. Note that e_p, e_q are adjacent nodes in G. Using the nine subsets in the

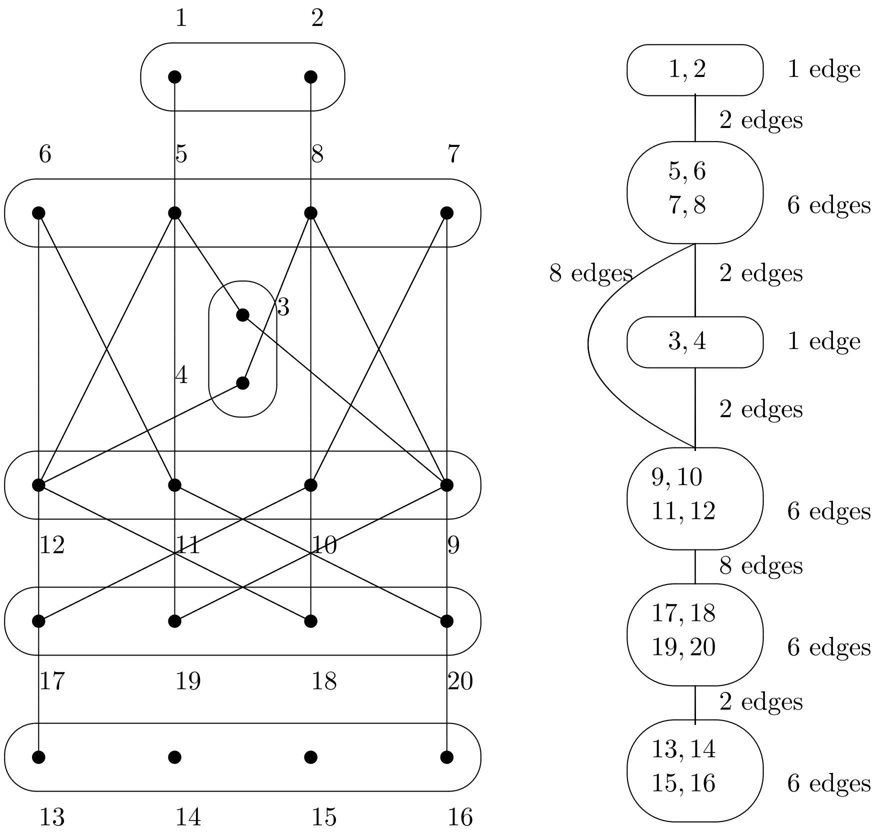

Fig. 1. On the left is the conflict graph G in Example 1. Each distinct two among the elements of $\{5, 6, 7, 8\}$ are adjacent. the same holds for the sets $\{9, 10, 11, 12\}$, $\{13, 14, 15, 16\}$, $\{17, 18, 19, 20\}$, $\{1, 2\}$, $\{3, 4\}$. In order to avoid an overly cluttered picture these edges are not drawn. On the right is a condensed form of the conflict graph. The nodes inside each ovals are pair-wise adjacent. An edge between ovals represents several edges. The number of the edges are given near to the ovals and near to the edges.

definition of the adjacency in G we get that there is a hyperedge e_s of H such that $u, v \in e_s$ and $e_s \subseteq C$. Note that e_r, e_s are adjacent nodes in G. We get that there is a hyperedge of H that contains u, v, w. Therefore each three element subset of C is a hyperedge of H. This means that H has a hyperclique of size k.

3. The auxiliary hypergraph

We pick a hyperedge e of the hypergraph H. We partition e into the subsets $T(e, 1), \dots, T(e, r)$. In other words we choose the subsets $T(e, 1), \dots, T(e, r)$ such that they satisfy the following conditions.

1. $T(e, i) \neq \emptyset$ for each i, $1 \leq i \leq r$.
2. $T(e, 1) \cup \cdots \cup T(e, r) = e$.
3. $T(e, i) \cap T(e, j) = \emptyset$ for each i, j, $1 \leq i < j \leq r$.

We will refer to the subsets $T(e, 1), \dots, T(e, r)$ as tiles associated with the hyperedge e.

