

SUBGROUPS OF TWISTED WREATH PRODUCTS

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Abstract

By determining subdirect products invariant under the action of a regular permutation group of the components we provide a natural motivation for the definition of twisted wreath products. Then—based on papers of R. Baddeley, A. Lucchini, F. Börner, and M. Aschbacher—we explain how twisted wreath products play a fundamental role in the problem of representing finite lattices as intervals in subgroup lattices of finite groups.

1 Introduction

Our first goal is to provide a natural motivation for the definition of twisted wreath products. Originally, the twisted wreath product was introduced by Bernhard H. Neumann [17] in 1963. At first glance his definition looks quite complicated. Michio Suzuki [22, Chapter 2, §10] presented a more elegant treatment of this construction. In Section 2 we will determine all those subdirect products in a direct product of isomorphic non-abelian simple groups that are invariant under a regular permutation group of the components. This naturally leads to the definition of the twisted wreath product.

Twisted wreath products occur in the O’Nan–Scott–Aschbacher Theorem on the classification of primitive finite permutation groups. They were erroneously omitted from the first version [20] of the theorem, and were only added later to the list in the paper of Michael Aschbacher and Leonard Scott [5], and independently by László Kovács [14]. (See also [16].)

Although in the original paper of B. H. Neumann [17], as well as in several later developments, twisted wreath products were used for the construction of infinite groups with certain peculiar properties, in the present paper we will restrict our attention to twisted wreath products of finite groups.

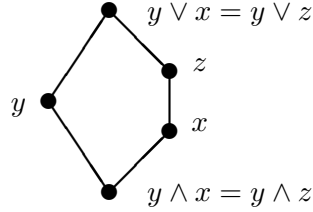
Our second goal is to explain the role twisted wreath products play for the problem of representing finite lattices as intervals in subgroup lattices of finite groups. This was explicitly or implicitly observed in the papers of Robert Baddeley and Andrea Lucchini [7], Baddeley [6], Ferdinand Börner [8], and Michael Aschbacher [2]. Based on their results we present in Section 3 a simplified proof showing that this representation problem can be reduced either to the case of almost simple groups or to the case of twisted wreath products. In Section 4 we give proper credits to the original papers and make further comments on the related literature.

We tried to make the paper as much self-contained as possible. However, at several places the proof makes use of **Schreier's Hypothesis** claiming that the outer automorphism group $\text{Out}(T) = \text{Aut}(T)/\text{Inn}(T)$ of every finite non-abelian simple group T is solvable. This is a well-known consequence of the classification of finite simple groups.

As for many questions in finite group theory it would be desirable to reduce the problem to the case of **almost simple groups** (groups G with a simple normal subgroup T with $\mathbf{C}_G(T) = 1$). However, it seems inevitable to consider also certain twisted wreath products in the context of representing finite lattices as intervals in subgroup lattices of finite groups.

The basic group theoretic notions do not need explanation for the readership of this proceedings. As lattice theory is concerned, let us recall that a **lattice** L is a partially ordered set where any two elements x, y have a greatest lower bound (called their meet, denoted by $x \wedge y$) and a least upper bound (their join, denoted by $x \vee y$). Finite lattices have a smallest and a largest element; these will be denoted by 0_L and 1_L . By a **filter** F in a finite lattice L we mean a non-empty subset of the form $\{x \in L \mid x \geq a\}$ for some $a \in L$. We obtain the **dual** of a lattice when we reverse the ordering, so the meet in the dual lattice is the same as the join in the original lattice, and similarly, the new join is the old meet.

A lattice L is called **modular** if $\forall x, y, z \in L : x \leq z \Rightarrow (x \vee y) \wedge z = x \vee (y \wedge z)$. The subgroup lattice of an abelian group is always modular. We will call a lattice L consisting of more than two elements **strongly non-modular** if for every $y \in L$, $y \neq 0_L, 1_L$, there exists a pair of elements $x < z \in L$ such that $y \vee x = y \vee z$ and $y \wedge x = y \wedge z$.



2 Invariant subdirect products

Let T (the “target”) be a finite non-abelian simple group and let D (the “domain”) be an arbitrary finite group. Consider the group F of all functions $D \rightarrow T$ with pointwise multiplication. Then $F \cong T \times \cdots \times T = T^{|D|}$. Let D act on F by translation, that is for $f \in F$, $d \in D$ let $f^d \in F$ be the function

$$f^d(x) = f(xd^{-1}) \quad (x \in D).$$

Clearly, this defines an action of the group D on F , as we have $f^{d_1 d_2}(x) = f(x(d_1 d_2)^{-1}) = f((x d_2^{-1}) d_1^{-1}) = f^{d_1}(x d_2^{-1}) = (f^{d_1})^{d_2}(x)$. The semidirect product $F \rtimes D$ is the **regular wreath product** $T \wr D$.

Recall that a subgroup $H \leq G_1 \times \cdots \times G_n$ of a direct product is said to be a **subdirect product**, if the projection of H to each factor G_i ($i = 1, \dots, n$) is

surjective. If the factors are pairwise isomorphic non-abelian simple groups, then the structure of any subdirect product is described by the following lemma (see, e.g., [9, Exercise 4.3]).

