Decomposition of a graph into two disjoint odd subgraphs

Gyula Y. Katona*

Department of Computer Science
and Information Theory
Budapest University of Technology
and Economics, Hungary
and
MTA-ELTE Numerical Analysis and
Large Networks Research Group, Hungary
kiskat@cs.bme.hu

Mikio Kano†

Ibaraki University
Hitachi, Ibaraki, Japan
mikio.kano.math@vc.ibaraki.ac.jp

Kitti Varga‡

Alfred Rényi Institute of Mathematics
Hungarian Academy of Sciences, Hungary
vkitti@cs.bme.hu

Abstract: An odd (resp. even) subgraph in a multigraph is its subgraph in which every vertex has odd (resp. even) degree. We say that a multigraph can be decomposed into two odd subgraphs if its edge set can be partitioned into two sets so that both form odd subgraphs. In this paper we give a necessary and sufficient condition for the decomposability of a multigraph into two odd subgraphs. We also present a polynomial time algorithm for finding such a decomposition or showing its non-existence. We also deal with the case of the decomposability into an even subgraph and an odd subgraph.

(The autors started to work on this problem on the 10th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications in Budapest, 2017)

Keywords: decomposability, edge decomposition, odd subgraphs, odd factor

1   Introduction

In this paper we mainly consider *multigraphs*, which may have multiple edges but have no loops. A graph without multiple edges or loops is called a *simple graph*. Let G be a multigraph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is called its *order* and denoted by $|G|$, and the number of edges in G is called its *size* and denoted by $e(G)$. Let $\mathrm{EvenV}(G)$ denote the set of vertices of even degree and $\mathrm{OddV}(G)$ denote the set of vertices of odd degree. For a vertex set U of G, the subgraph of G induced by U is denoted by $\langle U \rangle_G$. For two disjoint vertex sets U_1 and U_2 of G, the number of edges between U_1 and U_2 is denoted by $e_G(U_1, U_2)$. For a vertex v of G, the degree of v in G is denoted by $\deg_G(v)$. Moreover, when some edges of G are colored with red and blue, for a vertex v, the number of red edges incident with v is denoted by $\deg_{\mathrm{red}}(v)$, and the number of red edges in G is denoted by $e_{\mathrm{red}}(G)$. Analogously, $\deg_{\mathrm{blue}}(v)$ and $e_{\mathrm{blue}}(G)$ are defined.

An *odd (resp. even) subgraph* of G is a subgraph in which every vertex has odd (resp. even) degree. An *odd factor* of G is a spanning odd subgraph of G. It is obvious by the handshaking lemma that every connected multigraph containing an odd factor has even order. This condition is also sufficient as follows.

Proposition 1 (Problem 42 **of §7 in [2])** *A multigraph G has an odd factor if and only if every component of G has even order.*

Moreover, such an odd factor, if it exists, can be found in polynomial time (Problem 42 of §7 in [2]). Consider a connected multigraph of even order on the vertices $v_1, \ldots, v_{2m}$ and for any $i \in \{1, 2, \ldots, m\}$, fix a path P_i connecting v_i and v_{i+m}. Then the edges appearing odd number of times in the paths $P_1, \ldots, P_m$ form an odd factor. The above proposition also follows from the fact that for a tree T of even order, the set

$$\{e \in E(T) : T - e \text{ consists of two odd components}\}$$

forms an odd factor of T.

We say that G can be *decomposed into n odd subgraphs* if its edge set can be partitioned into n (possibly empty) sets $E_1, \ldots, E_n$ so that for every $i \in \{1, \ldots, n\}$, E_i forms an odd subgraph. Some authors say that in this case G is odd n-edge-colorable.

Our main result gives a criterion for a multigraph to be decomposed into two odd subgraphs, and proposes a polynomial time algorithm for finding such a decomposition or showing its non-existence.

We begin with some known results related to ours.

Theorem 2 ([6]) *Every simple graph can be decomposed into four odd subgraphs.*

This upper bound is sharp, for example, the wheel of four spokes (W_4, see Figure 1) cannot be decomposed into three odd subgraphs. In [4] Mátrai constructed an infinite family of graphs with the same property.