We pick a tile $T(e, i)$ and color its elements with the k colors in all possible ways. If $T(e, i)$ has t elements, then the number of possible colorings is equal to k^t. We will denote this number by $\alpha(e, i)$. We will denote a colored tile by $[T(e, i), C(e, i, j)]$. Here $C(e, i, j)$ is a coloring of the elements of $T(e, i)$, that is, $C(e, i, j)$ is a map from $T(e, i)$ to the set of colors $\{1, \dots, k\}$.

We define an ordinary graph Γ_1. The nodes of Γ_1 are the colored tiles we have just constructed. Two distinct colored tiles

$$[T(e_1, i_1), C(e_1, i_1, j_1)], \quad [T(e_2, i_2), C(e_2, i_2, j_2)]$$

will be adjacent in Γ_1 if the colorings $C(e_1, i_1, j_1), C(e_2, i_2, j_2)$ do not agree on the intersection of the tiles $T(e_1, i_1), T(e_2, i_2)$.

Next we define an r-uniform hypergraph Γ_2. The nodes of Γ_2 are the colored tiles. The pair-wise distinct colored tiles

$$[T(e, 1), C(e, 1, j_1)], \dots, [T(e, r), C(e, r, j_r)]$$

form a hyperedge of Γ_2 if all the nodes of the hyperedge e of H receive the same color at the colorings $C(e, 1, j_1), \dots, C(e, r, j_r)$ of the tiles. We call Γ_1, Γ_2 conflict graphs. Both represent situations that obstruct legal coloring of the nodes of the hypergraph

Table 4
The adjacency matrix of the conflict graph in Example 1.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	×	•			•															
2	•	×						•												
3			×	•	•				•											
4			•	×				•			•									
5	•		•		×	•	•	•			•	•								
6					•	×	•	•			•	•								
7					•	•	×	•	•	•										
8		•		•	•	•	•	×	•	•										
9			•				•	•	×	•	•	•							•	•
10							•	•	•	×	•	•					•	•		
11				•	•	•			•	•	×	•							•	•
12					•	•			•	•	•	×					•	•		
13													×	•	•	•	•			
14													•	×	•	•				
15													•	•	×	•				
16													•	•	•	×				•
17										•		•	•				×	•	•	•
18										•		•					•	×	•	•
19									•		•						•	•	×	•
20									•		•					•	•	•	•	×

H. As it turns out the information contained by the conflict graphs Γ_1, Γ_2 is sufficient to locate legal coloring of the nodes of the hypergraph H. We will state the results formally in two lemmas. Suppose the given hypergraph H has m hyperedges. The m hyperedges are partitioned into mr tiles.

Lemma 1. *If the nodes of the hypergraph H can be legally colored using k colors, then the conflict graphs Γ_1, Γ_2 contain an independent set I of size mr simultaneously.*

Proof. Let us assume that the nodes of the hypergraph H are legally colored using k colors and suppose that the map $f : V \to \{1, \ldots, k\}$ defines this coloring. Note that the colored tiles

$$[T(e, i), C(e, i, j)], \quad 1 \le j \le \alpha(e, i)$$

that are all the colored versions of the tile $T(e, i)$ are pair-wise adjacent in the conflict graph Γ_1. This means that only one of them can be an element of an independent set in Γ_1. It follows that an independent set in Γ_1 can have at most mr elements.

The map f restricted to the tile $T(e, i)$ provides a colored tile $[T(e, i), C(e, i, j)]$ for some j, $1 \le j \le \alpha(e, i)$. There are m choices for e and there are r choices for i. Therefore Γ_1 has an independent set I of size mr.

The colored tiles that form a hyperedge of the conflict graph Γ_2 are all associated one fixed hyperedge e of H. Further all these tiles are colored with one fixed color. But the map f cannot assign the same color to each node of e. This shows that the set I is an independent set in the conflict graph Γ_2. □

Lemma 2. *If the conflict graphs Γ_1, Γ_2 contain an independent set I of size mr simultaneously, then the nodes of the hypergraph H can be legally colored using k colors.*

Proof. Let us assume that the conflict graphs Γ_1, Γ_2 have an independent set I of size mr simultaneously. As in the previous proof note that the colored tiles

$$[T(e, i), C(e, i, j)], \quad 1 \le j \le \alpha(e, i)$$

are pair-wise adjacent in Γ_1. It follows that exactly one of these colored tiles must be an element of I. This means that each tile is colored, that is, no tile remains uncolored. Consequently, each node of the hypergraph H receives a color. The conflict graph Γ_1 guarantees that a node can receive only one color. The conflict graph Γ_2 makes sure that all the nodes of a hyperedge of H cannot receive the same color. □