Lemma 2.1 *Let T be a non-abelian simple group and let $H \leq T^n$ be a subdirect product. Then $H \cong T^m$ for some $1 \leq m \leq n$. Moreover, there exist a map $\nu : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ and automorphisms $\varphi_i \in \text{Aut}(T)$ such that $f : \{1, \dots, n\} \rightarrow T$ belongs to H iff $f(i) = \varphi_i(t_{\nu(i)})$ with $t_1, \dots, t_m \in T$.*

Now we are going to determine which subdirect products in $F \cong T \times \dots \times T$ are invariant under the action of D . Let $H \leq F$ be a subdirect product. By the lemma we have $H \cong T^m$ for some $1 \leq m \leq |D|$, and

$$H = \{f : D \rightarrow T \mid f(x) = \varphi_x(t_{\nu(x)}), t_1, \dots, t_m \in T\},$$

with appropriate $\nu : D \rightarrow \{1, \dots, m\}$ and $\varphi_x \in \text{Aut}(T)$ ($x \in D$). For $f \in H$, $b \in D$, the invariance of H means that $f^{b^{-1}}$ also belongs to H , hence with some $u_1, \dots, u_m \in T$ we have

$$f^{b^{-1}}(x) = \varphi_x(u_{\nu(x)}).$$

By the definition of the action of D we obtain

$$f^{b^{-1}}(x) = f(xb) = \varphi_{xb}(t_{\nu(xb)}).$$

Clearly, D preserves the partition given by the kernel of the map ν , hence it is a partition into the cosets of some subgroup $D_0 \leq D$: $D = D_0x_1 \cup D_0x_2 \cup \dots \cup D_0x_m$ with $\nu(d) = i$ iff $d \in D_0x_i$. Without loss of generality we may assume that $x_1 = 1$ and that for every $i = 1, \dots, m$ we have $\varphi_{x_i} = \text{id}$. If $b \in D_0x_i$ and $a \in D_0$, then $ab \in D_0x_i$ as well. Then $f(ab) = f^{b^{-1}}(a) = \varphi_a(u_{\nu(a)}) = \varphi_a(u_1)$. For $a = 1$ this yields $u_1 = f(b)$, hence $\forall a \in D_0, \forall b \in D : f(ab) = \varphi_a(f(b))$. If $b = 1$ we get $\forall a \in D_0 : f(a) = \varphi_a(f(1))$. If $a, b \in D_0$, then $\varphi_{ab}(f(1)) = f(ab) = \varphi_a(f(b)) = \varphi_a(\varphi_b(f(1)))$. Since H is a subdirect product, $f(1)$ can be any element of T , so $\varphi : D_0 \rightarrow \text{Aut}(T)$ is a homomorphism. Furthermore, $\varphi_{ax_i}(t_i) = f(ax_i) = \varphi_a(f(x_i)) = \varphi_a(t_i)$, so $\forall a \in D_0, \forall i \in \{1, \dots, m\} : \varphi_{ax_i} = \varphi_a$.

Conversely, it is easy to verify that if $D_0 \leq D$, $D = D_0x_1 \cup \dots \cup D_0x_m$, and $\varphi : D_0 \rightarrow \text{Aut}(T)$ is a homomorphism, then by defining

$$\text{Sdp}(D_0, \varphi) = \{f : D \rightarrow T \mid f(ax_i) = \varphi_a(t_i), a \in D_0, t_i \in T (i = 1, \dots, m)\}$$

we obtain a D -invariant subdirect product in $F \cong T^{|D|}$. Indeed, if $b \in D$, then multiplication by b from the right permutes the right cosets of D_0 and so we have $x_i b = a_i x_j$ with $a_i \in D_0$, $1 \leq j = j(i, b) \leq m$. If $f \in \text{Sdp}(D_0, \varphi)$ let us denote $f(x_i b)$ by u_i . With this notation we have $f^{b^{-1}}(ax_i) = f(ax_i b) = f(aa_i x_j) = \varphi_{aa_i}(t_j) = \varphi_a(\varphi_{a_i}(t_j)) = \varphi_a(f(a_i x_j)) = \varphi_a(f(x_i b)) = \varphi_a(u_i)$, so $f^{b^{-1}} \in \text{Sdp}(D_0, \varphi)$, as we wanted.

Thus we have proved the following proposition.

Proposition 2.2 *If T is a non-abelian simple group, D is any finite group, and $F = \{f : D \rightarrow T\}$, then the D -invariant subdirect products in $F \cong T^{|D|}$ are precisely the subgroups of the form $\text{Sdp}(D_0, \varphi)$ for subgroups $D_0 \leq D$ and homomorphisms $\varphi : D_0 \rightarrow \text{Aut}(T)$.*

We can define $\text{Sdp}(D_0, \varphi)$ even without assuming the simplicity of T , the construction makes sense for any T . Then the semidirect product $\text{Sdp}(D_0, \varphi) \rtimes D$ is the **twisted wreath product** $\text{Twr}(T, D, D_0, \varphi)$ of T and D with respect to the subgroup $D_0 \leq D$ and the homomorphism $\varphi : D_0 \rightarrow \text{Aut}(T)$.

If we want to compare invariant subdirect products, the following is obvious.

Lemma 2.3 *A subdirect product $\text{Sdp}(D_1, \varphi_1)$ is contained in another subdirect product $\text{Sdp}(D_2, \varphi_2)$ iff D_1 contains D_2 and φ_2 is the restriction of φ_1 to D_2 .*

Now we return to analyzing the subgroup structure of F when T is a non-abelian simple group.