Theorem 3 ([6]) *Every forest can be decomposed into two odd subgraphs.*

Theorem 4 ([6]) *Every connected multigraph of even order can be decomposed into three odd subgraphs.*

Since every connected multigraph G of even order has an odd factor, if we take an odd factor F with maximum size, then $G - E(F)$ becomes a forest, and it can be decomposed into two odd subgraphs by Theorem 3. Thus Theorem 4 follows.

Theorem 5 ([3]) *Every connected multigraph can be decomposed into six odd subgraphs. Moreover, with the exception of the Shannon triangles of type $(2, 2, 2)$ (see Figure 1), a decomposition into five odd subgraphs always exists.*

The following theorem strengthens both Theorems 2 and 5.

Theorem 6 ([5]) *Every connected multigraph can be decomposed into four odd subgraphs except for the Shannon triangles of types $(2, 2, 2)$ and $(2, 2, 1)$ (see Figure 1).*

In this paper we study the decomposability of a multigraph into an even subgraph and an odd subgraph, and into two odd subgraphs. We also remark that the case of decomposing into two even subgraphs is trivial.

Proposition 7 *A multigraph G can be decomposed into an even subgraph and an odd subgraph if and only if every component of $\langle \mathrm{OddV}(G) \rangle_G$ has even order.*

PROOF: Such a decomposition exists if and only if there is an odd factor in $\langle \mathrm{OddV}(G) \rangle_G$, since all edges incident with any vertex of even degree must belong to the even subgraph. So by Proposition 1, the proposition follows. $\square$

Since an odd factor can be found in polynomial time, we can conclude the following.
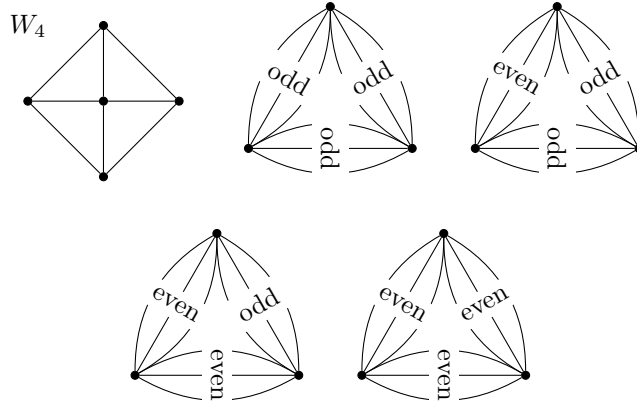
Figure 1: The wheel W_4 and the Shannon triangles of type (1,1,1), (2,1,1), (2,2,1), (2,2,2).

Corollary 8 *There is a polynomial time algorithm for decomposing a multigraph into an odd subgraph and an even subgraph or showing the non-existence of such a decomposition.*

Remark 9 *The case of the decomposability into two even subgraphs is trivial: a multigraph can be decomposed into two even subgraphs if and only if every vertex of the multigraph has even degree.*

The following two theorems are our main results.

Theorem 10 *Let G be a multigraph and let $\mathcal{X}$ denote the set of components of $\langle \mathrm{OddV}(G) \rangle_G$, and let $\mathcal{Y}$ and $\mathcal{Z}$ denote the sets of components of $\langle \mathrm{EvenV}(G) \rangle_G$ with odd order and even order, respectively. Now G can be decomposed into two odd subgraphs if and only if for every $\mathcal{S} \subseteq \mathcal{Y} \cup \mathcal{Z}$ with $|\mathcal{S} \cap \mathcal{Y}|$ odd, there exists a component $X \in \mathcal{X}$ such that $e_G(X, S)$ is odd, where S is the set of vertices that belong to the components in $\mathcal{S}$.*

Theorem 11 *There is a polynomial time algorithm for decomposing a multigraph into two odd subgraphs or showing the non-existence of such a decomposition.*

2 Proofs of Theorems 10 and 11

We begin with a definition and a proposition on it.

