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# Organizing Time Banks: Lessons from Matching Markets[*]

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**Abstract**

A time bank is a group of people that set up a common platform to trade services among themselves. There are several well-known problems associated with time banks, e.g., high overhead costs and difficulties to identify feasible trades. This paper constructs a non-manipulable mechanism that selects an individually rational and time-balanced allocation which maximizes exchanges among the members of the time bank (and those allocations are efficient). The mechanism works on a domain of preferences where agents classify services as unacceptable and acceptable (and for those services agents have specific upper quotas representing their maximum needs).

*Keywords*: market design; time banking; priority mechanism; non-manipulability.
*JEL Classification*: D82; D47.

## 1 Introduction

Time banks have now been established in at least 34 countries. In the United Kingdom, for example, there are more than 300 time banks, and time banks are operating in at least 40 states in the United States (Cahn, 2011).[1] A time bank is a group of individuals and/or organizations in a local community that set up a common platform to trade services among themselves. Members of a time bank earn time credit for each time unit they supply to members of the bank and the earned credit can be spent to receive services from other members of the bank. Very few time banks are not based on a "one-for-one" time system, meaning that members of the time bank need not get one unit of time back for each unit of time they supply (see also Footnote 7 or Croall, 1997). Therefore, we consider the most commonly used "one-for-one" time banks. For example, a gardener who supplies two hours of time

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[1]A more thorough description of timebanking and time banks will be provided in Section 2.

may, for example, receive two hours of child care in return for his gardening services. Ozanne (2010) reported that the most commonly exchanged services included gardening, giving lifts, befriending, do-it-yourself jobs, dog walking, and computer training. Even if time banks traditionally have had a very simple organization, most of the nowadays existing time banks take advantage of computer databases for record keeping and a broker (physical coordinator) that keeps track of transactions and match requests for services with those who can provide them (Seyfang, 2003, 2004; Williams, 2004).

A critical factor for a time bank to function smoothly is the coordination device that matches requests for services with those who can provide them. Our basic observation is that this type of service exchange shares many features with some classical markets previously considered in the matching literature, including, e.g., housing markets (Scarf and Shapley, 1974; Abdulkadiroğlu and Sönmez, 1999; Aziz, 2016b), organ markets (Roth et al., 2004; Biró et al., 2009; Ergin et al., 2017), one-to-one matching problems (Gale and Shapley, 1962), and markets for school seats (Abdulkadiroğlu and Sönmez, 2003; Kesten and Ünver, 2015). In particular, if a time bank is organized as a matching market, the time bank will have a structure of what in the matching literature is known as a many-to-many matching market. This follows since any member of a time bank can trade services with any other member of the very same time bank and there are no obstacles that prevent a member of a time bank to supply and receive multiple services from members of the very same time bank. Such matching markets have previously been considered by, e.g., Echenique and Oviedo (2006), Konishi and Ünver (2006), and Hatfield and Kominers (2016).

The above mentioned classical matching markets are centralized as the agents in the system (e.g., tenants, patients, or students) report their preferences over the items to be allocated (e.g., houses, organs, or school seats) to a clearing house and a mechanical procedure (or mechanism) determines the final allocation based on the reported preferences and a set of predetermined axioms. As will be described in more detail in Section 2, even if time banks often take advantage of computer databases, there is no mechanical procedure that determines the trade of services among the members in the bank based on reported preferences, and it is exactly in this respect that time banks can learn from insights in classical matching markets.

By organizing a time bank as a matching market, it is possible to solve a number of problems which have been associated with time banks. For example, time banks typically encounter long run organizational sustainability problems since they experience high overhead costs, e.g., as staff is needed to keep the organization running and, in particular, to help out in the coordination process (Seyfang, 2004). Moreover, independently of if possible exchanges are identified manually by a broker or if members propose exchanges through an internet-based software, it is challenging to identify and coordinate longer trading cycles.[2] Time bank members sometimes experience that time credits are comparatively easy to earn but harder to spend (Ozanne, 2010), i.e., the reverse situation compared to conventional money which generally is hard to earn, but easy to spend. The consequence of the latter problem is that potential members never join time banks simply because there is a risk that they provide more time than they get back.

A computer-based clearing house, e.g., an internet-based interface for reporting needs and requests together with an algorithm for matching needs and requests, on the other hand, can help in reducing

[2] In fact, some members have experienced difficulties to understand the mechanism for making trades (Ozanne, 2010; Seyfang, 2004) and, consequently, the role of the broker is important even when an internet-based software is used.

costs related to coordination. Such an algorithm can also identify and coordinate longer trading cycles which in turn will lead to a maximal number of exchanged time units and, consequently, to an efficient outcome. In addition, problems related to participation concerns can be solved by restricting the algorithm to only propose individually rational time-balanced exchanges, i.e., exchanges where all members of the bank only receive services they have requested and get exactly as much time back as they supply to the bank.

The above discussions and observations also motivate the interest in (time) allocations that are individually rational, maximal, and time-balanced. A first observation is that such allocations always exist on the general preference domain. This follows since the allocation in which all agents receive their initial time endowments is individually rational and satisfies time-balance. The conclusion then follows directly from the observation that the number of individually rational allocations that satisfy time-balance is finite and, consequently, that there exists an allocation among those which maximizes trade in the time bank.

However, even if an allocation satisfying these specific properties can be identified, two new problems arise. First, it is often natural to require that the algorithm should be designed in such fashion that it is in the best interest for all agents to report their preferences truthfully (non-manipulability). This property is incompatible with individual rationality, efficiency and time-balance on a general preference domain (Sönmez, 1999, Corollary 1).[3] Second, because members of a time bank can exchange multiple time units, it is not clear that it is easy for members to generally rank any two "consumption bundles". For example, is two hours of hairdressing, two hours of gardening and one hour of babysitting strictly better, equally good, or less preferred to one hour of hairdressing, one hour of gardening and three hours of housekeeping? Hence, it may be an obstacle for members to report their preferences if multiple time units are on stake and if multiple agents are allowed to be involved in a longer trade.

We show that if agents' preferences satisfy certain conditions, the above two problems are no longer present. In some settings, the considered preference domain is clearly unrealistic (e.g., in the school choice problem by Abdulkadiroğlu and Sönmez, 2003). In the case of timebanking, however, they provide a reasonable approximation as will be explained below. The considered restricted domain is an extension of the dichotomous domain popularized by Bogomolnaia and Moulin (2004).[4] In the considered domain, individual preferences are completely described by (i) partitioning the members of the bank (or, equivalently, the services that the members provide) into two disjoint subsets containing acceptable and unacceptable members, and by (ii) specifying a member specific upper time bound for each acceptable member. The former condition reflects that an agent is not necessarily interested in all services provided in the bank (an agent's "horizontal" preference) whereas the latter condition captures the idea that an agent may, for example, be interested in at most one haircut but can accept up to 10 hours of babysitting (an agent's "vertical" preference). One advantage of adopting this preference domain is that it facilitates for agents to report their preferences as they just need to report all unacceptable members and all acceptable members with their upper bounds in contrast to

[3]This impossibility should come as no surprise given the results in, e.g., Hurwicz (1972), Green and Laffont (1979), Roth (1982), Alcalde and Barberà (1994), Barberà and Jackson (1995), or Schummer (1999).

[4]In fact, Bogomolnaia and Moulin (2004) and a series of subsequent papers, argue that it is natural to consider a dichotomous domain in problems involving "time sharing".

reporting a ranking over all possible bundles.[5] Agents then strictly prefer receiving more time units from acceptable services to receiving fewer time units from acceptable services (without exceeding upper bounds or receiving unacceptable services). In this sense, an agent may have many different indifference classes and preferences are dichotomous over single services and polychotomous over bundles of services.

We define and apply a priority mechanism to solve the problem of exchanging time units between members in a time bank. It is demonstrated that the priority mechanism can be formulated as a min-cost flow problem (Proposition 1). Consequently, it is not only possible to identify time-balanced trades, it is also computationally feasible. The definition of the priority mechanism is flexible as it can be adopted on the restricted preference domain or the general domain. Our main result shows that the priority mechanism is non-manipulable on the restricted preference domain and that it always makes a selection from the set of individually rational, maximal, and time-balanced allocations (Theorem 1). To prove this result, a number of novel graph theoretical techniques are needed. In particular, Appendix B demonstrates an equivalence result between the min-cost flow problem and a circulation-based maximization problem.[6] Using graph theoretical tools and in particular min-cost/max-weight formulations to solve matching problems is common in the literature. In the house allocation problem with dichoutomous preferences Aziz (2016b) formalizes a bipartite graph and solves for a max-weight matching. His graph construction is based on having houses on one side and agents on the other side. This is in contrast to our approach where we make copies of agents. Furthermore, our graph construction and the solution is more intricate since agents' can have more than one object. Because finding a maximal alloaction is more involved in our problem and a potential manipulation is more complex we use the graph theoretical tools in proving non-manipulability of the mechanism. This is not needed in Aziz (2016b) because of the less complex optimization problem.

A variety of real-life problems have previously been considered in the matching literature including the above mentioned house allocation problem, kidney exchange problem and school choice problem. There are, however, several differences between these problems and the time banking problem. For example, in the time banking problem, an agent may receive and supply multiple time units. In the school choice problem and the kidney exchange problem, on the other hand, students are allocated at most one school seat and a patient is involved in at most one kidney exchange, respectively. Furthermore, in many matching problems including, e.g., the school choice problem and the house allocation problem, agents' (reported) preferences are typically strict and indifference relations are consequently not allowed (while the kidney exchange problem is often defined on a dichotomous domain). Generalizations to allow for a weak preference structure have recently been proposed by Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012). However, both these papers only allow agents to trade at most one object.

The papers closest to the model investigated in this paper are Athanassoglou and Sethuraman (2011), Aziz (2016a), Biró et al. (2017) and Manjunath and Westkamp (2018), which we describe below.

[5]The strict preference domain is often considered in the matching literature. However, the dichotomous domain is much smaller in size than the strict preference domain, but is not a subset of the strict domain since indifference relations are allowed in the former but not in the latter domain.

[6]The min-cost flow problem is considered in the main part of the paper since it is more intuitive and, moreover, can be introduced using minimal notation.

Athanassoglou and Sethuraman (2011) and Aziz (2016a) consider a housing market where initial endowments as well as allocations are described by a vector of fractions of the houses in the economy. The fractional setting makes it possible to analyze, e.g., efficiency based on (first-order) stochastic dominance, and it is demonstrated that the efficiency and fairness notions of interest conflict with non-manipulability. Even if a similar impossibility is present in the model considered in this paper, the fractional setting is analyzed using different axioms and mechanisms. In addition, Athanassoglou and Sethuraman (2011) and Aziz (2016a) are unable to find any positive results related to non-manipulability in their considered reduced preference domains.