Lemma 2.4 *If T is a non-abelian simple group, and $D_0 \leq D$, $\varphi : D_0 \rightarrow \text{Aut}(T)$ satisfy $\varphi(D_0) \geq \text{Inn}(T)$, then every non-trivial D -invariant subgroup of $\text{Sdp}(D_0, \varphi)$ is a subdirect product.*

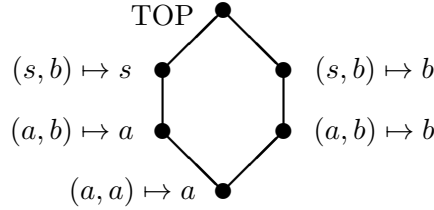
Proof. Let $H \leq \text{Sdp}(D_0, \varphi)$ be any D -invariant subgroup. Consider the image $U = \{f(1) \mid f \in H\}$ of H under the projection to the first coordinate. Let $u \in U$ (so $u = f(1)$ for some $f \in H$), and take an $a \in D_0$. Then $\varphi_a(u) = \varphi_a(f(1)) = f(a) = f^{a^{-1}}(1) \in U$, hence U is a $\varphi(D_0)$ -invariant subgroup of the simple group T . By assumption, $\varphi(D_0) \geq \text{Inn}(T)$, so U is a normal subgroup of T , hence by the simplicity of T , either $U = 1$ or $U = T$. Since D acts transitively on the components of the direct product, either all projections of H have trivial image, and so $H = 1$, or all projections map onto T , and so H is a subdirect product.

Thus the D -invariant subgroups of $\text{Sdp}(D_0, \varphi)$ apart from the trivial subgroup are all of the form $\text{Sdp}(D_1, \varphi_1)$ with $D_0 \leq D_1$ and $\varphi_1|_{D_0} = \varphi$. Combining the previous two lemmas, we obtain the following.

Proposition 2.5 *Let T be a non-abelian simple group, and assume that $D_0 \leq D$, $\varphi : D_0 \rightarrow \text{Aut}(T)$ satisfy $\varphi(D_0) \geq \text{Inn}(T)$. Then the lattice of D -invariant subgroups of $\text{Sdp}(D_0, \varphi)$ is isomorphic to the dual of the lattice of all extensions of φ to subgroups containing D_0 , together with an additional top element (what corresponds to the trivial subgroup via the dual isomorphism).*

Remark 2.6 Note that in the setting of (2.5) if $\varphi_1 : D_1 \rightarrow \text{Aut}(T)$ extends $\varphi : D_0 \rightarrow \text{Aut}(T)$, then φ_1 is uniquely determined by its kernel. Indeed, if $K = \ker(\varphi_1)$, then $D_1/K \cong \varphi_1(D_1) \leq \text{Aut}(T)$ is an almost simple group, and the image of D_0 contains $\text{Inn}(T) \cong T$, hence the action of D_1/K on $\varphi^{-1}(\text{Inn}(T))/K \cong T$ determines the homomorphism $\varphi_1 : D_1 \rightarrow \text{Aut}(T)$.

Example 2.7 Let A_5 and S_5 denote the alternating and the symmetric group of degree 5, and let $T = A_5$, $D = S_5 \times A_5$, $D_0 = \text{diag}(A_5) = \{(x, x) \mid x \in A_5\} < D$, and $\varphi : D_0 \cong A_5 \rightarrow \text{Aut}(T) \cong S_5$ an embedding. It is easy to see that the subgroups of D containing D_0 are $D_0 = \text{diag}(A_5)$, $A_5 \times A_5$, and $D = S_5 \times A_5$. Now φ has two extensions to $A_5 \times A_5$, corresponding to the first and the second projection. Likewise, there are two extensions to $S_5 \times A_5$. Together with the additional top element this gives a hexagon lattice (here $a, b \in A_5$, $s \in S_5$):



Hence by (2.5) the lattice of D -invariant subgroups of $\text{Sdp}(D_0.\varphi)$ is the hexagon lattice.

If M is a minimal normal subgroup of a finite group, then M is characteristically simple, so it is either an elementary abelian p -group for some prime number p , or it is isomorphic to a direct power of a non-abelian simple group T . We consider the latter case, when $M = T_1 \times \cdots \times T_k$ ($k \geq 1$) and each $T_i \cong T$. Now let a group A act on $T_1 \times \cdots \times T_k$ in such a way that A permutes the direct factors transitively. Let $A_1 = \{a \in A \mid T_1^a = T_1\}$ and denote by α the homomorphism $A_1 \rightarrow \text{Aut}(T_1)$ determined by the action of A_1 . Furthermore, choose a set of coset representatives $x_1 = 1, x_2, \dots, x_k$ of A_1 in A and fix the isomorphism $t \mapsto t^{x_i}$ ($t \in T_1$) between $T_1 = T$ and T_i ($i = 1, \dots, k$). Then $\text{Sdp}(A_1, \alpha) \cong T^k$ with the isomorphism given by the projection onto the group of functions $\{f : \{x_1, \dots, x_k\} \rightarrow T\}$ and this isomorphism is compatible with the action of A . In the rest of the paper we will freely use this identification of $\text{Sdp}(A_1, \alpha)$ and T^k . (Cf. [15].) The following lemmas will be needed in the proof of the main result (3.2).

Lemma 2.8 *Every A -invariant subgroup of T^k is one of the following types:*

1. a *subdirect product* in T^k ;
2. a **box**, that is, U^k for some A_1 -invariant subgroup $1 < U < T$;
3. a **skew subgroup**, i.e., a nontrivial subgroup properly contained in a box;
4. the *trivial subgroup*.