Definition 12 *Let G be a multigraph and $T \subseteq V(G)$. A subgraph J of G is called a T-join if $\mathrm{OddV}(J) = T$.*

Proposition 13 ([1]) *Let G be a multigraph and $T \subseteq V(G)$. There exists a T-join in G if and only if every component of G contains an even number of vertices of T.*

The following theorem gives another necessary and sufficient condition for a multigraph to be decomposed into two odd subgraphs.

Theorem 14 *Let G be a multigraph and $\mathcal{Y}$ and $\mathcal{Z}$ denote the sets of components of $\langle \mathrm{EvenV}(G) \rangle_G$ with odd order and even order, respectively. Then G can be decomposed into two odd subgraphs if and only if there exists a partition $\mathcal{R} \cup \mathcal{B}$ of the components of $\langle \mathrm{OddV}(G) \rangle_G$ such that*

(i) $e_G(R, Y)$ and $e_G(B, Y)$ are both odd for every $Y \in \mathcal{Y}$, and

(ii) $e_G(R, Z)$ and $e_G(B, Z)$ are both even for every $Z \in \mathcal{Z}$,
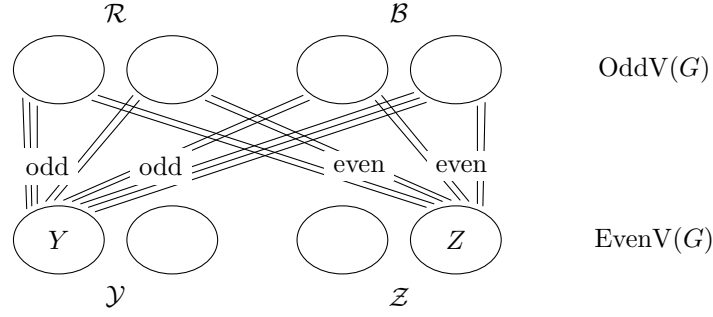
Figure 2: The structure of the multigraph.

where R and B are the sets of vertices that belong to the components in $\mathcal{R}$ and $\mathcal{B}$, respectively. (See Figure 2.)

PROOF: Suppose that G can be decomposed into two odd subgraphs, and color the edges of one with red and with blue the other. Obviously, if a vertex of G has odd degree, then all edges incident with it must have the same color. Consider an arbitrary component X of $\langle \text{OddV}(G) \rangle_G$. Then all edges that have at least one endpoint in X have the same color. Let $\mathcal{R}$ and $\mathcal{B}$ denote the sets of those components of $\langle \text{OddV}(G) \rangle_G$ in which the edges are red and blue, respectively. Let $Y \in \mathcal{Y}$. Then

$$\sum_{v \in Y} \deg_{\text{red}}(v) = 2e_{\text{red}}(\langle Y \rangle_G) + e_G(R, Y).$$

Since $|Y|$ is odd and $\deg_{\text{red}}(v)$ is odd for every $v \in Y$, the left side of the equation is odd, and so $e_G(R, Y)$ must be odd. Similarly, $e_G(B, Y)$ is also odd, and $e_G(R, Z)$ and $e_G(B, Z)$ are both even. Therefore, conditions (i) and (ii) hold.

Next assume that there exists a partition $\mathcal{R} \cup \mathcal{B}$ satisfying (i) and (ii). Then color all the edges incident with any vertex of R red and all the edges incident with any vertex of B blue. Note that no edge of $\langle \text{EvenV}(G) \rangle_G$ is colored yet, and there exist no edges between R and B. Let $T \subseteq \text{EvenV}(G)$ be the set of vertices having even red-degree at this stage.

Now we show that every component of $\langle \text{EvenV}(G) \rangle_G$ contains an even number of vertices of T. Let $Y \in \mathcal{Y}$. Then by condition (i),

$$\sum_{v \in Y} \deg_{\text{red}}(v) = \underbrace{\sum_{v \in Y \cap T} \underbrace{\deg_{\text{red}}(v)}_{\text{even}}}_{\text{even}} + \sum_{v \in Y \setminus T} \underbrace{\deg_{\text{red}}(v)}_{\text{odd}} = \underbrace{e_G(R, Y)}_{\text{odd}}.$$

Hence $|Y \setminus T|$ is odd, and since $|Y|$ is odd, $|Y \cap T|$ must be even. By the same argument given above, for any $Z \in \mathcal{Z}$ since $|Z|$ is even and by condition (ii) $e_G(R, Z)$ is even, it follows that $|Z \setminus T|$ is even and $|Z \cap T|$ is even. So by Proposition 13, there exists a T-join in $\langle \text{EvenV}(G) \rangle_G$. Color all the edges of this T-join red, and all the remaining edges blue. Now the resulting red subgraph and blue subgraph are odd subgraphs and form a partition of $E(G)$. □

Now we prove Theorem 10.