Let W be the set of all colored tiles. The edges of the conflict graph Γ_1 are two element subsets of W. The hyperedges of the conflict graph Γ_2 are r element subsets of W. We add $r - 2$ new nodes $\beta_1, \ldots, \beta_{r-2}$ to W to get W'. We construct a new conflict graph $\Gamma = (W', F)$. If the unordered pair $\{u, v\}$ is an edge of Γ_1, then we add the hyperedge $\{u, v, \beta_1, \ldots, \beta_{r-2}\}$ to Γ. We add the hyperedges of Γ_2 to Γ without any modification. The conflict graph Γ carries exactly the same information as the conflict graphs Γ_1, Γ_2. In order to find a legal coloring of the nodes of H we should locate an independent set I of size $mr + r - 2$ in the conflict graph Γ. Or equivalently we should look for a hyperclique of size $mr + r - 2$ in the complement of the conflict graph Γ.

Table 5
The edges of the conflict graph in Example 1. The 9th row of the table codes the information that the unordered pairs {9, 19} and {9, 20} are edges of the conflict graph G.

1	5	
2	8	
3	5	9
4	8	12
5	11	12
6	11	12
7	9	10
8	9	10
9	19	20
10	17	18
11	19	20
12	17	18
13	17	
14		
15		
16	20	
17		
18		
19		
20		

4. Toy examples

Let us consider the hypergraph $H = (V, E)$ with $V = \{1, 2, \ldots, 8\}$ and $E = \{e_1, \ldots, e_4\}$, where

$$e_1 = \{1, 3, 6\}, \quad e_2 = \{1, 3, 8\},$$
$$e_3 = \{1, 5, 8\}, \quad e_4 = \{2, 4, 5, 7\}.$$

The hypergraph H has 8 nodes and 4 hyperedges. We ask if the nodes of H can be colored legally using two colors. The incidence matrix of the edges of H is in Table 1.

Example 1. Using the hypergraph H we construct an auxiliary hypergraph $G = (W, F)$. In this example we cut the hyperedges of H into two tiles. In other words we choose the number r in the construction to be 2.

Using the hyperedges of the hypergraph $H = (V, E)$ we construct certain subsets of V. As in Section 3 we will call the family of these subsets tiles. The list of pair-wise distinct tiles is the following

$$T_1 = \{6\}, \quad T_2 = \{8\}, \quad T_3 = \{1, 3\},$$
$$T_4 = \{1, 5\}, \quad T_5 = \{2, 4\}, \quad T_6 = \{5, 7\}.$$

The way we constructed the tiles is summarized in Table 2. There are many ways to divide the hyperedges of H into two tiles. Any of these can be used to construct an auxiliary graph. These auxiliary graphs need not to have the same number of nodes.

After the list of tiles is available we construct a list of colored tiles by assigning colors to the nodes in the tiles in all possible ways. If a tile has n nodes and we try to color the nodes of the hypergraph H using k colors, then from the uncolored tile we will construct k^n colored tiles. The colored tiles are the nodes of the auxiliary hypergraph G. The procedure of coloring the tiles can be followed in Table 3.

There is a conflict in the following cases.

1. Two tiles are not disjoint and the common part of the tiles is not colored in the same way in the two tiles.
2. The union of two tiles is equal to a hyperedge of the hypergraph H and all the nodes in the two tiles are receiving the same color.

We are looking for a conflict free collection of colored tiles. In other words we are looking for an independent set in the conflict graph. Or we are looking for a clique in the complement of the conflict graph. Only one colored version of each of the six uncolored tiles can enter into an independent set. On the other hand each uncolored tile must occur in one colored version in the independent set. As there are six uncolored tiles we are looking for an independent set of size six in the conflict graph. Equivalently, we are looking for a clique of size six in the complement of the conflict graph. Table 4 contains the adjacency matrix of the conflict graph. Table 5 lists the edges of the graph and Fig. 1 depicts a geometric representation of the graph.

Table 6
The tiles coincide with the hyperedges in Example 2. The hyperedges are cut into one tile.