Proof. Let $H \leq T^k$. Let the projection to the first component map H onto $U \leq T$. Since H is A -invariant, each projection maps H onto U in the corresponding component (isomorphic to T via the fixed isomorphism). So $H \leq U^k$. If $U = T$, then H is a subdirect product. If $1 < U < T$ and $H = U^k$, then H is a box. If $1 < U < T$ and $H < U^k$, then H is a skew subgroup. (Note that the box U^k is also A -invariant in this case.) If $U = 1$, then $H = 1$ as well.

Corollary 2.9 *A maximal A -invariant subgroup is either a subdirect product or a box, unless the only proper A -invariant subgroup is the trivial one.*

The following is obvious.

Lemma 2.10 *The box subgroups in $\text{Sub}_A(T^k)$ form a sublattice isomorphic to $\text{Sub}_{A_1}(T)$.*

The next result is a well-known consequence of Schreier's Hypothesis, see [6, Proposition 2.4].

Lemma 2.11 *If $\alpha(A_1) \not\leq \text{Inn}(T)$, then there exists a proper non-trivial A_1 -invariant subgroup of T .*

Lemma 2.12 *A box cannot be contained in a proper subdirect product.*

Proof. For any proper subdirect product H there exist a pair of indices $1 \leq i < j \leq k$ and $\varphi \in \text{Aut}(T)$ such that for any $(x_1, \dots, x_k) \in H$ we have $x_j = \varphi(x_i)$. Hence if $x_i = 1$, then $x_j = 1$ as well. In the contrary, any box subgroup contains elements with $x_i = 1$, $x_j \neq 1$.

Lemma 2.13 *Let $H_1 > H_2$ be A -invariant subdirect products and let U be an A_1 -invariant subgroup. Then either $H_1 \cap U^k > H_2 \cap U^k$ or $H_2 \cap U^k = 1$.*

Proof. Let $H_2 = \text{Sdp}(B_2, \beta)$ and $H_1 = \text{Sdp}(B_1, \beta|_{B_1})$ with $B_1 < B_2$ (see (2.3)). Suppose that $H_2 \cap U^k \neq 1$. By the definition of $H_2 = \text{Sdp}(B_2, \beta)$ this means that there exists $t_1, \dots, t_m \in T$ ($m = |A : B_2|$) not all equal to 1, such that $\beta_b(t_i) \in U$ for every $b \in B_2$ and each $i = 1, \dots, m$. In particular, we can choose $1 \neq u \in U$ such that $\beta_b(u) \in U$ for every $b \in B_2$. Then the function $f : A \rightarrow T$ given by $f(a) = \beta_a(u)$, if $a \in B_1$, and $f(a) = 1$, otherwise, belongs to $H_1 = \text{Sdp}(B_1, \beta|_{B_1})$, but not to H_2 , hence $H_1 \cap U^k > H_2 \cap U^k$.

3 Intervals in subgroup lattices of finite groups

Let $\text{Sub}(G)$ denote the subgroup lattice of the group G , and for a subgroup $H < G$ we denote by $\text{Int}(H, G) = \{X \mid H \leq X \leq G\}$ the lattice of intermediate subgroups (in other words: overgroups of H), and call it the **interval** between H and G in the subgroup lattice. If a group A acts by automorphisms on G , then the A -invariant subgroups form a sublattice in $\text{Sub}(G)$; it will be denoted by $\text{Sub}_A(G)$. Likewise, we will use the notation $\text{Int}_A(H, G)$, whenever H is an A -invariant subgroup of G .

Intervals of subgroup lattices occur in various contexts. If $F \subset E$ is a finite separable field extension and E^* is the splitting field containing E , then the lattice of intermediate fields $\{X \mid F \subseteq X \subseteq E\}$ is dually isomorphic to the interval $\text{Int}(\text{Gal}(E^*|E), \text{Gal}(E^*|F))$ in the Galois group of E^* .

In the theory of operator algebras it is an open problem whether every finite lattice is isomorphic to the lattice of intermediate subfactors of a von Neumann algebra. Yasuo Watatani [24] proved that whenever a lattice can be represented as

an interval in a subgroup lattice of a finite group, then it also occurs as a lattice of intermediate subfactors of a von Neumann algebra. With the exception of two lattices, he was able to find intervals isomorphic to every lattice with at most six elements. One of the missing cases was the hexagon lattice. M. Aschbacher [2] gave a general construction whose particular cases provided examples for the hexagon and for the other six-element lattice Watatani was not able to handle.

In universal algebra a well-known open problem asks whether every finite lattice is isomorphic to the congruence lattice of a finite algebra. This problem is motivated by the fundamental result of George Grätzer and E. Tamás Schmidt [11] stating that every algebraic lattice is isomorphic to the congruence lattice of some algebraic structure. (A lattice is called algebraic iff it is complete and every element is a join of compact elements. In particular, every finite lattice is algebraic.) For almost all finite lattices each known proof of the Grätzer–Schmidt Theorem constructs infinite algebras to represent the lattice as a congruence lattice. So it is a natural question to ask, if finite algebras with prescribed finite congruence lattice can be constructed. In a joint paper with Pavel Pudlák [19] we proved that the problem about general algebraic structures is actually equivalent to a group theoretic problem.