PROOF: Let $\mathcal{X}$ denote the set of components of $\langle \text{OddV}(G) \rangle_G$, and let $\mathcal{Y}$ and $\mathcal{Z}$ denote the sets of components of $\langle \text{EvenV}(G) \rangle_G$ with odd order and even order, respectively.

Consider the bipartite graph G^*, whose vertices correspond to the elements of $\mathcal{X}$ and $\mathcal{Y} \cup \mathcal{Z}$, and an element of $\mathcal{X}$ and that of $\mathcal{Y} \cup \mathcal{Z}$ are joined by an edge if and only if there are odd number of edges of G

between the corresponding components. Then it is easy to see that every vertex $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$ has even degree in G^*.

Our goal is to give a system of linear equations that is solvable if and only if G is decomposable into two odd subgraphs and its solutions describe partitions satisfying the properties of Theorem 14. For every $X_i \in \mathcal{X}$, we assign a binary variable x_i which decides whether $X_i \in \mathcal{R}$ or not. If $x_i = 1$, then $X_i \in \mathcal{R}$, and if $x_i = 0$, then $X_i \in \mathcal{B}$. Since we want $e_G(R, Y)$ to be odd for every $Y \in \mathcal{Y}$ and $e_G(R, Z)$ to be even for every $Z \in \mathcal{Z}$, consider the following system of linear equations over the binary field $GF(2) = \{0, 1\}$:

$$\sum_{X_i \in N_{G^*}(Y)} x_i = 1 \qquad \text{for all } Y \in \mathcal{Y},$$

$$\sum_{X_i \in N_{G^*}(Z)} x_i = 0 \qquad \text{for all } Z \in \mathcal{Z}.$$

By Theorem 14, the multigraph G is decomposable into two odd subgraphs if and only if this system has a solution. The system is solvable if and only if one of the following three equivalent statements holds.

(i) There is no collection of equations such that the sum of the left-hand sides is 0 and the sum of the right-hand sides is 1 (over the binary field).

(ii) For any subset of the equations if the sum of the right-hand sides is 1, then there exists a variable x_i which appears odd number of times in these equations.

(iii) For any $\mathcal{S} \subseteq \mathcal{Y} \cup \mathcal{Z}$ for which $|\mathcal{S} \cap \mathcal{Y}|$ is odd, there exists $X \in \mathcal{X}$ such that $|N_{G^*}(X) \cap \mathcal{S}|$ is odd.

Note that statement (iii) is a graph presentation of statement (ii). $\square$

Since a system of linear equations over the binary field can be solved in polynomial time, Theorem 11 follows.

However, it is worth translating the algorithm to the language of graphs. The steps of the Gaussian elimination can be followed in the auxiliary bipartite graph $\widehat{G}^*$ which is a slight modification of the graph G^* used in the proof of Theorem 10, see Figure 3 for an example. In the following, we use $*$ as an operation that contracts components into single vertices and takes the number of resulting multiple edges modulo 2. So the color classes of G^* are the vertex sets $\mathcal{X}^*$ and $\mathcal{Y}^* \cup \mathcal{Z}^*$, and our goal is to partition $\mathcal{X}^*$ into $\mathcal{R}^*$ and $\mathcal{B}^*$. To obtain $\widehat{G}^*$ a new vertex b is added to G^* and is connected to all vertices in $\mathcal{Y}^*$. This vertex b corresponds to the constant 1 on the right sides of the linear equations.