Biró et al. (2017) consider, as this paper, a model where agents are endowed with multiple units of an indivisible and agent-specific good, and search for balanced allocations. In their reduced preference domain, agents have responsive preferences over consumption bundles. On this reduced domain, they demonstrate that, for general capacity configurations, no mechanism satisfies individual rationality, efficiency, and non-manipulability. Given this negative finding, they characterize the capacity configurations for which individual rationality, efficiency and non-manipulability are compatible. They also demonstrate that for these capacity configurations, their defined Circulation Top Trading Cycle Mechanism is the unique mechanism that satisfies all three properties of interest. Hence, the main difference between this paper and Biró et al. (2017) is that they consider a different preference domain and, consequently, need a different mechanism to escape the impossibility result.

Manjunath and Westkamp (2018) have independently considered a model closely related to the one considered here. In their model, an agent can supply distinct services but at most one time unit of each service (recall that agents in our model supply one service but, possibly, several time units of it). They also require time-balance and consider a preference domain classifying services as unacceptable and acceptable (there is no need to specify upper bounds on services since each service is available in one unit). Given this, Manjunath and Westkamp (2018) define a priority mechanism over the set of individually rational and efficient allocations. The main differences between their work and ours is that (i) they allow agents having distinct services whereas each agent in our model has a specific service that comes in multiple copies, (ii) their priority mechanism chooses from the set of individually rational and efficient allocations whereas ours chooses from the set of individually rational and maximal allocations (and as we show in Example 3, any priority mechanism may choose different allocations in their setting and in ours), and (iii) for the non-manipulability result they use a bipartite graph approach whereby capacities for unacceptable services are reduced one-by-one (following the priority order) whereas we use a direct circulation based graph with upper capacities on edges (where the min-cost flow corresponds to the allocation chosen by the priority mechanism).

The remaining part of the paper is outlined as follows. Section 2 gives a more detailed introduction to timebanking and provides some descriptive statistics of the time banks associated with *TimeBanks USA*. Section 3 introduces the theoretical framework and some basic definitions. The priority mechanism is presented in Section 4. The main results are presented in Section 5. Section 6 discusses the main findings of the paper and some extensions of the considered timebanking model. Section 7 concludes the paper. All proofs are relegated to the Appendix.

## 2 Timebanking

As already explained in the Introduction, a time bank is a group of individuals and/or organizations in a local community that set up a common platform to trade services among themselves. This section gives a more detailed description of the fundamental ideas underlying time banks and how time banks attempt to integrate in a larger society. This section also provides some details of *TimeBanks USA* (the largest time bank operating in the United States).

### 2.1 Time Banks and the Society

Even if concepts closely related to timebanking dates back to the 19th century, timebanking was popularized and pioneered in the 1990s by Edgar Cahn and Martin Simon in the United States and the United Kingdom, respectively. One of the fundamental ideas in timebanking is that one hour of service generates one time credit regardless of the provider or the nature of the service performed. This rule is deeply rooted in the philosophical view that even if services are valued differently, human beings share fundamental equality.[7]

Because time bank members exchange services among themselves in local communities, potential positive external welfare effects of timebanking includes resilient local communities, extended social networks and informal neighbourhood support (this is also part of the core values of timebanking, see footnote 7). Even if there is limited research on timebanking, there exists empirical evidence that time banks indeed help in building strong local networks. For example, in a UK based case study, Boyle et al. (2006) showed that time banks not only help their members to extend their social networks but also that time banks are an effective way of developing reciprocal relationships between members in the bank. In another UK based case study, Seyfang and Smith (2002) demonstrated that time banks are successful in attracting participants both from socially excluded groups (people on benefit programs, from low income households, etc.) and from groups that normally not are involved in traditional volunteering. For example, 16 percent of traditional volunteers have an annual income below $\pounds 10,000$ but the corresponding number for the time bank members in their survey was 58 percent. Furthermore, 40 percent of traditional volunteers are not in formal employment compared to 72 percent of the time bank members.

The above findings show that a majority of the time bank members belong to socially excluded groups and low income communities are also supported in other studies in both the United Kingdom (Seyfang, 2003) and the United States (Collom, 2007).[8] This could explain the existence of time banks in a world where monetary transfers are available. Namely, because most members have small social networks and in many cases also lack both income and employment, timebanking is one way to be included in a social network and to increase welfare. Collom (2007) also finds that the single most important reason for joining a time bank is to expand purchasing power through an alternative

[7] The philosophy of timebanking rests on five core values: assets, redefining work, reciprocity, social networks, and respect. The idea that one hour of service always generates one time credit, the so-called one-for-one time system, is embedded in these core values, see Cahn (2000). It should also be noted that a small minority of all time banks are not based on a one-for-one time system, meaning that members of the time bank need not get one unit of time in return for each unit of time they supply (Croall, 1997). This paper, however, restricts attention to one-for-one time systems.

[8] Most of the studies in the literature, consequently, focuses focus on socially excluded and low income groups. An exception is Ozanne (2010) where it is demonstrated that time banks has provided high benefits in the form of social, human, physical and cultural capital also within affluent groups.

currency. Seyfang (2003) found that persons also join time banks, e.g., to meet other participants, to help other members and to get more involved in the local neighbourhood. Similar motives are recoded by Collom (2007) and Caldwell (2000). Joining a time bank may also have other positive spill-overs. For example, when the concept of time banking was used within health care provisions, it was found that there were tangible benefits, both practical and motivational, to health services users (Boyle and Bird, 2014; Simon, 2003).

### 2.2 Organization and Descriptive Statistics

In most time banks, a broker is employed to manage the bank, maintain the database, record transactions, recruit new members, etc. (Seyfang, 2004; Williams, 2004). A "matching mechanism" helps the broker to coordinate requests for services with those who can provide them. In some time banks, this mechanism is simple and the broker manually matches requests with offers (Seyfang, 2003). The obvious drawback for such a mechanism is the difficulty to identity and coordinate longer trading cycles. Hence, this type of matching device naturally restricts trade to bilateral exchanges. A few large time banks, e.g., *TimeBanks USA* and *Timebanking UK*, have developed their own computer software where members can see what other members offer and keep track of their own activity. As will be described below, the members themselves then make requests and offers through the computer software. Also in this case, however, it is difficult to coordinate longer trading cycles as members only can see their own activity.

An example of computer software for timebanking is *Community Weaver 3* which is the most recent software launched by *TimeBanks USA*. This software allows members to register their talents in 11 different categories including, e.g., education, transportation, business services, recreation, and companionship.[9] Each of these categories also have subcategories. The category "education", for example, contains, e.g., the subcategories advocacy, computers, languages, finances, and tutoring. When a member have registered her talents, she can formally offer her services and start making requests. An offer is essentially a registration on the online platform that enables other members to see and request her talents. If a member approves a request, she receives the agreed amount of time credits and the member that receives the service is credited by the same amount of time credits. The software also keeps track of the time credit balance for each member.

Even if the first time banks saw the light of the day in the 1990s, it took another 20 years before the concept of timebanking had a serious impact in society. Dash and Sandhu (2018) report that the first time bank in the United Kingdom was set up in 1996 but that only 2,200 persons had joined a time bank in 2003. Eight years later after additional experimentation, learning, and expansion, there were around 30,000 registered members in the United Kingdom, 30,000 registered members in the United States and an additional 100,000 members scattered across 34 countries (Cahn, 2011). This number has continued to grow. In 2014, there were around 35,000 members in the United Kingdom and even more in the United States.[10]

To the best of our knowledge, there exists no public database that provides detailed information about time banks worldwide. Instead, this section will end with some descriptive statistics of *TimeBanks USA* (the largest time bank operating in the United States). This bank currently has 107

[9] All data and documentation related to *TimeBanks USA* stated in this section is available in the Online Appendix.

[10] These figures are from Boyle and Bird (2014) and `www.timebanks.org` (retrieved 2019–02–05).

branches in the United States spread out over 33 states, and it also operates in Australia, Canada, France, Greece, Guatemala, Israel, New Zealand, South Africa, South Korea, and United Kingdom (see Table 1). Even if not all registered branches are active, Table 2 provides some more detailed information about the active branches. As can be seen from the table, a time bank located in the United States have on average around 100 members and have on average performed 1,958 trades since April 2015. These trades involved on average 7,736 time units meaning that each time exchange was on average for 3.95 hours. As also can be seen from Table 2, the average time bank in the United States had 115.1 registered active offers and 115.5 active requests on January 15, 2019. The US figures from Table 2, therefore, roughly translates to that each member, in an active branch, on average had one active offer and one active request on January 15, 2019.

Table 1: Descriptive data of *TimeBanks USA*.

| Country | Number of branches | Active branches | Represented in states/provinces/regions |
|---|---|---|---|
| USA | 107 | 84 | 33 out of 50 |
| New Zealand | 30 | 28 | 7 out of 16 |
| Canada | 11 | 9 | 5 out of 10 |
| Other countries | 8 | 7 | – |

* The data was collected from `www.timebanks.org` on 2019–01–15 and it is available in the Online Appendix.

Table 2: Mean summary statistics for the active time banks in Table 1.

| Country | Number of members | Number of exchanges | Number of hours exchanged | Active offers | Active requests |
|---|---|---|---|---|---|
| USA | 98.9 (105.9) | 1,957.8 (5,034.9) | 7,736.0 (25,509.6) | 115.1 (748.9) | 115.5 (746.6) |
| New Zealand | 158.7 (189.7) | 1,913.4 (2,798.6) | 10,568.9 (29,138.0) | 25.9 (23.7) | 28.2 (28.1) |
| Canada | 64.1 (64.9) | 187.1 (255.3) | 600.9 (1,019.7) | 34.4 (41.6) | 27.9 (38.9) |
| Other countries | 113.7 (233.5) | 1,464.3 (3,623.4) | 5,401.7 (13,694.3) | 1.6 (2.1) | 2.0 (3.0) |

* All values are mean values (standard deviation within brackets).

## 3 The Model and Basic Definitions

This section introduces the time banking problem together with some definitions and axioms.

### 3.1 Agents, Bundles, and Allocations

Let $N = \{1, \ldots, n\}$ denote the finite set of agents. Each agent $i \in N$ is endowed with $t_i \in \mathbb{N}$ units of time which can be used to exchange services with agents in $N$. Let $t = (t_1, \ldots, t_n)$ denote the vector of time endowments. Because the exact nature of the services is of secondary interest, the problem will be described in terms of the time that an agent receives from and provides to other agents in $N$. Let $x_{ij}$ denote the time that agent $i \in N$ receives from agent $j \in N$, or, equivalently, the time that agent $j$ provides to agent $i$. Here, $x_{ii}$ represents the time that agent $i \in N$ receives from or, equivalently, spends with himself. It is assumed that $x_{ij}$ belongs to the set $\mathbb{N}_0$ of non-negative integers (including 0) representing standardized time units (e.g., 0 minutes for zero units, 30 minutes for one unit, 60 minutes for two units, etc.)