Problem 3.1 Is every finite lattice isomorphic to an interval in the subgroup lattice of a finite group?

One direction of the equivalence is obvious. Let G act on the set of right cosets of the subgroup H , and consider each permutation in G as an operation with one variable. Then the congruences are exactly the partitions into cosets of subgroups belonging to the interval $\text{Int}(H, G)$, hence the congruence lattice of this multi-ary algebra is isomorphic to this interval. Concerning the reverse implication, it should be emphasized that we do not claim that the congruence lattices of finite algebras are (up to isomorphism) the same as the intervals in subgroup lattices of finite groups. What we proved is that if *all* finite lattices can be represented as congruence lattices of finite algebras then *all* finite lattices can be represented as intervals in subgroup lattices of finite groups. In fact, we embed any finite lattice into a finite lattice with some useful properties, and then we show that the smallest algebra with a congruence lattice having these properties is a transitive permutation group considered as a multi-ary algebra.

It was shown by Jiří Tůma [23] that every algebraic lattice is isomorphic to an interval in the subgroup lattice of an infinite group. So it is the finiteness of the group what seems to constitute a severe restriction. Therefore, it is generally believed that the answer to the finite representation problem is negative.

Making use of ideas from the fundamental papers of R. Baddeley and A. Lucchini [7], R. Baddeley [6], F. Börner [8], and M. Aschbacher [2] we present here a simplified proof for a slightly modified version of the main result of F. Börner [8] giving a reduction of the problem to almost simple groups and to twisted wreath products.

Theorem 3.2 *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group if and only if one of the following is true:*

(1) Every finite lattice consisting of more than one element is isomorphic to an interval $\text{Int}(H, G)$ in the subgroup lattice of an **almost simple** finite group G with a core-free subgroup H (that is, $\bigcap_{g \in G} g^{-1}Hg = 1$).

(2) Every finite lattice consisting of more than one element is isomorphic to an interval $\text{Int}(D, G)$ in the subgroup lattice of a **twisted wreath product** $G = \text{Tw}(T, D, D_0, \varphi)$ of a non-abelian finite simple group T and a finite group D with respect to a subgroup $D_0 < D$ and a homomorphism $\varphi : D_0 \rightarrow \text{Aut}(T)$ satisfying $\varphi(D_0) \geq \text{Inn}(T)$.

In the proof we will make use of the following lemmas.

Lemma 3.3 Every finite lattice L can be embedded as a filter into a finite lattice \hat{L} generated by its **coatoms** (maximal elements in $\hat{L} \setminus \{1_{\hat{L}}\}$).

Proof. We define $\hat{L} = L \cup \{c_x^1, c_x^2, a_x \mid x \in L \setminus \{0_L, 1_L\}\} \cup \{c^{11}, c^{12}, c^{21}, c^{22}, a^1, a^2, 0^*\}$ and we extend the order on L to \hat{L} in the following way: each new element is $< 1_L$; $a_x < y$ if $x \leq y$ in L ; $a^1, a^2 < y$ for all $y \in L$; 0^* is smaller than every other element of \hat{L} ; $a_x < c_x^1, c_x^2$, $a^1 < c^{11}, c^{12}$, $a^2 < c^{21}, c^{22}$. It is straightforward to check that \hat{L} is a lattice and L is the filter in \hat{L} consisting of the elements above 0_L . Moreover, the elements $c_x^1, c_x^2, c^{11}, c^{12}, c^{21}, c^{22}$ are coatoms and they generate the whole lattice \hat{L} , since $a_x = c_x^1 \wedge c_x^2$, $a^1 = c^{11} \wedge c^{12}$, $a^2 = c^{21} \wedge c^{22}$, $0^* = a^1 \wedge a^2$, $0_L = a^1 \vee a^2$, $x = a_x \vee 0_L$ ($x \in L \setminus \{0_L, 1_L\}$), and $1_L = c^{11} \vee c^{12}$.

Lemma 3.4 If $N \triangleleft G$ and $N \leq H$, then $\text{Int}(H, G) \cong \text{Int}(H/N, G/N)$.

Lemma 3.5 If $N \triangleleft G$, $H < G$ with $NH = G$, then $\text{Int}(H, G) \cong \text{Int}_H(H \cap N, N)$.

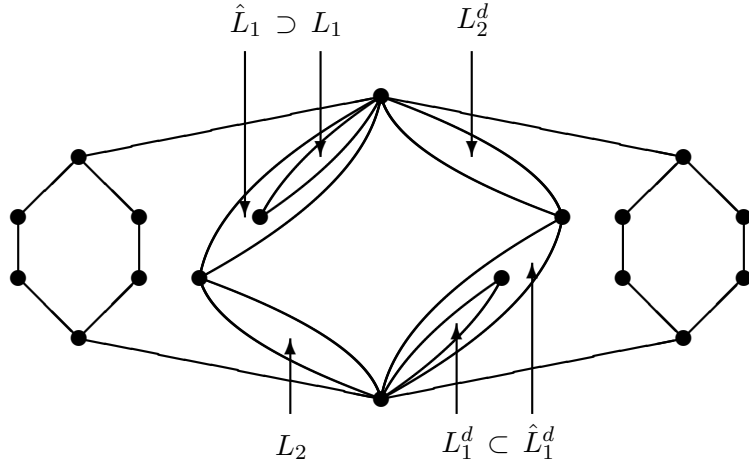
Proof. It is easy to check that the maps $U \mapsto U \cap N$ ($U \in \text{Int}(H, G)$) and $V \mapsto VH$ ($V \in \text{Int}_H(H \cap N, N)$) are order-preserving, and are inverses to each other, showing the isomorphism of the two intervals.