To start the Gaussian elimination we need to select a variable that has a non-zero coefficient (i.e. 1) in at least two equations and pick one of these equations. Therefore in $\widehat{G}^*$ we choose an edge $x_i w$ with $|N_{\widehat{G}^*}(x_i)| \geq 2$, $x_i \in \mathcal{X}^*$ and $w \in \mathcal{Y}^* \cup \mathcal{Z}^*$. Now in the Gaussian elimination, we add the equation corresponding to w to all the equations corresponding to any element of $N_{\widehat{G}^*}(x_i) - \{w\}$ to make the coefficient of x_i zero in these equations. Then the resulting system of linear equations corresponds to the bipartite graph $\widehat{G}_1^*$ that is obtained from $\widehat{G}^*$ by replacing the induced subgraph $\langle N_{\widehat{G}^*}(w) \cup (N_{\widehat{G}^*}(x_i) - \{w\}) \rangle_{\widehat{G}^*}$ with its bipartite complement: $x' \in N_{\widehat{G}^*}(w)$ and $w' \in N_{\widehat{G}^*}(x_i) - \{w\}$ are adjacent in $\widehat{G}_1^*$ if and only if x' and w' are not adjacent in $\widehat{G}^*$. The other edges are not changed. Notice that the degree of x_i in $\widehat{G}_1^*$ is one.

Next we repeat this procedure by choosing another edge $x_j w'$ in $\widehat{G}_1^*$ that satisfies the same conditions. Since the degree of x_i is already one, x_j will automatically differ from the previously chosen vertices, but we also choose w' to be different from all previously chosen vertices. If there are no more such edges then the procedure stops.

Consider the graph of the final stage. At this point we can obtain the desired partition of the edge set into two odd subgraphs or show the non-existence of such a partition as follows.

- If a vertex $w \in \mathcal{Y}^* \cup \mathcal{Z}^*$ is adjacent only to the vertex b, then the graph G cannot be decomposed into two odd subgraphs, since this means that adding up some equations results in 0 on the left-hand side and 1 on the right-hand side.

So we may assume that no vertex $w \in \mathcal{Y}^* \cup \mathcal{Z}^*$ is adjacent only to the vertex b. In this case we obtain a solution as follows.

- If a vertex $x_i \in \mathcal{X}^*$ has degree at least two, then let $x_i \in \mathcal{B}^*$ and remove all the edges incident with x_i. This means that the variable x_i is a free variable, so it can be set to 0. Thus we may assume that every $x \in \mathcal{X}^*$ is adjacent to at most one vertex of $\mathcal{Y}^* \cup \mathcal{Z}^*$. Removing these edges makes x_i an isolated vertex, but note that other vertices in $\mathcal{X}^*$ cannot be isolated.

- If there is a vertex in $\mathcal{Y}^* \cup \mathcal{Z}^*$ that is adjacent to b and has more than one neighbor in $\mathcal{X}^*$ (that are all leaves), then let one of these neighbors be in $\mathcal{R}^*$ and all the others in $\mathcal{B}^*$. This means that we set one variable to 1 and all the others to 0, so their sum is equal to 1.

- Otherwise, if x_i is in the same component as b, then let $x_i \in \mathcal{R}^*$, meaning that x_i was set to 1 in the solution.

- If x_i is not in the component of b, then let $x_i \in \mathcal{B}^*$, meaning that x_i was set to 0 in the solution.

The above graph operation gives us a partition of $\mathcal{X}^*$ into $\mathcal{R}^* \cup \mathcal{B}^*$ and the corresponding partition of $\mathcal{X}$ satisfies the conditions in Theorem 14, and hence G is decomposed into two odd subgraphs.

The example on Figure 3: The steps of the Gaussian elimination in an auxiliary bipartite graph $\widehat{G}^*$.
First, we eliminate the variable x_6 with the help of equation (w_4), i.e. we take the bipartite complement of $\langle x_4, x_6; w_2, w_3 \rangle_{\widehat{G}^*}$ to obtain $\widehat{G}_1^*$.
Second, we eliminate the variable x_4 with the help of equation (w_3'), i.e. we take the bipartite complement of $\langle x_4, x_5; w_4 \rangle_{\widehat{G}_1^*}$ to obtain $\widehat{G}_2^*$.