The time that agent $i \in N$ receives from the agents in $N$ can be described by the bundle (or vector) $x_i = (x_{i1}, \ldots, x_{in})$. The bundle where agent $i \in N$ spends all time with himself is denoted by $\omega_i$ (where $\omega_{ii} = t_i$ and $\omega_{ij} = 0$ for $j \neq i$). An allocation $x = (x_1, \ldots, x_n)$ is a collection of $n$ bundles (one for each agent in $N$). An allocation is *feasible* if

$$\sum_{j=1}^{n} x_{ij} = t_i \text{ for all } i \in N, \tag{1}$$

$$\sum_{j=1}^{n} x_{ji} = t_i \text{ for all } i \in N. \tag{2}$$

This means any agent $i$ receives the same amount of time from other agents that the agent supplies to other agents (recall that an agent can receive time from and spend time with himself). In this sense, any feasible allocation satisfies the time-balance conditions (1) and (2). In the remaining part of the paper, it is understood that any allocation is feasible.

### 3.2 Preferences and Preference Domains

A preference relation for agent $i \in N$ is a complete and transitive binary relation $R_i$ over feasible bundles such that $x_i R_i x_i'$ whenever agent $i$ finds bundle $x_i$ at least as good as bundle $x_i'$. Let $P_i$ and $I_i$ denote the strict and the indifference part of $R_i$, respectively. Let $\mathcal{R}_i$ denote the set of all preference relations of agent $i \in N$. A (preference) profile $R$ is a list of individual preferences $R = (R_1, \ldots, R_n)$. The general domain of profiles is denoted by $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$. A profile $R \in \mathcal{R}$ may also be written as $(R_i, R_{-i})$ when the preference relation $R_i$ of agent $i \in N$ is of particular importance.

A restricted preference domain $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_1 \times \cdots \times \tilde{\mathcal{R}}_n \subset \mathcal{R}$ will be considered for our main results. As explained in the Introduction, this restricted domain is based on the idea that any preference relation $R_i \in \tilde{\mathcal{R}}_i$:

(a) partitions the set of agents $N \backslash \{i\}$ into two disjoint sets containing acceptable and unacceptable agents, denoted by $A_i(R_i) \subseteq N \setminus \{i\}$ and $U_i(R_i) = N \setminus (A_i(R_i) \cup \{i\})$, respectively, and;

(b) associates with each acceptable agent $j \in A_i(R_i)$ an upper bound $\bar{t}_{ij} \in \mathbb{N}_0$ on how much time agent $i$ at most would like to receive from agent $j$.

Here, one may interpret (a) as agent $i$'s "horizontal preference" over acceptable and unacceptable services and (b) as agent $i$'s "vertical preference" of how much agent $i$ needs at most of each service. Then, for agent $i \in N$, the preference relation $R_i$ belongs to $\tilde{\mathcal{R}}_i$ if for any allocations $x$ and $y$:

(i) $\omega_i P_i x_i$ if $x_{ik} > 0$ for some $k \in U_i(R_i)$ or $x_{ij} > \bar{t}_{ij}$ for some $j \in A_i(R_i)$,

(ii) $x_i I_i y_i$ if both $\omega_i P_i x_i$ and $\omega_i P_i y_i$,

(iii) $y_i P_i x_i$ if $y_i R_i \omega_i$, $x_i R_i \omega_i$ and $\sum_{j \in A_i(R_i)} y_{ij} > \sum_{j \in A_i(R_i)} x_{ij}$, or

(iv) $y_i I_i x_i$ if $y_i R_i \omega_i$, $x_i R_i \omega_i$ and $\sum_{j \in A_i(R_i)} y_{ij} = \sum_{j \in A_i(R_i)} x_{ij}$.

The first condition states that an agent strictly prefers not to be involved in any trade rather than receiving time from an unacceptable agent or exceeding his upper bound from an acceptable agent. The second condition means that an agent is indifferent between any two bundles containing an unacceptable agent or exceeding his upper bound from an acceptable agent. The last two conditions reflect a monotonicity property and state that an agent weakly prefers bundles with weakly more acceptable agents whenever bundles do not contain any unacceptable agents and as long as the time bounds $\bar{t}_{ij}$ are not exceeded for acceptable agents.

In the Introduction we motivate the preference domain further. Our assumptions captures certain aspects of existing time banks, such as that you can classify your services supplied in various categories and then the other agents can request a certain services. There are clearly other assumptions on preferences that one can make, and eventually it is a matter of whether a particular application fits to the model or not. In Section 7 we also discuss two other applications different from the timebanking one. One being shift-exchange among workers, for instance at a hospital, and the second is seminar exchange for PhD students.

**Remark 1.** For the restricted domain $\tilde{\mathcal{R}}$, a report $R_i$ for agent $i \in N$ is given by a set of acceptable agents $A_i(R_i)$ together with an upper time bound $\bar{t}_{ij}$ for each $j \in A_i(R_i)$. An equivalent formulation of the reported preference for agent $i \in N$ is a vector $\bar{t}_i = (\bar{t}_{i1}, \ldots, \bar{t}_{in}) \in \mathbb{N}_0^n$ where $\bar{t}_{ii} = t_i$. Then $\bar{t}_{ij} = 0$ stands for $j \in U_i(R_i)$, i.e., agent $i$ is willing to accept at most zero time units from agent $j$. Whether the first or the second formulation is used is just a matter of choice. □

**Remark 2.** For any agent $i \in N$ and $R_i \in \tilde{\mathcal{R}}_i$, the preference $R_i$ is dichotomous over single services because they are partitioned into acceptable services and unacceptable services. The preference $R_i$ is polychotomous over bundles of services in the following way: for any $h = 0, 1, \ldots, \min\{t_i, \sum_{j \in A_i(R_i)} \bar{t}_{ij}\} = m$, all allocations $x$ and $y$ such that for all $j \in A_i(R_i)$ $x_{ij} \leq \bar{t}_{ij}$ and $y_{ij} \leq \bar{t}_{ij}$ for all $j \in A_i(R_i)$, $x_{ik} = 0 = y_{ik}$ for all $k \in U_i(R_i)$ and $\sum_{j \in A_i(R_i)} y_{ij} = h = \sum_{j \in A_i(R_i)} x_{ij}$ are ranked indifferent by $R_i$. Let $\mathcal{I}(h)$ denote this indifference class. Then under $R_i$ all allocations in $\mathcal{I}(m)$ are strictly preferred to all allocations in $\mathcal{I}(m-1)$, and in general, for $h = 1, \ldots, m$, under $R_i$ all allocations in $\mathcal{I}(h)$ are strictly preferred to all allocations in $\mathcal{I}(h-1)$. Thus, $R_i$ contains $m+2$ indifference classes (where $\mathcal{I}(0) = \{\omega_i\}$ and $\omega_i$ is strictly preferred to all allocations which are positive for some unacceptable service or exceeds the time bound for an acceptable service). In this sense, preferences belonging to $\tilde{\mathcal{R}}_i$ are polychotomous over bundles of services (where the upper bounds are incorporated) and at the same time dichotomous over single services. □

### 3.3 Axioms and Mechanisms

Let $\mathcal{F}(R)$ denote the set of all feasible allocations at profile $R \in \tilde{\mathcal{R}}$. Allocation $x \in \mathcal{F}(R)$ is *individually rational* if, for all $i \in N$, $x_i R_i \omega_i$. Allocation $x \in \mathcal{F}(R)$ *Pareto dominates* allocation $x' \in \mathcal{F}(R)$ if $x_i R_i x'_i$ for all $i \in N$ and $x_j P_j x'_j$ for some $j \in N$. An allocation is *efficient* if it is not Pareto dominated by any feasible allocation. An allocation $x$ is *maximal* at $R$ if $\sum_{i \in N} \sum_{j \in A_i(R_i)} x_{ij} \geq \sum_{i \in N} \sum_{j \in A_i(R_i)} x'_{ij}$ for all individually rational allocations $x'$. All individually rational and maximal allocations at profile $R \in \tilde{\mathcal{R}}$ are gathered in the set $\mathcal{X}(R) \subset \mathcal{F}(R)$. Note that $\mathcal{X}(R) \neq \emptyset$ for all $R \in \tilde{\mathcal{R}}$ and that any $x \in \mathcal{X}(R)$ is efficient.[11]

[11] If $x$ is not efficient, then there exists an individually rational allocation $x'$ such that $x'_i R_i x_i$ for all $i \in N$ and $x'_j P_j x_j$ for some $j \in N$. But then $\sum_{i \in N} \sum_{j \in A_i(R_i)} x_{ij} < \sum_{i \in N} \sum_{j \in A_i(R_i)} x'_{ij}$ meaning that $x$ is not maximal, a contradiction.

A mechanism $\varphi$ with domain $\tilde{\mathcal{R}}$ chooses for any profile $R \in \tilde{\mathcal{R}}$ a feasible allocation $\varphi(R) \in \mathcal{F}(R)$. Mechanism $\varphi$ is manipulable at profile $R \in \tilde{\mathcal{R}}$ by an agent $i \in N$ if there exists $R'_i$ such that $R' = (R'_i, R_{-i}) \in \tilde{\mathcal{R}}$, and for $x = \varphi(R)$ and $x' = \varphi(R')$ we have $x'_i P_i x_i$. If mechanism $\varphi$ is not manipulable by any agent $i \in N$ at any profile $R \in \tilde{\mathcal{R}}$, then $\varphi$ is *non-manipulable* (on the domain $\tilde{\mathcal{R}}$).

## 4 Priority Mechanisms

Often in real life, the chosen allocation is based on a priority mechanism: any such mechanism uses a priority-ordering, which may be deduced from a lottery or from a schematic update based on previous allocation rounds. Let $\pi : N \mapsto N$ be an exogenously given priority-ordering where the highest ranked agent is $i \in N$ with $\pi(i) = 1$, the second highest ranked agent is $i' \in N$ with $\pi(i') = 2$, and so on.

Given $R \in \tilde{\mathcal{R}}$, $i \in N$ and $\mathcal{Z}^* \subseteq \mathcal{X}(R)$, allocation $x \in \mathcal{Z}^*$ belongs to the set $\mathcal{X}^{i,\mathcal{Z}^*}(R)$ if $x_i R_i x'_i$ for all $x' \in \mathcal{Z}^*$, i.e., if allocation $x$ is weakly preferred to any allocation in the set $\mathcal{Z}^*$ under preference $R_i$. In the special case where the set $\mathcal{Z}^*$ is based on the choice made by some agent $i' \neq i$ for some profile $R \in \tilde{\mathcal{R}}$, i.e., where $\mathcal{Z}^* = \mathcal{X}^{i',\mathcal{Z}^{**}}(R)$ for some $\mathcal{Z}^{**} \subseteq \mathcal{X}(R)$, the set $\mathcal{X}^{i,\mathcal{Z}^*}(R)$ is denoted by $\mathcal{X}^{i,i'}(R)$.

**Definition 1.** An allocation $x \in \mathcal{X}(R)$ is agent-$i$-optimal at profile $R \in \tilde{\mathcal{R}}$ if $x \in \mathcal{X}^{i,\mathcal{X}(R)}(R)$.