Lemma 3.6 Assume that $\text{Int}(H, G)$ is a strongly non-modular lattice. Then for any normal subgroup $N \triangleleft G$ either $N \leq H$ or $NH = G$. Moreover, if H is core-free, then G has a unique minimal normal subgroup M and M is not abelian.

Proof. Suppose that $H < NH < G$. Since the interval is strongly non-modular, there exist subgroups $H < X < Z < G$ with $NH \wedge X = NH \wedge Z$ and $NH \vee X = NH \vee Z$. Now NX is a subgroup and $NX = N \vee X = N \vee H \vee X = NH \vee X = NH \vee Z \geq Z$, hence every $z \in Z$ can be written as $z = nx$ with $n \in N$ and $x \in X$. Then $n = zx^{-1} \in NH \wedge Z = NH \wedge X \leq X$, so $z = nx \in X$, that is $Z \leq X$, a contradiction.

Suppose now that H is core-free, and let $M \triangleleft G$ be a minimal normal subgroup. Then $MH = G$, and by (3.5) $\text{Int}(H, G) \cong \text{Int}_H(H \cap M, M)$. This lattice is not modular, hence M cannot be abelian. If M^* is another minimal normal subgroup, then M and M^* elementwise commute, so any H -invariant subgroup of M is also M^*H -invariant, i.e., normal in G (since $M^*H = G$). Then the minimality of M gives that $\text{Int}_H(H \cap M, M)$ can have at most two elements, although we have

Proof of the Theorem. We have to prove that if both (1) and (2) fail, then there exists a finite lattice that cannot be represented as an interval in the subgroup lattice of a finite group. Let L_1 be a lattice that is not isomorphic to $\text{Int}(H, G)$ in the subgroup lattice of any finite almost simple group G with a core-free subgroup H , and let L_2 be a lattice that is not isomorphic to $\text{Int}(D, G)$ in the subgroup lattice of any twisted wreath product $G = \text{Twr}(T, D, D_0, \varphi)$ of a non-abelian finite simple group T and a finite group D with respect to a subgroup $D_0 < D$ and a homomorphism $\varphi : D_0 \rightarrow \text{Aut}(T)$ satisfying $\varphi(D_0) \geq \text{Inn}(T)$. Let us embed L_1 as a filter into a finite lattice \hat{L}_1 that is generated by its coatoms (see **(3.3)**). Now let L be the following lattice assembled together using the parts \hat{L}_1 , L_2 , their duals \hat{L}_1^d , L_2^d , and two hexagons:



Clearly, H is a core-free subgroup in G (cf. **(3.4)**). Let M be a minimal normal subgroup of G . Since the interval $\text{Int}(H, G)$ is a strongly non-modular lattice, **(3.6)** implies that $MH = G$, M is the unique minimal normal subgroup in G , and $M = T_1 \times \cdots \times T_k$ ($k \geq 1$) with pairwise isomorphic non-abelian simple groups T_1, \dots, T_k . Now H permutes the direct factors of M transitively. Let $H_1 = \mathbf{N}_H(T_1)$ and let $\alpha : H_1 \rightarrow \text{Aut}(T_1)$ give the automorphisms of T_1 induced by conjugation by elements of H_1 . Furthermore, take coset representatives $x_1 = 1, x_2, \dots, x_k$ of H_1 in H so that $x_i^{-1}T_1x_i = T_i$ ($i = 1, \dots, k$). For each i fix the isomorphism $t \mapsto x_i^{-1}tx_i$ ($t \in T_1$) between T_1 and T_i , so that M becomes T^k (where $T = T_1$). As in Section 2 we may identify M with $\text{Sdp}(H_1, \alpha) \leq T^{|H|}$. Now $\text{Int}(H, G) \cong \text{Int}_H(H \cap M, M)$ (see **(3.5)**).

Step 1. We show that $H \cap M = 1$.

Following (2.8) we distinguish four cases for the subgroup $H \cap M$:

1. $H \cap M$ is a subdirect product in $M = T^k$;
2. $H \cap M$ is a box, i.e., $H \cap M = U^k$ for some H_1 -invariant subgroup $1 < U < T$;
3. $H \cap M$ is a skew subgroup, i.e., a nontrivial subgroup properly contained in a box;
4. $H \cap M = 1$.

If $H \cap M$ is a subdirect product, then every subgroup containing it is also a subdirect product. In virtue of (2.2) let $H \cap M$ be determined by a subgroup $H_0 \leq H$ containing H_1 and a homomorphism $\beta : H_0 \rightarrow \text{Aut}(T)$ extending α . Then (2.5) implies that $\text{Int}_H(H \cap M, M)$ is isomorphic to the dual of the interval $\text{Int}(H_1, H_0)$, so this latter lattice is also isomorphic to L , contrary the minimal choice of G .

If $H \cap M = U^k$ is a box, then it follows from (2.9) and (2.12) that all maximal invariant subgroups in $\text{Int}_H(H \cap M, M)$, i.e., all coatoms are also boxes. Since \hat{L}_1 is generated by coatoms, we obtain that all invariant subgroups in this subinterval are boxes, hence $\hat{L}_1 \cong \text{Int}_{H_1}(V, T)$ for an appropriate H_1 -invariant subgroup V . As L_1 is a filter in \hat{L}_1 , we obtain that L_1 occurs as an interval in the subgroup lattice of the almost simple group $TH_1/\mathbf{C}_{TH_1}(T)$, contrary to our assumption on L_1 .