In the graph $\widehat{G}_2^*$ there is only one vertex belonging to the color class $\mathcal{X}^*$ that has degree at least two, namely x_5, but since both of its neighbors have been already chosen before, the procedure stops.
Since in $\widehat{G}_2^*$ no vertex is adjacent only to b, there exists a decomposition of G into two odd subgraphs. Since the vertex x_5 is a free variable, i.e. it has degree (at least) two after the procedure stopped, we can set it to zero, i.e. we can put it into $\mathcal{B}^*$ (here in the figure we indicate this by coloring it blue) and remove all the edges incident with it.
In the so obtained graph the vertex w_1 is adjacent to b and has (at least) two neighbors in $\mathcal{X}^*$, so we put one of its neighbors, namely x_1, into $\mathcal{R}^*$ (which we indicate in the figure by coloring it red), and all the others, namely x_2, into $\mathcal{B}^*$. Finally, considering the still "uncolored" vertices of $\mathcal{X}^*$, since x_3 belongs to the same component as b, we put it into $\mathcal{R}^*$, and since x_4 and x_6 do not belong to the component of b, we put them into $\mathcal{B}^*$.
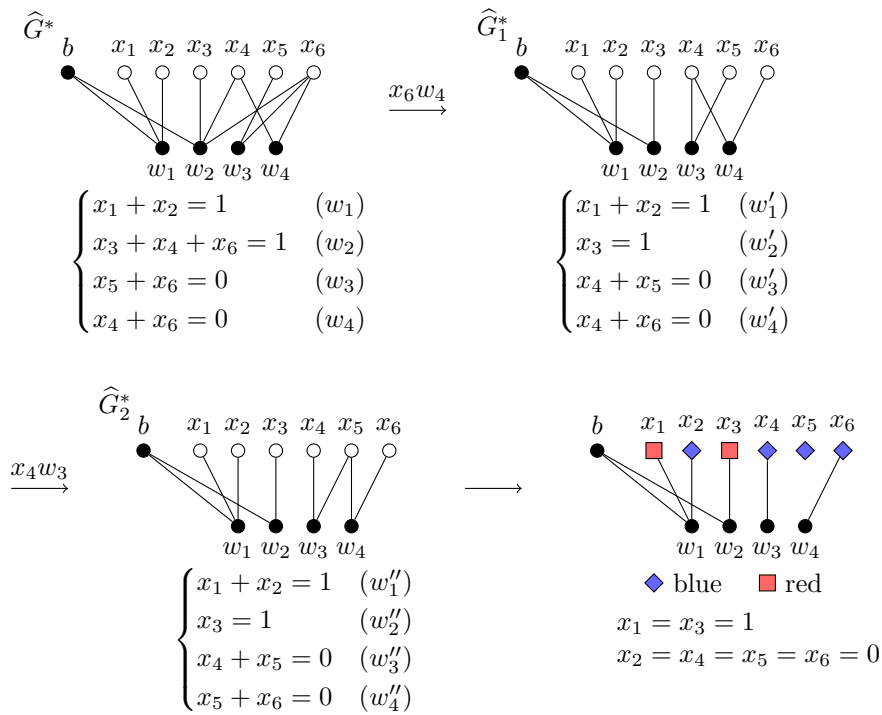
Figure 3: An example for the algorithm.

References

[1] J. Edmonds, E.L. Johnson: Matching, Euler tours and the Chinese postman, Math. Programming, **5(1)**, 88–124 (1973)

[2] L. Lovász: Combinatorial problems and exercises (second edition), North-Holland Publishing Co., Amsterdam, 61–61 (1993)

[3] B. Lužar, M. Petruševski, R. Škrekovski: Odd edge coloring of graphs, Ars Math. Contemporanea, **9**, 277–287 (2015)

[4] T. Mátrai, Covering the edges of a graph by three odd subgraphs, J. Graph Theory, **53**, 75–82 (2006)

[5] M. Petruševski, Odd 4-edge-colorability of graphs, J. Graph Theory, **87**, 460–474 (2018)

[6] L. Pyber, Covering the edges of a graph by..., Sets, Graphs and Numbers, Colloquia Mathematica Societatis János Bolyai, **60**, 583–610 (1991)