Note the difference between the sets $\mathcal{X}^{i,\mathcal{X}(R)}(R)$ and $\mathcal{X}^{i,\mathcal{Z}^*}(R)$. The former set contains all agent *i*'s most preferred allocations in the set $\mathcal{X}(R)$ whereas the latter set contains all agent *i*'s most preferred allocations in a subset $\mathcal{Z}^*$ of $\mathcal{X}(R)$.

**Definition 2.** Let $\pi$ be a priority ordering and $N = \{i_1, \ldots, i_n\}$ be such that $\pi(i_k) = k$ for all $k = 1, \ldots, n$. Then $x \in \mathcal{X}(R)$ is a $\pi$-priority allocation at profile $R \in \tilde{\mathcal{R}}$ if:

(i) $x$ belongs to $\mathcal{X}^{i_1,\mathcal{X}(R)}(R)$,

(ii) $x$ belongs to $\mathcal{X}^{i_k,i_{k-1}}(R)$ for all $k = 2, \ldots, n$.

One way to think about the set of priority allocations is the following. First, the highest ranked agent identifies all his most preferred allocations in the set $\mathcal{X}(R)$. Then the agent with the second highest priority identifies all his most preferred allocations in the set identified by the highest ranked agent, then the agent with the third highest priority identifies all his most preferred allocations in the set identified by the second highest ranked agent, and so on. Formally, this means that if $x$ is a $\pi$-priority allocation, then:

$$x \in \mathcal{X}^{i_n,i_{n-1}}(R) \subseteq \mathcal{X}^{i_{n-1},i_{n-2}}(R) \subseteq \ldots \subseteq \mathcal{X}^{i_2,i_1}(R) \subseteq \mathcal{X}^{i_1,\mathcal{X}(R)}(R) \subseteq \mathcal{X}(R). \tag{3}$$

Note that a priority allocation is agent-$i$-optimal for the agent $i \in N$ with $\pi(i) = 1$. Moreover, all agents in $N$ are, by construction, indifferent between all allocations in the set $\mathcal{X}^{i_n,i_{n-1}}(R)$.

**Definition 3.** A mechanism $\varphi$ is a priority mechanism if there exists a priority ordering $\pi$ such that for all profiles $R \in \tilde{\mathcal{R}}$ the mechanism $\varphi$ selects a $\pi$-priority allocation from the set $\mathcal{X}(R)$.

Since a priority mechanism always makes a selection from the set $\mathcal{X}(R)$, it chooses an individually rational, maximal, and time-balanced allocation (which is efficient).

## 5 Results

As we show in Section 6, it is impossible to construct an individually rational, efficient, and non-manipulable mechanism on the general domain $\mathcal{R}$. Our first main result demonstrates that this impossibility can be avoided on the restricted domain $\tilde{\mathcal{R}}$ if trades are based on a priority mechanism.

**Theorem 1.** Any priority mechanism with domain $\tilde{\mathcal{R}}$ is non-manipulable.

In most settings proving non-manipulability of a priority mechanism is rather straight forward, e.g., Svensson (1994). In our setting with multiple objects and potentially different number of objects the scope for manipulation is much larger. The maximal set of allocations changes in a complex manner if one agent reports something slightly different. To prove non-manipulability we formulate a circulation flow network corresponding to the allocation in the priority mechanism. This enables us to keep track of changes from potential manipulations using the network formulation. The proof boils down to showing that no agent can ever gain by reducing the number of desired copies from an acceptable agent. For details of the argument and the construction of the network see Appendix B.

In Proposition 1 below, it is demonstrated that a priority mechanism can be formulated as a min-cost flow problem. To formulate this problem, a bipartite graph needs to be defined and specific values must be attached to the vertices and the edges in the graph.

**Definition 4.** For any profile $R \in \tilde{\mathcal{R}}$, the bipartite graph $g = (N, M, E, u)$ is defined by two disjoint sets of vertices, $N$ and $M$, a set of edges, $E$, and a profile of upper bounds $u = (u(i,l))_{(i,l)\in E}$ on the flow between any two edges, defined by:

(i) $N = \{1, \ldots, n\}$,

(ii) $M = \{n+1, n+2, \ldots, n+n\}$,

(iii) $E = \{(i, n+j) \in N \times M : j \in A_i(R_i) \text{ or } j = i\}$, and

(iv) for all $i \in N$ and each edge $(i, n+j) \in E$ where $j \in A_i(R_i)$ we set $u(i, n+j) = \bar{t}_{ij}$ and $u(i, n+i) = t_i$.

**Example 1.** Let $N = \{1, 2, 3, 4\}$, $t_1 = t_2 = 1$ and $t_3 = t_4 = 2$. Let $R \in \tilde{\mathcal{R}}$ be such that $A_1(R_1) = A_2(R_2) = \{3, 4\}$ (with $\bar{t}_{13} = \bar{t}_{14} = \bar{t}_{23} = \bar{t}_{24} = 1$) and $A_3(R_3) = A_4(R_4) = \{1, 2\}$ (with $\bar{t}_{31} = \bar{t}_{32} = \bar{t}_{41} = \bar{t}_{42} = 2$). The constructed graph $g$ is depicted in Figure 1. □

The interpretation of the graph $g$ is that the agents in $M$ should be regarded as copies of the agents in $N$ and in particular, agent $n+i \in M$ is the copy of agent $i \in N$. Furthermore, agents $i \in N$ and $n+j \in M$ are connected by an edge if agent $j$ is acceptable for agent $i$ or if $j = i$. Because an allocation will be defined by the flows between the agents in $N$ and $M$, the above construction guarantees that $n+j \in M$ can only provide time for an agent $i \in N$ if agent $i$ finds agent $j$ acceptable or if agent $j$ is his own copy. Finally, the upper bound on flow from $n+j$ to $i$ where $j \in A_i(R_i)$ is equal to the upper bound of how much time agent $i$ wants from agent $j$. A flow $x$ specifies for each $(i, l) \in E$ a non-negative integer $x_{il} \in \mathbb{N}_0$.[12] Any flow $x$ is equivalent to an allocation in the usual sense: $x_{ii} = x_{i(n+i)}$, $x_{ij} = x_{i(n+j)}$ for all $j \in A_i(R_i)$, and $x_{ij} = 0$ for all $j \in U_i(R_i)$.

[12] In general, flows may assign real numbers to edges, but for our purpose we restrict flows to assign integers.

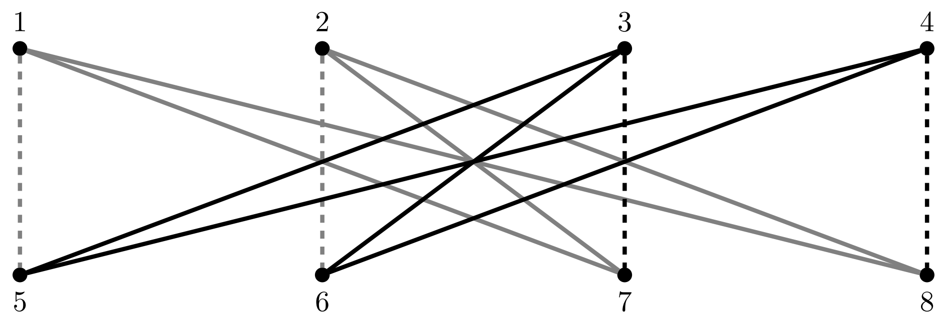


Figure 1: Edge capacity 1 is color-coded by gray, while capacity 2 is denoted by black edges. The edges connecting two copies of the same agent are marked by dashed lines.

Recall that the time-balance conditions (1) and (2) must hold for any allocation. In the language of min-cost flow problems, this means that the required flow (between the vertices in the bipartite graph $g$) is dictated by equations (1) and (2) which must be reformulated for the bipartite setting as follows:

$$\sum_{j \in A_i(R_i) \cup \{i\}} x_{i(n+j)} = t_i \text{ for all } i \in N, \tag{1'}$$

$$\sum_{i \in A_j(R_j) \cup \{i\}} x_{j(n+i)} = t_i \text{ for all } i \in N. \tag{2'}$$

A natural interpretation of the bipartite graph is therefore that agents in $M$ supply time to the demanding agents in $N$. To obtain a maximal outcome, it is important to prevent flows between agents in $N$ and their own copies in $M$ whenever there are other feasible flows or, equivalently, to prevent agents to supply time to their own copies whenever it is feasible to supply time to other distinct agents (by the time-balance conditions, any agent supplying time to other agents also receives in return more time from acceptable agents). This can be achieved by introducing an artificial cost whenever agents supply time to themselves. Let, for this purpose, $c_{il}$ denote the cost associated when $l \in M$ is supplying time to agent $i$, and let, in particular, for each $(i, l) \in E$:

$$c_{il} = \begin{cases} -1 & \text{if } l = n + i \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For a given profile $R \in \tilde{\mathcal{R}}$, a given graph $g = (N, M, E, u)$ and given costs $c = (c_{il})_{(i,l)\in E}$, the (artificial) cost is minimized at any allocation $x \in \mathcal{F}(R)$ that solves the following maximization problem:[13]

$$\max \sum_{(i,l)\in E} c_{il} x_{il} \text{ s.t. conditions (1'), (2'), } x_{il} \in \mathbb{N}_0 \text{ and } x_{il} \leq u(i,l) \text{ for all } (i,l) \in E. \tag{5}$$

An allocation $x \in \mathcal{F}(R)$ is a *maximizer* if it is a solution of the maximization problem (5). Let $\mathcal{V}(R, c) \subseteq \mathcal{F}(R)$ denote the set of all maximizers at profile $R \in \tilde{\mathcal{R}}$ for given costs $c = (c_{il})_{(i,l)\in E}$.

[13] Note that costs of edges are non-positive and the max-cost flow problem is equivalent to the usual min-cost flow problem.

For notational convenience, the value of an allocation $x$ at cost $c$ is given by $V(x,c) = \sum_{(i,l)\in E} c_{il}x_{il}$.

**Lemma 1.** If allocation $x$ belongs to $\mathcal{V}(R,c)$ at profile $R \in \tilde{\mathcal{R}}$, then $x \in \mathcal{X}(R)$.

The set of maximizers $\mathcal{V}(R,c)$ is non-empty for any profile $R \in \tilde{\mathcal{R}}$ since $\mathcal{V}(R,c) \subseteq \mathcal{X}(R)$ and $\mathcal{X}(R)$ is non-empty and finite for all $R \in \tilde{\mathcal{R}}$. However, as stated above, agents need not be indifferent between all allocations in the set $\mathcal{V}(R,c)$ since $\mathcal{V}(R,c) \subseteq \mathcal{X}(R)$. Hence, in order to define a priority mechanism based on a solution to maximization problem (5), a refined selection from the set $\mathcal{V}(R,c)$ is necessary which will be based on the priority-ordering $\pi$.