Hyperedge	Tile
$\{1, 3, 6\}$	$\{1, 3, 6\}$
$\{1, 3, 8\}$	$\{1, 3, 8\}$
$\{1, 5, 8\}$	$\{1, 5, 8\}$
$\{2, 4, 5, 7\}$	$\{2, 4, 5, 7\}$

An inspection shows that the colored tiles numbered 1, 3, 6, 10, 13, 19 form an independent set in the conflict graph. From this we can read off a coloring of the node of the given hypergraph H.

node	1	2	3	4	5	6	7	8
color	1	1	2	1	2	1	1	1

This coloring of the nodes is a legal coloring of the nodes of H using two colors.

Example 2. Using the hypergraph H we construct an auxiliary hypergraph $G = (W, F)$. In Section 3 we did not cover the case when the tiles are identical with the hyperedges of H. In this example we choose the number r to be 1.

In this case the tiles coincide with the hyperedges of the hypergraph H. For the sake of a unified treatment we listed the tiles in Table 6. The list of pair-wise distinct tiles is the following

$$T_1 = \{1, 3, 6\}, \quad T_2 = \{1, 3, 8\},$$
$$T_3 = \{1, 5, 8\}, \quad T_4 = \{2, 4, 5, 7\}.$$

Table 7 lists the colored tiles. We dropped the colored tiles whose elements are colored with one color. The remaining 32 colored tiles are the nodes of the conflict hypergraph G.

There is a conflict in the following case.

1. Two tiles are not disjoint and the common part of the tiles is not colored in the same way in the two tiles.

The conflict hypergraph is a 2-uniform hypergraph, that is, an ordinary graph. We are looking for an independent set of size 4 in the conflict graph. Or a clique of size 4 in the complement of the conflict graph. The 4 tiles we constructed must be colored in same way and the corresponding 4 colored tiles must be conflict free. Table 8 contains the adjacency matrix of the conflict graph. Fig. 2 depicts a geometric representation of the graph.

For the sake of practice we work out one more example in detail.

Example 3. Using the hypergraph H we construct an auxiliary hypergraph $G = (W, F)$. In this example we cut the hyperedges of H into three tiles, that is, we deal with the $r = 3$ case.

We displayed the tiles in Table 9. The list of pair-wise distinct tiles is the following

$$T_1 = \{1\}, \quad T_2 = \{3\}, \quad T_3 = \{5\}, \quad T_4 = \{6\}$$
$$T_5 = \{7\}, \quad T_6 = \{8\}, \quad T_7 = \{2, 4\}.$$

Table 10 lists the colored tiles. There are 16 colored tiles. Together with one additional node they are the nodes of the conflict hypergraph G.

There is a conflict in the following cases.

1. Two tiles are not disjoint and the common part of the tiles is not colored in the same way in the two tiles.
2. The union of three tiles is equal to a hyperedge of the hypergraph H and all the nodes in the three tiles are receiving the same color.

The conflict hypergraph is a 3-uniform hypergraph. If three colored tiles are in conflict by condition (2), then the unordered triple formed by these three tiles is a hyperedge of the conflict graph. If two colored tiles are in conflict by condition (1), then we add node 17 to form a three element subset. All these unordered triples are the hyperedges of the conflict hypergraph. The conflict hypergraph G has 17 nodes. A 3-uniform hypergraph with 17 nodes may have at most $\binom{17}{3} = 680$ hyperedges. The complement of G has $680 - 20 = 660$ hyperedges. We are looking for an independent set of size 8 in the conflict graph. Equivalently we are looking for a clique of size 8 in the complement of the conflict graph. The nodes of the 7 tiles we picked must receive colors and the coloring of corresponding 7 colored tiles cannot be conflicting. Table 12 contains the adjacency matrix of the conflict graph. We used Table 11 for the construction of the adjacency matrix.

5. An application

Voloshin [13] introduced the following type of coloring of the nodes of a hypergraph. The edges of the given hypergraph H are labeled as C type or D type hyperedges. An edge may belong to both types or may belong to neither. We color the nodes of the hypergraph H in the following way.

Table 7
The colored tiles assigned to the hyperedges in Example 2. The first rows of the matrices contain the tiles and the second rows contain the colors.