If $H \cap M$ is a skew subgroup, then let U^k be the box containing it, i.e., the direct product of the images of the projections of $H \cap M$. Then $H \cap M < U^k < M = T^k$, and at least one of the hexagons in $\text{Int}_H(H \cap M, M)$, say, $\text{Int}(X, Z)$ (where $X < Z \in \text{Int}_H(H \cap M, M)$) does not contain U^k . The top element of the hexagon, Z is a maximal H -invariant subgroup in T^k , hence it is either a subdirect product or a box (see (2.9)). If it were a box, then $Z \cap U^k = H \cap M$ would be a box as well, which is not the case. So Z is a subdirect product, and then (2.12) implies that X is also a subdirect product. In this case, however, $Z \cap U^k = X \cap U^k = H \cap M \neq 1$ cannot hold by (2.13).

We conclude that $H \cap M = 1$, so $G = MH$ is a semidirect product, in fact, it is the twisted wreath product $\text{Twr}(T, H, H_1, \alpha)$.

Step 2. We exclude the possibility that the top elements of both hexagons in $\text{Int}_H(1, M)$ are boxes.

Assume that these coatoms of L correspond to the subgroups U_1^k and U_2^k . Then U_1 and U_2 are maximal H_1 -invariant subgroups of T . Since the normalizer of a H_1 -invariant subgroup is also H_1 -invariant and T is a simple group, it follows that U_1 and U_2 are self-normalizing. As $U_1^k \cap U_2^k = H \cap M = 1$, we get $\mathbf{N}_T(U_1) \cap \mathbf{N}_T(U_2) = 1$. Therefore, if $h \in H_1$ induces an inner automorphism on T , then it is the trivial automorphism, that is, $\alpha(H_1) \cap \text{Inn}(T) = 1$.

We distinguish two cases, whether $\alpha(H_1) = 1$ or not, and show that in both cases at least one of the subgroups U_1, U_2 is a p -group for some prime number p . If $\alpha(H_1) = 1$, then all subgroups of T are H_1 -invariant. Now U_1 is a p -group, otherwise two Sylow subgroups corresponding to different prime divisors of $|U_1|$ would provide two H -invariant subgroups of U_1^k with trivial intersection, which is

not the case as U_1^k is the top element of one of the hexagons in L . Now consider the case when $\alpha(H_1)$ is non-trivial. We have seen that $\alpha(H_1) \cap \text{Inn}(T) = 1$, hence $\alpha(H_1)$ is isomorphic to a subgroup of $\text{Out}(T)$, which is a solvable group according to Schreier's Hypothesis. Let A be a minimal normal subgroup in $\alpha(H_1) \leq \text{Aut}(T)$. This is an elementary abelian q -group for some prime q . Now $\mathbf{C}_T(A) < T$ is a proper H_1 -invariant subgroup. (It may be the trivial subgroup.) We can choose one of $i \in \{1, 2\}$ so that the H -invariant box subgroup $\mathbf{C}_T(A)^k$ intersects U_i^k trivially. Then we have $\mathbf{C}_T(A) \cap U_i = 1$, that is, A acts fixed point freely on U_i . It follows that q does not divide $|U_i|$, and for every prime divisor p of $|U_i|$ there is a unique A -invariant Sylow p -subgroup of T (see [1, 18.7]). The uniqueness implies that this Sylow subgroup is also H_1 -invariant. It follows, as before, that U_i is a p -group.

We have shown that in both cases U_i ($i = 1$ or 2) is a p -group. Observe that the Frattini subgroup $\Phi(U_i^k)$ is also H -invariant, as it is a characteristic subgroup of U_i^k . Since $U_i^k/\Phi(U_i^k)$ is abelian, the interval $\text{Int}_H(\Phi(U_i^k), U_i^k)$ is a modular lattice consisting of at least two elements. Moreover, by the basic property of the Frattini subgroup, if $W \vee \Phi(U_i^k) = U_i^k$ for some subgroup W , then $W = U_i^k$. However, there is no element in the lattice L that has these properties required from $\Phi(U_i^k)$. This contradiction shows that it is not possible that the top elements of both hexagons are boxes.

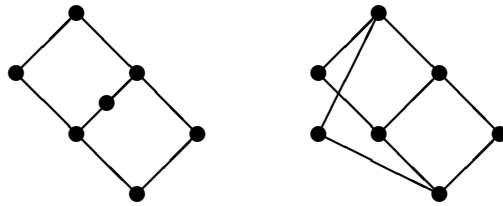
Step 3. Conclusion of the proof.