We will modify the costs $c$ in order to take the priority-ordering $\pi$ into account, let $\varepsilon_0 \in (0,1)$ and $\varepsilon_{i-1} = (1+t_i)\varepsilon_i$ for each $i \in \{1,\dots,n\}$. By construction of $\varepsilon_i$, it follows that:[14]

$$1 > \varepsilon_0 \geq \varepsilon_i > \sum_{k=i+1}^{n} t_k\varepsilon_k > 0 \text{ for all } i \in \{0,\dots n-1\}. \tag{6}$$

To guarantee a larger flow to agents with higher priorities, the value associated with a flow will be monotonically increasing with higher priorities. More specifically, let for each $(i,l) \in E$:

$$\tilde{c}_{il} = \begin{cases} -1 & \text{if } l = n+i \\ \varepsilon_{\pi(i)} & \text{otherwise.} \end{cases}$$

The above construction means that the agent with the highest priority (i.e., the agent with $\pi(i) = 1$) will receive the highest edge weight (for edges $(i,l) \in E\backslash\{(i,n+i)\}$), the agent with the second highest priority (i.e., the agent with $\pi(i) = 2$) will receive the second highest edge weight, and so on.

Our second main result demonstrates that a mechanism that selects an allocation from the set of maximizers for each profile in $\tilde{\mathcal{R}}$ and any given priority-ordering is a priority mechanism. From Theorem 1, it is already known that such a mechanism is non-manipulable on the domain $\tilde{\mathcal{R}}$.

**Proposition 1.** For a given priority-ordering $\pi$, a mechanism $\varphi$ selecting for each profile $R \in \tilde{\mathcal{R}}$ an allocation from $\mathcal{V}(R,\tilde{c})$ is a priority mechanism based on $\pi$.

## 6 Discussion and Extensions

This section discusses essentially single-valued cores and random mechanisms under two separate headings.[15]

### 6.1 Essentially Single-Valued Cores

Theorem 1 establishes that, in the considered time bank problem, there exist mechanisms which are individually rational, efficient, and non-manipulable on the domain $\tilde{\mathcal{R}}$. This is surprising as a number of previous impossibility results for the combination of these axioms have been established by applying a essentially single-valued cores result by Sönmez (1999). Below we connect his result to time banking.

[14] To see this, note that $\varepsilon_{n-1} = (1+t_n)\varepsilon_n > t_n\varepsilon_n$ since $\varepsilon_n > 0$ and, consequently, $\varepsilon_{n-2} = (1+t_{n-1})\varepsilon_{n-1} = \varepsilon_{n-1} + t_{n-1}\varepsilon_{n-1} > t_n\varepsilon_n + t_{n-1}\varepsilon_{n-1}$. Condition (6) then follows by repeating these arguments.

[15] We are grateful to the referees and the coeditor for bringing our attention to random mechanisms.

Given $R \in \tilde{\mathcal{R}}$, the core of $R$, denoted by $\mathcal{C}(R)$, consists of all feasible allocations $x \in \mathcal{F}(R)$ which are not dominated via some coalition and some allocation meaning that there exists no $\emptyset \neq S \subseteq N$ and $y \in \mathcal{F}(R)$ such that (i) $y_i R_i x_i$ for all $i \in S$, (ii) $y_j P_j x_j$ for some $j \in S$ and (iii) $\{j \in N : y_{ij} \neq 0\} \subseteq S$ for all $i \in S$. The core of $R$ is essentially single-valued if for all $x, y \in \mathcal{C}(R)$ we have $x_i I_i y_i$ for all $i \in N$. Note that if $\mathcal{C}(R) = \emptyset$, then the core of $R$ is essentially single-valued.

Let $\tilde{\mathcal{R}}^1$ denote the set of all profiles $R \in \tilde{\mathcal{R}}$ such that for all $i \in N$ and all $j \in A_i(R_i)$ we have $\bar{t}_{ij} = 1$ and $t_i = 1$ (i.e., any agent demands at most one time unit of any acceptable service and any agent provides at most one unit of time). This corresponds to the classical dichotomous domain by Bogomolnaia and Moulin (2004). Then it is easy to check that the domain $\tilde{\mathcal{R}}^1$ satisfies Assumption A and B of Sönmez (1999).[16] Hence, his main result applies, which shows the following: if there exists an individually rational, efficient, and non-manipulable mechanism, then for any profile where the core is non-empty we have (i) the core is essentially single-valued and (ii) the mechanism chooses a core allocation. However, here for any $R \in \tilde{\mathcal{R}}^1$, if the core of $R$ is non-empty, then the set of individually rational and efficient allocations is essentially single-valued (and the core is essentially single-valued).[17] But then any priority mechanism chooses a core allocation. Note that Proposition 1 of Sönmez (1999) shows that when the core of each profile is externally stable, then any selection from the core correspondence is non-manipulable.[18] External stability implies that the core is non-empty for any profile, but here, if the core is non-empty, then the set of individually rational and efficient allocations is essentially single-valued. As this is often not the case, the core is often empty and Proposition 1 of Sönmez (1999) cannot be used to show the non-manipulability of priority mechanisms.

Once non-unitary endowments are allowed (as it is the case for time banks), the domain $\tilde{\mathcal{R}}$ does not satisfy Assumption B of Sönmez (1999). This is illustrated in the next example.

**Example 2.** We use the instance introduced in Example 1, i.e., $N = \{1, 2, 3, 4\}$, $t_1 = t_2 = 1$, $t_3 = t_4 = 2$, and $R \in \tilde{\mathcal{R}}$ is such that $A_1(R_1) = A_2(R_2) = \{3, 4\}$ (with $\bar{t}_{13} = \bar{t}_{14} = \bar{t}_{23} = \bar{t}_{24} = 1$) and $A_3(R_3) = A_4(R_4) = \{1, 2\}$ (with $\bar{t}_{31} = \bar{t}_{32} = \bar{t}_{41} = \bar{t}_{42} = 2$). If agent 3 comes before agent 4 in the priority order $\pi$, then $(3, 3, 12, 0)$ is the unique $\pi$-priority allocation (where this stands for agent 1 receiving one time unit from agent 3, agent 2 receiving one time unit from agent 3, agent 3 receiving one time unit from both agent 1 and agent 2, and agent 4 keeping his endowment). If agent 4 comes before agent 3 in the priority order $\pi$, then $(4, 4, 0, 12)$ is the unique $\pi$-priority allocation. Note that $(3, 3, 12, 0) P_3 (3, 4, 1, 2) P_3 \omega_3$ but there exists no $R'_3$ such that $(3, 3, 12, 0) P'_3 \omega_3 P'_3 (3, 4, 1, 2)$. The latter conclusion follows since $(3, 3, 12, 0) P'_3 \omega_3$ implies $1 \in A_3(R'_3)$ and $\bar{t}'_{31} \geq 1$, and thus $(3, 4, 1, 2) P'_3 \omega_3$. Hence, Assumption B is violated for the domain $\tilde{\mathcal{R}}$ and at the same time any priority mechanism is individually rational, efficient and non-manipulable. $\square$

[16] In our framework (without externalities), Assumption A says that for any allocation $x$ we have $x_i I_i \omega_i$ if and only if $x_i = \omega_i$ and Assumption B says that whenever for two allocations $x$ and $y$ with $x_i P_i y_i$ and $x_i R_i \omega_i$, there exists a preference relation $R'_i$ such that $x_i R'_i \omega_i R'_i y_i$.

[17] Note that for any $R \in \tilde{\mathcal{R}}^1$, if the set of individually rational and efficient allocations is not essentially single-valued, then any two individually rational and efficient allocations, which are not regarded indifferent by all agents, dominate (via some coalition) each other and the core must be empty: more formally, for $R \in \tilde{\mathcal{R}}$ and any two individually rational and efficient allocations $x$ and $y$ for which not $x_i I_i y_i$ for all $i \in N$, for $S = \{i \in N : x_{ii} = 0\}$ we have for all $i \in S$, $x_i R_i y_i$, and for some $j \in S$, $x_j P_j y_j$, i.e., $x$ dominates $y$ with the coalition $S$ (and the same argument applies for $y$ in the role of $x$ and $x$ in the role of $y$). Thus, the core of $R$ is empty.

[18] See also Demange (1987) for an important study of non-manipulable cores.

The above example also shows that in general we do not have dichotomous preferences in the domain $\tilde{\mathcal{R}}$. We may have many distinct indifference classes for preferences in the domain $\tilde{\mathcal{R}}$ and yet by Theorem 1, there exists an individually rational, efficient, and non-manipulable mechanism.

Finally, we show that a priority mechanism with the same order may select different allocations when choosing from the set of individually rational and efficient allocations (as in Manjunath and Westkamp, 2018).

**Example 3.** Let $N = \{1, 2, 3, 4\}$ and $t_1 = t_2 = t_3 = t_4 = 1$. Let $R \in \tilde{\mathcal{R}}$ be such that $A_1(R_1) = \{2\}$, $A_2(R_2) = \{3\}$, $A_3(R_3) = \{1, 4\}$, and $A_4(R_4) = \{3\}$ (with $\bar{t}_{12} = \bar{t}_{23} = \bar{t}_{31} = \bar{t}_{34} = \bar{t}_{43} = 1$). Then $\mathcal{X}(R) = \{(2, 3, 1, 0)\}$, i.e., there is a unique individually rational and maximal allocation which is chosen by any priority mechanism. However, the allocation $(0, 0, 4, 3)$ is individually rational and efficient which is selected by any priority mechanism which chooses from the whole set of individually rational and efficient allocations and where agent 4 occupies the first position in the priority order (and such a priority mechanism would not necessarily result in a maximal allocation). Note that the same argument applies if a priority mechanism chooses from the set of all feasible allocations. □

### 6.2 Random Mechanisms

Priority mechanisms are unfair in the sense that the agent in first position of the priority ordering receives for any profile $R$ his most preferred bundle among all allocations in $\mathcal{X}(R)$ (but this is not the case for the agent in last position of the priority ordering). To establish fairness, one may consider random allocations and random mechanisms, which we define briefly below.

A random allocation for $R$ is a probability distribution $p$ over $\mathcal{F}(R)$. For all $x \in \mathcal{F}(R)$, let $p(x)$ denote the probability of allocation $x$. The support of $p$ is given by the allocations which are chosen with positive probability by $p$, i.e. $supp(p) = \{x \in \mathcal{F}(R) : p(x) > 0\}$. Then $p$ is ex-post individually rational for $R$ if for all $x \in supp(p)$, $x$ is individually rational. Analogously, ex-post maximality and ex-post efficiency are defined. For two random allocations $p$ and $q$, we say that $p$ stochastically $R_i$-dominates $q$ (where we write equivalently $p_i$ stochastically $R_i$-dominates $q_i$), denoted by $p_i R_i^{sd} q_i$, if for all $y \in \mathcal{F}(R)$ we have

$$\sum_{x \in \mathcal{F}(R): x_i R_i y_i} p(x) \geq \sum_{x \in \mathcal{F}(R): x_i R_i y_i} q(x).$$

Then $p_i P_i^{sd} q_i$ if $p_i R_i^{sd} q_i$ and not $q_i R_i^{sd} p_i$. A random mechanism $\phi$ chooses for any profile $R \in \tilde{\mathcal{R}}$ a random allocation for $R$. The random mechanism $\phi$ is ex-post individually rational if for any profile $R$ the random allocation is ex-post individually rational for $R$. Analogously, ex-post maximality and ex-post efficiency are defined for random mechanisms.