$\begin{bmatrix} 1 & 3 & 6 \\ 1 & 1 & 1 \end{bmatrix}$	$1 : \begin{bmatrix} 1 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix}$	$2 : \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 1 \end{bmatrix}$	$3 : \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 2 \end{bmatrix}$
$4 : \begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 1 \end{bmatrix}$	$5 : \begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 2 \end{bmatrix}$	$6 : \begin{bmatrix} 1 & 3 & 6 \\ 2 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$	$7 : \begin{bmatrix} 1 & 3 & 8 \\ 1 & 1 & 2 \end{bmatrix}$	$8 : \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 1 \end{bmatrix}$	$9 : \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 2 \end{bmatrix}$
$10 : \begin{bmatrix} 1 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}$	$11 : \begin{bmatrix} 1 & 3 & 8 \\ 2 & 1 & 2 \end{bmatrix}$	$12 : \begin{bmatrix} 1 & 3 & 8 \\ 2 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 8 \\ 2 & 2 & 2 \end{bmatrix}$
$\begin{bmatrix} 1 & 5 & 8 \\ 1 & 1 & 1 \end{bmatrix}$	$13 : \begin{bmatrix} 1 & 5 & 8 \\ 1 & 1 & 2 \end{bmatrix}$	$14 : \begin{bmatrix} 1 & 5 & 8 \\ 1 & 2 & 1 \end{bmatrix}$	$15 : \begin{bmatrix} 1 & 5 & 8 \\ 1 & 2 & 2 \end{bmatrix}$
$16 : \begin{bmatrix} 1 & 5 & 8 \\ 2 & 1 & 1 \end{bmatrix}$	$17 : \begin{bmatrix} 1 & 5 & 8 \\ 2 & 1 & 2 \end{bmatrix}$	$18 : \begin{bmatrix} 1 & 5 & 8 \\ 2 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 5 & 8 \\ 2 & 2 & 2 \end{bmatrix}$
$\begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$19 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 1 & 1 & 2 \end{bmatrix}$	$20 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 1 & 2 & 1 \end{bmatrix}$	$21 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 1 & 2 & 2 \end{bmatrix}$
$22 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 2 & 1 & 1 \end{bmatrix}$	$23 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$	$24 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 2 & 2 & 1 \end{bmatrix}$	$25 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 1 & 2 & 2 & 2 \end{bmatrix}$
$26 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 1 & 1 & 1 \end{bmatrix}$	$27 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 1 & 1 & 2 \end{bmatrix}$	$28 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 1 & 2 & 1 \end{bmatrix}$	$29 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 1 & 2 & 2 \end{bmatrix}$
$30 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 2 & 1 & 1 \end{bmatrix}$	$31 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 2 & 1 & 2 \end{bmatrix}$	$32 : \begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 2 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 4 & 5 & 7 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

Table 8
The edges of the conflict graph in Example 2. The 7th row of the table holds the information that the unordered pairs {7, 14}, {7, 16}, {7, 17}, {7, 18} are edges of the conflict graph G.

1	8	9	10	11	12	16	17	18
2	7	10	11	12	16	17	18	
3	7	10	11	12	16	17	18	
4	7	8	9	12	13	14	15	
5	7	8	9	12	13	14	15	
6	7	8	9	10	11	13	14	15
7	14	16	17	18				
8	13	15	16	17	18			
9	14	16	17	18				
10	13	14	15	17				
11	13	14	15	16	18			
12	13	14	15	17				
13	20	21	24	25	28	29	32	
14	19	22	23	26	27	30	31	
15	19	22	23	26	27	30	31	
16	20	21	24	25	28	29	32	
17	20	21	24	25	28	29	32	
18	19	22	23	26	27	30	31	
19								
.								
.								
.								
32								

1. Each node receives exactly one color.
2. All the nodes of a D type hyperedge cannot receive the same color.
3. The nodes of a C type hyperedge cannot receive all different colors.

It is easy to see that the proposed construction of Γ_1 and Γ_2 conflict graphs can easily be carried out for this type of graph coloring too.