Now we may assume that the top element of one of the hexagons in $\text{Int}_H(1, M) \cong L$ is a subdirect product. Let this hexagon be $\text{Int}(X, Z)$. By (2.12) we see that X is also a subdirect product. Hence (2.2) and (2.3) yield that $X = \text{Sdp}(H_X, \beta)$ for a subgroup $H_X \leq H$ and a homomorphism $\beta : H_X \rightarrow \text{Aut}(T)$, and the subgroups of M containing X are exactly the subdirect products $\text{Sdp}(W, \beta|_W)$ corresponding to subgroups $W \in \text{Int}(H_1, H_X)$. (In particular, we have $\alpha = \beta|_{H_1}$.) Let $Z = \text{Sdp}(H_Z, \beta|_{H_Z}) \cong T^{|H:H_Z|}$. Since $\text{Int}_H(1, Z)$ does not contain any box or skew subgroup, we must have $\beta(H_Z) \geq \text{Inn}(T)$ (see (2.11)). Let K be the kernel of β . Since $\text{Int}(H_Z, H_X)$ is a hexagon, and that is a strongly non-modular lattice, it follows from (3.6) that either $K \leq H_Z$ or $KH_Z = H_X$. In the first case $\text{Inn}(T) \leq \beta(H_Z) \cong H_Z/K < H_X/K \cong \beta(H_X) \leq \text{Aut}(T)$, so the hexagon $\text{Int}(H_Z, H_X) \cong \text{Int}(\beta(H_Z)/\text{Inn}(T), \beta(H_X)/\text{Inn}(T))$ is an interval in the subgroup lattice of the outer automorphism group of T . However, the outer automorphism group is solvable by Schreier's Hypothesis, but the subgroup lattice of a solvable group cannot contain a hexagon as an interval by (3.6). Thus we have $KH_Z = H_X$, and so $KH_1 = H_X$ as well, thus $\beta(H_1) = \beta(H_Z) \geq \text{Inn}(T)$. By (2.4) this means that with the exception of the trivial subgroup, $\text{Int}_H(1, M)$ consists of subdirect products only. In particular, all elements apart from the trivial subgroup in the interval corresponding to L_2 are subdirect products, hence L_2 occurs as an interval in the subgroup lattice of a twisted wreath product as in (2), contrary to our assumption. This finishes the proof of Theorem 3.2.

4 Comments on related literature

The definition of the twisted wreath product is given in the books by Bertram Huppert [13] and by Michio Suzuki [22]. Huppert [13, Definition I.15.10] gives the same definition as in the original paper of B. H. Neumann [17], and uses the German name *verschränktes Kranzprodukt*. Suzuki's definition [22, Chapter 2, Definition 10.3] is essentially the same as we have formulated it in Section 2. It is worth mentioning the analogy between twisted wreath products and induced representations, noticed by Dan Haran [12, Section 1].

The problem of representing finite lattices as intervals in subgroup lattices has raised considerable interest. For a survey see [18]. Although a negative answer is expected, only some deep reduction theorems and solutions for particular classes of lattices have been achieved so far. Recently Michael Aschbacher devoted several voluminous works to this problem. Here we can mention only two of these: one dealing with overgroups of root subgroups in classical groups [4], another investigating intervals in the subgroup lattice of alternating and symmetric groups [3]. John Shareshian [21] suggested some candidates for lattices that may not be representable as intervals in subgroup lattices of finite groups. William DeMeo [10] found representations of all lattices consisting of at most 7 elements, with two exceptions:



So currently these are the smallest lattices for which no representation as an interval in the subgroup lattice of a finite group is known.

In the present paper our goal was to combine the ideas of Aschbacher, Baddeley, Börner, and Lucchini ([2], [6], [7], [8]) in order to give an accessible proof of the reduction theorem (3.2). The statement of the theorem slightly differs from Börner's version. On one hand, we have improved case (1) by stating it for all lattices not just for those generated by coatoms. This was made possible by the embedding lemma (3.3). (Börner [8, Lemma 1.1] found a less useful embedding.) On the other hand, our version of (2) is slightly weaker than his, we do not get that D_0 is a core-free subgroup of D . That was achieved by Börner with the help of a more complex lattice than our L and using some additional arguments. The main ideas are also present in the other papers. However, Baddeley [6] gives a reduction that can somewhat alter the lattice to be represented, and Aschbacher [2] formulates the alternative only for a special class of lattices what he calls CD-lattices. Baddeley and Lucchini [7] study lattices of height 2.

Finally, let us point out some substantial parts from these papers that were used in our presentation.

- **(2.5)** and **(2.6)**: [2, (7.1)]. On the basis of this remark Aschbacher concentrates on the kernels of the extensions of the homomorphism $\varphi : D_0 \rightarrow \text{Aut}(T)$. These are subgroups normalized by D_0 and together with an additional top element form what he calls (the dual of) a **signalizer lattice**.
- **(2.7)**: Aschbacher's example for the hexagon [2, Example 8.5] is slightly different, he takes $T = A_5$, $D = A_6 \times A_6$, $D_0 = \text{diag}(A_5)$.
- **(2.13)**: [8, Lemma 4.10]
- **(3.6)**: Baddeley [6, Definition 3.3] uses a weaker condition, what he calls QP-property. In contrast, Börner's LP-property [8, Definition 2.1] is stronger than ours, making the unnecessary requirement that $y \wedge x = 0_L$ and $y \vee x = 1_L$. Aschbacher's A-lattices [2, p. 810] are exactly those what we call strongly non-modular. The conclusion of **(3.6)** is obtained in each of these papers, and it is emphasized that this corresponds to the notion of **quasiprimitive permutation groups**.
- Step 1 in the proof of **(3.2)**: [8, Theorems 5.1, 5.2, 5.3]
- Step 2 in the proof of **(3.2)**: [8, Theorem 5.6], [6, part of Theorem 4.9]
- Step 3 in the proof of **(3.2)**: [8, Lemma 5.4]

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