Now let $\varphi^\pi$ denote a deterministic priority mechanism using $\pi$ as a priority ordering and $\Pi$ denote the set of all priority orderings. Then let $RP = \sum_{\pi \in \Pi} \frac{1}{n!} \varphi^\pi$ denote the random priority mechanism putting equal priority on each priority ordering. Because determinstic priority mechanisms are individually rational, maximal and efficient, the random priority mechanism is ex-post individually rational, ex-post maximal and ex-post efficient.

For random mechanisms, often axioms are defined in terms of stochastic dominance. The random mechanism is sd-non-manipulable if for all $R, R' \in \tilde{\mathcal{R}}$ such that $R' = (R'_i, R_{-i})$ for some $i \in N$ we have $\phi_i(R) R_i^{sd} \phi_i(R')$. The random mechanism is sd-efficient if for all $R \in \tilde{\mathcal{R}}$ there exists no random allocation $p$ for $R$ such that $p_i R_i^{sd} \phi_i(R)$ for all $i \in N$ and $p_j P_i^{sd} \phi_j(R)$ for some $j \in N$. The random mechanism is sd-fair if for all $R \in \tilde{\mathcal{R}}$ and all $i, j \in N$, $\phi_i(R) R_i^{sd} \phi_j(R)$.

Now from our results we obtain the following corollary.

**Corollary 1.** The random priority mechanism is sd-non-manipulable, sd-efficient and sd-fair.

In Corollary 1 sd-non-manipulability and sd-fairness are quite obvious, whereas sd-efficiency is more surprising and relies on the fact that preferences are dichotomous over single services (see also Bogomolnaia and Moulin, 2004). Besides random priority mechanisms, it would be interesting whether there are any other "nice" random mechanisms which are not simply a mixing of deterministic mechanisms. This question is left for future research.

### 6.3 Extensions

This section contains discusses three possible extensions of the considered model.[19]

#### 6.3.1 More General Preferences

One may argue that the upper bounds on how much time agent $i$ at most would like to receive from agent $j$ is extreme in the following sense. Suppose that there are two agents 1 and 2 such that $t_1 = t_2 = 3$. Now if for profile $R$ we have $\bar{t}_{12} = 2$, then $(22, 11) P_1 w_1 P_1 (222, 111)$ meaning that agent 1 would strictly prefer his endowment to receiving three time units of service from agent 2. One may argue that agent 1 has a preference such that $(22, 11) P_1 (222, 111) P_1 w_1$, i.e., receiving two time units of service from agent 2 is optimal, but receiving three time units is still better than his endowment. This would correspond to agent 1 having a peak at two time units and a maximum at three time units.

It is easy to see that including such preferences would result in a manipulable mechanism (if the mechanism is maximal and individually rational). If both agents have a peak at two time units and the maximum consumption at three time units, then the unique maximal allocation is $(222, 111)$. Now if agent 1 reduces his maximal consumption to two, then the unique maximal allocation is $(22, 11)$, which is strictly preferred by agent 1 to $(222, 111)$. This impossibility is not surprising, see for instance Konishi et al. (2001) where agents are endowed with multiple types of indivisible goods and have more general preferences.

Another possibility is that agents possess more general preferences but are only allowed to report profiles belonging to $\tilde{\mathcal{R}}$ to the priority mechanism. Then non-manipulability becomes meaningless as agents cannot report their true preferences and one would have to consider games induced by the mechanism and the general preferences. For instance, if all agents report no services as acceptable, then this is a Nash equilibrium outcome which is in general neither efficient nor maximal for the true preferences.

#### 6.3.2 Non-Integer Endowments and Upper Bounds

Our analysis can easily accommodate endowments and upper bounds given by rational numbers. Without going into the details, for any profile $R$ (where for any $i \in N$, $t_i$ and $\bar{t}_{ij}$ are rational numbers),

[19] We thank the referees and the coeditor for suggesting these extensions.

let $d$ denote the greatest common denominator of all constraints. Then for the profile $R$ in terms of unit $d$, agent $i$ is endowed with $dt_i$ time units and $d\bar{t}_{ij}$ is agent $i$'s upper bound for services from agent $j$. Our construction applies to the profile $R$ in terms of unit $d$ to obtain a priority allocation which is individually rational and maximal. For non-manipulability, for two profiles $R$ and $R'$, let $R$ be in terms of unit $d$ and $R'$ in terms of unit $d'$. But then we can express $R$ in terms of unit $dd'$ and the priority allocation for $R$ in terms of unit $d$ is also a priority allocation in terms of unit $dd'$. Similarly, we can express $R'$ in terms of unit $dd'$ and the priority allocation for $R'$ is also a priority allocation in terms of unit $dd'$. Because the same priority ordering is used, then non-manipulability follows from Theorem 1. Moreover, the algorithm computing a maximum weight circulation is strongly polynomial (Orlin, 1993), and thus, the running time does not depend on $d$ or $d'$.[20]

#### 6.3.3 Embedding in Trading Networks

We show that our model can be embedded in the general framework of trading networks by Hatfield et al. (2018).

In any allocation $x$, for any $i, j \in N$, the number $x_{ij}$ can be viewed as a contract between agent $i$ and agent $j$ where $j$ provides $x_{ij}$ units of service to $i$. For later purposes, the number $x_{ij}$ is decomposed into separate units $1_{ij}, 2_{ij}, \ldots, x_{ij}$ where $k_{ij}$ stands for the $k$th unit of service provided by agent $j$ to agent $i$. Then $i$ is the buyer in contract $k_{ij}$ and $j$ is the seller in contract $k_{ij}$. We write $i = b(k_{ij})$ and $j = s(k_{ij})$. We ignore prices and set them implicitly equal to one. We denote the set of all contracts by:

$$\mathcal{Y} = \{k_{ij} : i, j \in N \text{ with } i \neq j \text{ and } k \in \{1, \ldots, t_j\}\}.$$

An allocation is then simply a subset of contracts $Y \subseteq \mathcal{Y}$. Let $Y_{\to i} = \{y \in Y : b(y) = i\}$ denote the set of contracts in $Y$ where $i$ is a buyer and $Y_{i\to} = \{y \in Y : s(y) = i\}$ the set of contracts in $Y$ where $i$ is a seller. Let $Y_{ij} = Y_{\to i} \cap Y_{j\to}$ denote the set of contracts in $Y$ where $i$ is the buyer and $j$ is the seller. Let $Y_i = Y_{\to i} \cup Y_{i\to}$.

Given $R \in \tilde{\mathcal{R}}$, a set of contracts $Y$ is feasible (for $R$) if (i) for all $i \in N$, $|Y_{\to i}| = |Y_{i\to}| \leq t_i$, (ii) for all $i, j \in N$, if $k_{ij} \in Y$ and $k > 1$, then $(k-1)_{ij} \in Y$, and (iii) for all $i, j \in N$, if $k_{ji} \in Y$ and $k > 1$, then $(k-1)_{ji} \in Y$. Note that (i) corresponds to equations (1) and (2), and (ii) says that if agent $j$ provides to agent $i$ the $k$th unit of time, then agent $j$ provides to agent $i$ the $k-1$th unit of time (and similarly for (iii)). We say that $Y_i$ is feasible for $i$ if (i), (ii) and (iii) hold for agent $i$. Note that any feasible allocation $Y$ corresponds to an allocation $y$ in the original model by setting for all $i, j \in N$ with $i \neq j$, $y_{ij} = |Y_{ij}|$ and $y_{ii} = t_i - |Y_{i\to}|$ (and vice versa as above).

Given profile $R \in \tilde{\mathcal{R}}$, agent $i$'s utility function over subsets of contracts $Y \subseteq \mathcal{Y}_i$ which are feasible for $i$ is given by (i) $U_i(Y) = |Y_{i\to}|$ if $Y_{ij} = \emptyset$ for all $j \in U_i(R_i)$ and $|Y_{ij}| \leq \bar{t}_{ij}$ for all $j \in A(R_i)$, and (ii) $U_i(Y) = -\infty$ otherwise. Then an allocation $Y$ is feasible if for all $i \in N$, $U_i(Y) \neq -\infty$. Note that $U_i(\emptyset) = 0$.

Then agent $i$'s choice correspondence for subsets of contracts is defined as follows. For all $Y \subseteq$

[20] We leave the incorporation of irrational constraints for future research.

$\mathcal{Y}_i$, let:

$$C_i(Y) = \{X \subseteq Y : X \text{ is feasible for } i \text{ and } U_i(X) \geq U_i(X') \text{ for all } X' \subseteq Y \text{ feasible for } i\}.$$

We then write for $Y \subseteq \mathcal{Y}$, $C_i(Y) = C_i(Y_i)$.

Then allocation $Y$ is individually rational if for all $i \in N$, $Y_i \in C_i(Y)$. The allocation $Y$ is maximal if $Y$ is individually rational and there exists no other individually rational allocation $W$ such that $\sum_{i \in N} U_i(W) > \sum_{i \in N} U_i(Y)$. Then allocation $Y$ is stable if $Y$ is individually rational and there exists no $Z \subseteq \mathcal{Y} \backslash Y$ such that for all $i \in N(Z) = \{i \in N : Z_i \neq \emptyset\}$ and all $W \in C_i(Y \cup Z)$, $Z_i \subseteq W_i$.

**Corollary 2.** If $Y$ is individually rational and maximal, then $Y$ is stable.

Thus, our results establish that there exists a stable and non-manipulable mechanism. Again this is surprising as often there does not exist any stable and non-manipulable mechanism.

Below we verify that agents' preferences satisfy monotone substitutability. Hence, by Theorem 1 of Hatfield et al. (2018), stability is equivalent to "chain stability". Moreover, one may verify in Example 3 that $(0, 0, 4, 3)$ is stable, and the set of individually rational and maximal allocations is in general a strict subset of the set of stable allocations.