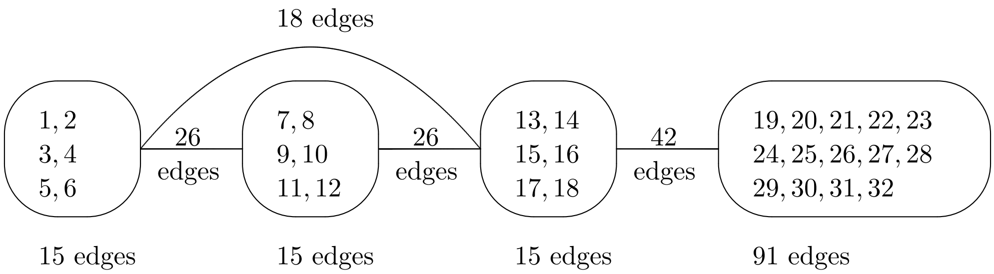

Fig. 2. The condensed form of the conflict graph *G* in Example 2. The nodes inside an oval are pair-wise connected by edges. Edges between ovals represent many edges. The number of edges are written near to the ovals and near to the edges.

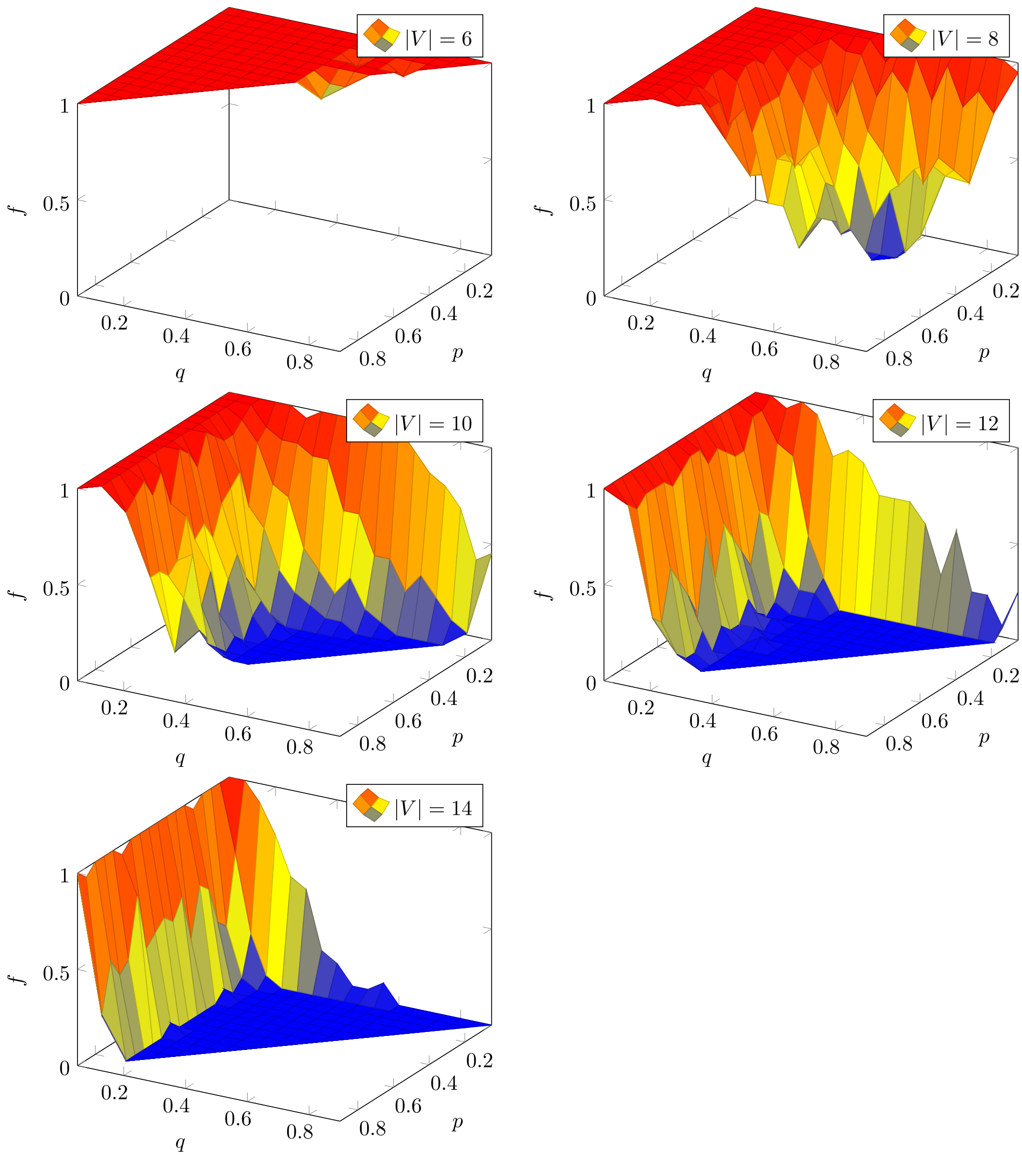

Fig. 3. Results of the A series, where an edge can be a *C* edge or a *D* edge but not both.