Then agent $i$'s choice function is monotone substitutable if (1) for all $Y, Z \subseteq \mathcal{Y}_i$ such that $Y_{i\rightarrow} = Z_{i\rightarrow}$ and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that (i) $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$, (ii) $(Y_{\rightarrow i} \backslash Y^*_{\rightarrow i}) \subseteq (Z_{\rightarrow i} \backslash Z^*_{\rightarrow i})$, and (iii) $|Z^*_{\rightarrow i}| - |Z^*_{i\rightarrow}| \geq |Y^*_{\rightarrow i}| - |Y^*_{i\rightarrow}|$; and (2) for all $Y, Z \subseteq \mathcal{Y}_i$ such that $Y_{\rightarrow i} = Z_{\rightarrow i}$ and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that (i) $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$, (ii) $(Y_{i\rightarrow} \backslash Y^*_{i\rightarrow}) \subseteq (Z_{i\rightarrow} \backslash Z^*_{i\rightarrow})$, and (iii) $|Z^*_{i\rightarrow}| - |Z^*_{\rightarrow i}| \geq |Y^*_{i\rightarrow}| - |Y^*_{\rightarrow i}|$

**Lemma 2.** For all $i \in N$ and all $R \in \tilde{\mathcal{R}}$, agent $i$'s choice function $C_i$ is monotone substitutable.

## 7 Concluding Remarks

This paper has modeled a time bank as a matching market. On a restricted but yet natural preference domain, it has been demonstrated that a priority mechanism can be formulated as a min-cost flow problem and, furthermore, that such mechanism is non-manipulable and always makes a selection from the set of individually rational, efficient, and time-balanced allocations. No mechanism with these properties exists on the general preference domain (Sönmez, 1999, Corollary 1).

Even if the considered priority mechanism has been demonstrated to satisfy all properties of interest on a restricted preference domain, the mechanism can be criticized from a fairness perspective as it discriminates low priority agents (see the discussion in Section 6.2). For this reason, it is important to characterize the entire class of mechanisms that satisfies the axioms of interest to see if such discrimination can be avoided or not. Moreover, even if the considered domain restriction is natural for the time banking problem, it may also be of importance to find a maximal domain result where the above mentioned impossibility can be escaped as this will give important information about how much more detailed preferences that may be reported to a time bank. Both these open problems are left for future research.

We would also like to point out that the model considered in this paper is not restricted to the timebanking problem. As already described, the model considered by Manjunath and Westkamp

(2018) is almost identical to the one considered in this paper. However, their leading example is shift-reallocation. This application is motivated by the fact that millions of people, in many different professions, engage in shift work (e.g., physicians, retail workers, etc.) but that shift workers sometimes are dissatisfied with their assigned time slots. Another problem that recently has been solved, using a version of the priority mechanism proposed in this paper, is the seminar exchange problem. This problem was initiated in Scandinavia in the fall of 2018 (by one of the authors of this paper) to help final year PhD students to practice their job market talks at external departments. To solve this problem, Economics departments classified 11 different research field as acceptable and unacceptable. Job market candidates, on the other hand, classified themselves by one of the 11 different research fields and all departments as either acceptable or unacceptable. A specific construction guaranteed that the job market candidates also played the role of their departments and could, therefore, be engaged in time-balanced seminar exchanges with other students. In total, 10 departments and 21 job market candidates from Denmark, Norway and Sweden participated in the centralized market for seminar exchange. In the end, all candidates were matched to some department in a balanced sense (i.e., each department organized exactly as many seminars as their own students was invited to).[21]

## Appendix A: Proofs

Appendix A contains the proofs of all results except Theorem 1, which is in Appendix B.

**Proof of Lemma 1.** Suppose that allocation $x$ belongs to $\mathcal{V}(R, c)$. The fact that $x$ is feasible and individually rational follows directly from the construction of the graph $g = (N, M, E, u)$ and by definition of the maximization problem (5), i.e., $n + j \in M$ is only connected to an agent $i \in N$ if agent $j \in A_i(R_i) \cup \{i\}$, all flows are between connected agents and the flow never exceeds the upper bounds $\bar{t}_{ij}$ on any edge $(i, n + j) \in E$.

To show that allocation $x$ is maximal, it will be demonstrated that $x$ minimizes the total flow between agents $i \in N$ and their respective clones $i + n \in M$. Because $x \in \mathcal{V}(R, c)$ is a maximizer, it follows that:

$$\sum_{(i,l)\in E} c_{il} x_{il} \geq \sum_{(i,l)\in E} c_{il} x'_{il} \text{ for any feasible allocation } x' \text{ in program (5).} \tag{7}$$

Given the construction of the costs in condition (4), it now follows from condition (7) that:

$$\sum_{i=1}^{n} c_{i(n+i)} x_{i(n+i)} \geq \sum_{i=1}^{n} c_{i(n+i)} x'_{i(n+i)}.$$

Because $c_{i(i+n)} = -1$ for all $i \in N$, by condition (4), the above inequality can be rewritten as:

$$\sum_{i=1}^{n} x'_{i(n+i)} \geq \sum_{i=1}^{n} x_{i(n+i)}.$$

But this condition means that allocation $x$ minimizes the total flow between agents $i \in N$ and their

[21] All details of the seminar exchange market are described at the Swedish Economics blog *Ekonomistas*. See `https://ekonomistas.se/2018/12/03/en-skandinavisk-matchningsmarknad-for-doktorandseminarier/`.

respective clones $i+n \in M$ among all feasible allocations, which is the desired conclusion. $\square$

**Proof of Proposition 1.** It is first demonstrated that $\mathcal{V}(R,\tilde{c}) \subseteq \mathcal{V}(R,c)$ for each profile $R \in \tilde{\mathcal{R}}$. Suppose now that $x \in \mathcal{V}(R,c)$ but $x' \notin \mathcal{V}(R,c)$ for some $x'$ that is feasible in the optimization program defined in (5). To reach the conclusion, it is sufficient to show $x' \notin \mathcal{V}(R,\tilde{c})$.

Note that $x \in \mathcal{V}(R,c)$ and $x' \notin \mathcal{V}(R,c)$ imply $V(x,c) > V(x',c)$. This conclusion together with $c_{il} \in \{-1,0\}$ and $x_{il} \in \mathbb{N}_0$ for all $(i,l) \in E$ and $\varepsilon_0 < 1$ gives $V(x,c) > V(x',c) + \varepsilon_0$. Because $\tilde{c}_{il} \geq c_{il}$ for all $(i,l) \in E$ by construction, it holds that $V(x,\tilde{c}) \geq V(x,c)$. This together with the above inequalities imply $V(x,\tilde{c}) > V(x',c) + \varepsilon_0$. To complete this part of the proof, we show that $V(x',c) + \varepsilon_0 \geq V(x',\tilde{c})$, since this condition together with the above conclusions then show $V(x,\tilde{c}) > V(x',\tilde{c})$, i.e., that $x' \notin \mathcal{V}(R,\tilde{c})$.

To demonstrate $V(x',c) + \varepsilon_0 \geq V(x',\tilde{c})$, we partition $E$ into two disjoint sets, $E^1$ and $E^2$, where the former set contains all edges $(i,l)$ in $E$ where $l \neq i+n$ and the latter contains all edges $(i,l)$ in $E$ where $l = i+n$. Consequently, $c_{il} = 0 < \tilde{c}_{il} = \varepsilon_i$ for all $(i,l) \in E^1$ and $c_{il} = \tilde{c}_{il} = -1$ for all $(i,l) \in E^2$. Hence, the inequality $V(x',c) + \varepsilon_0 \geq V(x',\tilde{c})$ can be rewritten as:

$$\begin{aligned}
V(x',c) + \varepsilon_0 &= \sum_{(i,l)\in E} c_{il}x'_{il} + \varepsilon_0, \\
&= \sum_{(i,l)\in E^1} c_{il}x'_{il} + \sum_{(i,l)\in E^2} c_{il}x'_{il} + \varepsilon_0, \\
&= \sum_{(i,l)\in E^2} \tilde{c}_{il}x'_{il} + \varepsilon_0, \\
&\geq \sum_{(i,l)\in E} \tilde{c}_{il}x'_{il} \\
&= \sum_{(i,l)\in E^1} \tilde{c}_{il}x'_{il} + \sum_{(i,l)\in E^2} \tilde{c}_{il}x'_{il}, \\
&= \sum_{(i,l)\in E^1} \varepsilon_i x'_{il} + \sum_{(i,l)\in E^2} \tilde{c}_{il}x'_{il}, \\
&= V(x',\tilde{c}).
\end{aligned}$$

or, equivalently, as:

$$\varepsilon_0 \geq \sum_{(i,l)\in E^1} \varepsilon_i x'_{il}. \tag{8}$$

Conditions (6) and (1') together with the fact that $\varepsilon_i x_{il} \geq 0$ for all $(i,l) \in N \times M$ now give:

$$\varepsilon_0 > \sum_{i\in N} \varepsilon_i t_i \geq \sum_{(i,l)\in E^1} \varepsilon_i x'_{il}.$$

But then condition (8) must hold. Hence, $\mathcal{V}(R,\tilde{c}) \subseteq \mathcal{V}(R,c)$. Thus, by Lemma 1 $\varphi(R) \in \mathcal{X}(R)$.

To conclude the proof, it needs only to be demonstrated that $\varphi$ is a priority mechanism. But this follows directly from the construction of the weights $\varepsilon_i$. To see this, recall from condition (6) that $\varepsilon_i > \sum_{k=i+1}^{n} t_k\varepsilon_k$ for all $i \in \{1,\ldots,n-1\}$. Hence, assigning *one* additional time unit to agent $i$

in maximization problem (5) is strictly preferred to assigning $t_j$ time units to each agent $j \in N$ with $\pi(i) < \pi(j)$. Thus, $\mathcal{V}(R, \tilde{c})$ is a selection from $\mathcal{V}(R, c) \subseteq \mathcal{X}(R)$ that first maximizes the number of time units that agent $i_1 \in N$ with $\pi(i_1) = 1$ exchanges with acceptable agents (i.e., a selection from the set $\mathcal{Z}^{i_1, \mathcal{V}(R,c)}(R)$), and then maximizes the number of time units that agent $i_2 \in N$ with $\pi(i_1) = 2$ exchanges with acceptable agents (i.e., a selection from the set $\mathcal{Z}^{i_2, i_1}(R)$), and so on. This is the definition of a priority mechanism. □

**Proof of Corollary 1.** Because any deterministic priority mechanism $\varphi^\pi$ is non-manipulable, for all $R, R' \in \tilde{\mathcal{R}}$ such that $R' = (R'_i, R_{-i})$ for some $i \in N$, we have $\varphi^\pi_i(R) R_i \varphi^\pi_i(R')$. But then the random priority mechanism is sd-non-manipulable.

For sd-efficiency, note that $\phi(R)$ is ex-post maximal, and for all $x, y \in supp(\phi(R))$ we have:

$$\sum_{i \in N} \sum_{j \in A_i(R_i)} x_{ij} = \sum_{i \in N} \sum_{j \in A_i(R_i)} y_{ij} \equiv m.$$

Now if for some random allocation $p$ for $R$ we have $p_i R_i^{sd} \phi_i(R)$ for all $i \in N$ and $p_j P_i^{sd} \phi_j(R)$ for some $j \in N$, then $p$ is ex-post individually rational. But then it follows that:

$$\sum_{i \in N} \sum_{x \in supp(p)} \sum_{j \in A_i(R_i)} x_{ij} > m.$$

But then there must exist $y \in supp(p)$ such that $\sum_{i \in N} \sum_{j \in A_i(R_i)} y_{ij} > m$, which is a contradiction to ex-post maximality of $\phi(R)$. Thus, $\phi$ is sd-efficient.