One remarkable property of mixed hypergraph coloring is that there are some mixed hypergraphs that cannot be colored properly at all. In his book Voloshin proposes some open questions and this particular one is among them. "Develop a

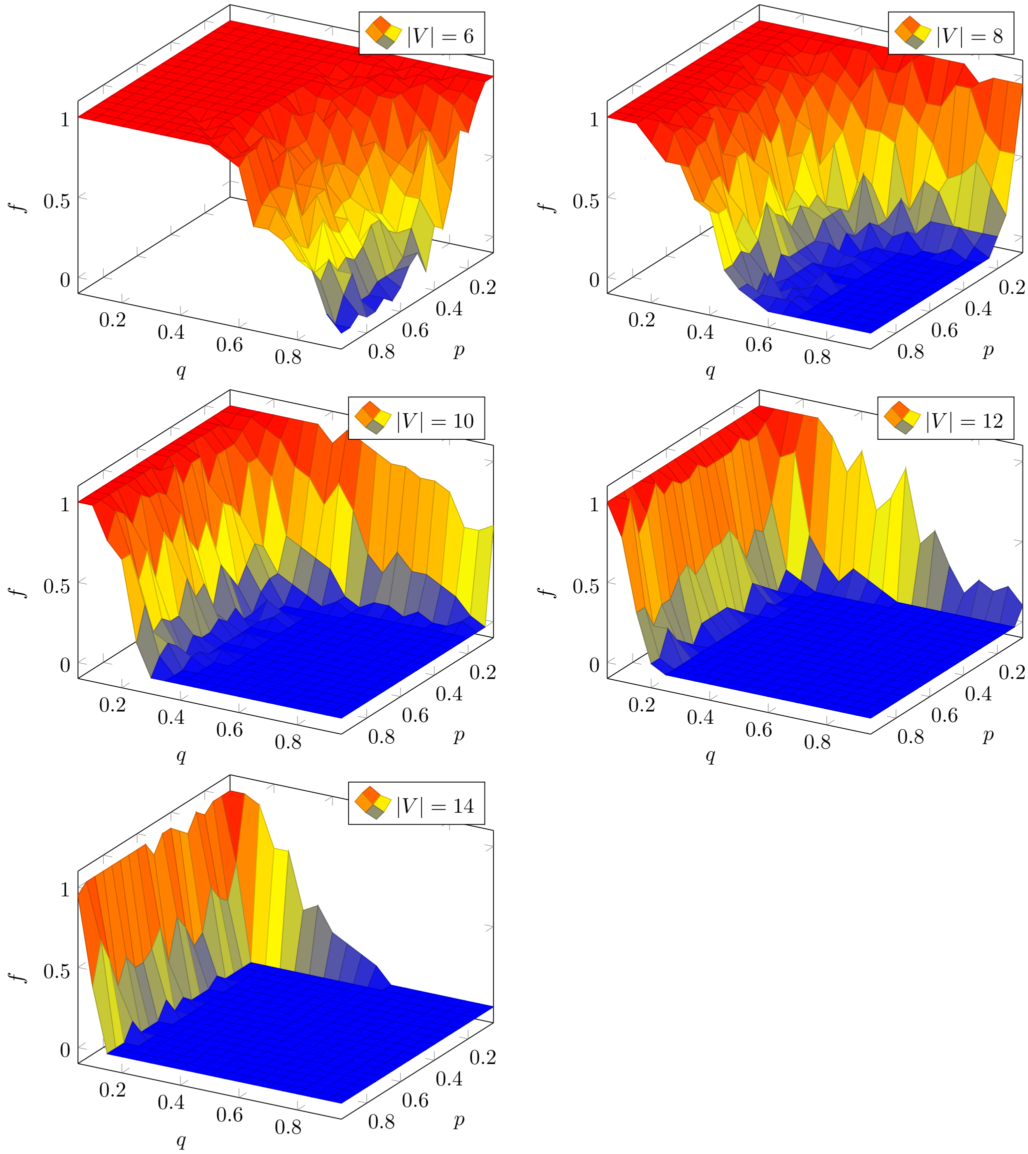

Fig. 4. Results of the B series, where an edge can be C edge and D edge at the same time.

Table 9
The hyperedges are cut into three tiles in Example 3.

Hyperedge	Tiles
{1, 3, 6}	{1}, {3}, {6}
{1, 3, 8}	{1}, {3}, {8}
{1, 5, 8}	{1}, {5}, {8}
{2, 4, 5, 7}	{2, 4}, {5}, {7}

probabilistic method for the colorability problem. Let $H = (V, C, D)$ be a mixed hypergraph with the probability of each C edge given by p and the probability of each D edge is given by q. What is the probability, as a function of p and q that H will be colorable". We are not going to solve the proposed problem, but can back up this question with some extended computational results. The question is ambiguous as it leaves open if the same subset can be a C and D edge at the same time or not. So we considered both possibilities.

We constructed a big series of 3-uniform mixed hypergraphs. In series A all 3 element subset of the nodes were either a C edge or a D edge or no edge. A random number $0 \leq z < 1$ was generated for each 3 nodes, and if $z < p$ these nodes became a C edge, if $p \leq z < q$ these nodes became a D edge. In series B we allowed the same 3 size subset to be a C edge

Table 10
The colored tiles assigned to the hyperedges in Example 3. The first rows of the matrices contain the tiles and the second rows contain the colors.

1: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	2: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$		
3: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	4: $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$		
5: $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$	6: $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$		
7: $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$	8: $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$		
9: $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$	10: $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$		
11: $\begin{bmatrix} 8 \\ 1 \end{bmatrix}$	12: $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$		
13: $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$	14: $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$	15: $\begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix}$	16: $\begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix}$

Table 11
Unordered pairs and unordered triplets expressing conflicts in Example 3.

Pairs	Triplets
{1, 2}, {3, 4} {5, 6}, {7, 8} {9, 10}, {11, 12}	{1, 3, 7}, {2, 4, 8} {1, 3, 11}, {2, 4, 12} {1, 5, 11}, {2, 6, 12}
{13, 14}, {15, 16} {13, 15}, {13, 16} {14, 15}, {14, 16}	{13, 5, 9}, {16, 6, 10}

Table 12
The incidence matrix of the conflict graph in Example 3.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
f_1	•	•															○
f_2			•	•													○
f_3					•	•											○
f_4							•	•									○
f_5									•	•							○
f_6											•	•					○
f_7													•	•			○
f_8															•	•	○
f_9													•		•		○
f_{10}													•			•	○
f_{11}														•	•		○
f_{12}														•		•	○
f_{13}	•		•				•										
f_{14}		•		•				•									
f_{15}	•		•								•						
f_{16}		•		•								•					
f_{17}	•				•						•						
f_{18}		•				•						•					
f_{19}					•				•				•				
f_{20}						•				•						•	

with probability of p and to be a D edge with probability of q. That means that the same subset can be either a C edge, or a D edge, or both, or not an edge. As in each graph at least one C edge was present that meant that the graph cannot be colored by $|V|$ colors. So we asked the question if it can be colored by $|V| - 1$ colors or less, and constructed an auxiliary graph Γ accordingly. There are few practical software for hyperclique or hyper independent set search. A recent publication [12] can deal with only small hypergraphs. We used $r = 2$, that is normal graphs for this construction. This also meant that we did not need two but only one auxiliary graph. We performed the k-clique search using the BBMCX maximum clique solver setting the upper and lower bounds (see [9,10].)

The nodes of the auxiliary graph Γ are all possible pairs of the set V colored by all possible colors. At this junction we would like to point that when we ask if the nodes of a hypergraph can be colored legally using k colors we actually mean k or less than k colors.

The series of experiments calculated the colorability of graphs of size 6,8,10,12 and 14. We set the values for p and q all possible ways by 5% steps. We generated 20 instances with the same p and q values, and checked the colorability of the resulting graphs. The results are the f frequency of the colorable ratio of these graphs, and pictured in Fig. 3 for A series, and Fig. 4 for B series.

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