For sd-fairness, note that for any $i, j \in N$, the probability that $i$ occupies the first position in a priority ordering is equal to the probability that $j$ occupies the first position in a priority ordering. Similarly, for any $l \in N \backslash \{i, j\}$, the probability that $i$ occupies the second position after $l$ in a priority ordering is equal to the probability that $j$ occupies the second position after $l$ in a priority ordering, and so on. But then it follows that the random priority mechanism is sd-fair. □

**Proof of Corollary 2.** Suppose that $Y$ is not stable. As $Y$ is individually rational, there exists $Z \subseteq \mathcal{Y} \backslash Y$ such that for all $i \in N(Z) = \{i \in N : Z_i \neq \emptyset\}$ and all $W \in C_i(Y \cup Z)$, $Z_i \subseteq W_i$. But then for all $i \in N(Z)$, $Y_i \notin C_i(Y \cup Z)$ and $U_i(Y \cup Z) > U_i(Y)$. Thus, $Z \neq \emptyset$. Let $\hat{Z} = \{k_{ij} \in Z : (k-1)_{ij} \notin Z\}$. Thus, $\hat{Z}$ collects the "additional" time units provided in $Z$ compared to $Y$.

Because $Z \neq \emptyset$ and $Z \cap Y = \emptyset$, we have $\hat{Z} \neq \emptyset$. Thus, there exist $i, j \in N(Z)$ such that $i \neq j$ and $k_{ij} \in \hat{Z}$. If $\hat{Z}_{\to j} = \emptyset$, then $Z_{\to j} = \emptyset$ and $Y_i \in C_i(Y \cup Z)$ (as $j$ does not get additional services in $Z$), a contradiction. Thus, $\hat{Z}_{\to j} \neq \emptyset$ and $h_{jl} \in \hat{Z}$ for some $l \in N(Z)$, and so on. But then we must a find a cycle $i_1, \ldots, i_m$ such that $k^1_{i_1 i_2} \in \hat{Z}, \ldots, k^m_{i_m i_1} \in \hat{Z}$. But now $Y \cup \{k^1_{i_1 i_2}, \ldots, k^m_{i_m i_1}\}$ is a feasible and individually rational allocation, a contradiction to the maximality of $Y$. □

**Proof of Lemma 2.** We only show (1) and (2) can be shown similarly. Let $Y, Z \subseteq \mathcal{Y}_i$ be such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Let $Y^* \in C_i(Y)$. Then (iii) is trivially satisfied as $|Y^*_{\to i}| - |Y^*_{i\to}| = 0 = |Z^*_{\to i}| - |Z^*_{i\to}|$ for all $Z^* \in C_i(Z)$.

If for some $Z^* \in C_i(Z)$, $U_i(Y^*) = U_i(Z^*)$, then $Y^* \in C_i(Z)$ and (i) and (ii) hold trivially. Otherwise, for some $Z^* \in C_i(Z)$, $U_i(Y^*) < U_i(Z^*)$. But then choose $U_i(Z^*) - U_i(Y^*)$ contracts from $Z^* \backslash Y^*$, say the set $\hat{Z}$, such that $|\hat{Z}_{\to i}| = |\hat{Z}_{i\to}|$, $k_{ij} \in \hat{Z}$ with $k > 1$ implies $(k-1)_{ij} \in \hat{Z} \cup Y^*$, and $k_{ji} \in \hat{Z}$ with $k > 1$ implies $(k-1)_{ji} \in \hat{Z} \cup Y^*$. Then we obtain $Y^* \cup \hat{Z} \in C_i(Z)$ (because $U_i(Y^* \cup \hat{Z}) = U_i(Z^*)$), and (i), (ii) and (iii) holds for $Y^*$ and $Y^* \cup \hat{Z}$. $\square$

## Appendix B: Proof of Theorem 1

This Appendix first introduces a graph theoretical tool, referred to as the circulation-based model (Appendix B.1). It will then be demonstrated that the circulation-based model, without loss of generality, can replace the min-cost flow problem when analysing the priority mechanism (Appendix B.2). These insights enable us to prove Theorem 1 (Appendix B.3).

### Appendix B.1: The Circulation-Based Model

Let $\mathbb{Z}$ denote the set containing all integers. For any profile $R \in \tilde{\mathcal{R}}$, construct a weighted directed graph $D_R = (V, A)$ with capacities $c : A \mapsto \mathbb{N}_0$ and weights $w : A \mapsto \mathbb{Z}$ on its arcs. For ease of notation, we write $D$ instead of $D_R$ whenever the profile $R$ is unambiguous. Each agent $i \in N$ is represented by two vertices, denoted by $i^{in}$ and $i^{out}$. These $2n$ vertices build the vertex set $V$ of the graph $D$. We draw a directed arc between each pair of type $(i^{in}, i^{out})$, pointing to $i^{out}$ and refer to this arc as the *inner arc* of agent $i \in N$. The inner arc has capacity $c(i^{in}, i^{out}) = t_i$. If agent $i$ finds agent $j$ acceptable, then $(j^{out}, i^{in})$ belongs to the (directed) arc set $A$ of the graph $D$. Any such arc is called *regular* and has capacity $c(j^{out}, i^{in}) = \bar{t}_{ij}$, i.e., the upper time bound on how much time agent $i$ wants from agent $j$. Note also that the vertices of type $i^{in}$ have incoming regular arcs and a single outgoing inner arc, while vertices of type $i^{out}$ have outgoing regular arcs and a single incoming inner arc. We define in Appendix B.2 the weights $w : A \mapsto \mathbb{Z}$ using a priority order. An instance of the model is illustrated in Figure 2 (the figure contains some concepts which only are explained later in the Appendix).

**Definition 5.** A *circulation* is a function $C : A \mapsto \mathbb{N}_0$ where:

(i) $C(u, v) \leq c(u, v)$ for every $(u, v) \in A$,

(ii) $\sum_{(u,v)\in A} C(u, v) = \sum_{(v,w)\in A} C(v, w)$ for every vertex $v \in V$.

Condition (i) is a capacity constraint which ensures that agents do not exchange services beyond their time endowment $t_i = c(i^{in}, i^{out})$, and that the upper time bound $\bar{t}_{ij}$ on how much time agent $i$ wants from agent $j$ is not exceeded. Condition (ii) is the classical flow conservation rule, stating that the total flow of the incoming arcs of a vertex equals the total flow of the outgoing arcs, i.e., that an agent provides and receives the same amount of time. The latter condition can also be formulated as:

$$C(i^{in}, i^{out}) = \sum_{(j^{out}, i^{in})\in A} C(j^{out}, i^{in}) = \sum_{(i^{out}, k^{in})\in A} C(i^{out}, k^{in}) \text{ for every agent } i \in N.$$

We call $C(i^{in}, i^{out})$ the *flow value at agent* $i$. Circulations in a graph $D$ are in one-to-one correspondence with allocations in the time banking problem, e.g., for an allocation $x$ the corresponding flow value of the inner arc at agent $i$ is $C(i^{in}, i^{out}) = t_i - x_{ii}$ and the flow value of any regular arc at agent $i$ is $C(j^{out}, i^{in}) = x_{ij}$ for all $j \in N$. The *allocation value* for agent $i$ is defined as $t_i - x_{ii}$. Another way of expressing this is that the allocation value $t_i - x_{ii}$ of agent $i$ in the time banking problem equals the flow value $C(i^{in}, i^{out})$ at agent $i$ in the circulation model.

**Appendix B.2: Replacement Result**

This section demonstrates that by placing appropriate weights on the arcs in the graph $D$, the maximum weight circulations correspond to the outcome of the min-cost flow problem used in Section 5 to identify the outcome of the priority mechanism (Proposition 2). This result implies that the circulation-based model can be adopted in the proof of Theorem 1. We remark that both maximum weight circulations and min-cost flows can be computed efficiently, or more precisely, in $O(|E|^2 \log |V|)$ time in a graph with $|V|$ vertices and $|E|$ arcs , which is strongly polynomial time (Orlin, 1993).

Let $\pi$ be a priority ordering. Let $t_{max}$ be the largest time endowment of any agent in $N$, and define the weight $w(u, v)$ on each arc $(u, v)$ in the directed graph $D = (V, A)$ by:

$$w(u,v) = \begin{cases} t_{max}^{2(n+1-\pi(i))} & \text{if } (u,v) = (i^{in}, i^{out}), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

We now illustrate our transformation from the priority mechanism to the circulation-based model by means of a simple example.

**Example 4.** Agents are denoted by $i, j, k$ and $l$. The upper bounds on the needed services are as follows: $\bar{t}_{ki} = 1, \bar{t}_{ji} = 2, \bar{t}_{kj} = 2, \bar{t}_{lj} = 1, \bar{t}_{ik} = 3, \bar{t}_{il} = 1, \bar{t}_{jl} = 3$. All other upper bounds are set to 0. Each agent has an endowment of 3, and their priority order is alphabetic.

In the circulation-based model, there are two vertex copies to each agent, connected by an inner arc. The amount of service each agent is willing to accept from another agent translates into an upper capacity on the regular arc connecting the out-vertex of the provider and the in-vertex of the receiver. In Figure 2, inner arcs are marked by horizontal lines, while regular arcs are bent and colored. Arc weights and capacities are written above and below each arc, respectively. Due to the alphabetic priority order and $t_{max}$ being 3 units, the arc weights of agents $i, j, k$ and $l$ on the inner arcs are given by $3^8$, $3^6$, $3^4$ and $3^2$, respectively. All arc weights on regular arcs are set to zero.

The max weight circulation in the network can be computed efficiently and it has weight $3 \cdot 3^8 + 3 \cdot 3^6 + 3 \cdot 3^4 + 1 \cdot 3^2$. It saturates all edges except the dotted $(l^{out}, i^{in})$ which is left empty, and the dashed $(l^{out}, j^{out})$ and $(l^{in}, i^{out})$, both of which carry one unit of flow. More precisely, agent $i$ sends 2 time units to agent $j$ and 1 time unit to agent $k$, agent $j$ sends 2 time units to agent $k$ and 1 time unit to agent $l$, agent $k$ sends 3 time units to agent $i$, and agent $l$ sends 1 time unit to agent $j$. $\square$

Let $w(C)$ denote the *weighted sum of flow values* of the agents in $N$ at circulation $C$, i.e., $w(C) = \sum_{i \in N} C(i^{in}, i^{out}) \cdot w(i^{in}, i^{out})$.

**Proposition 2.** For any given profile $R \in \tilde{\mathcal{R}}$, let $C$ be a maximum weight circulation where the weights are defined by condition (9). Let $C'$ be the circulation corresponding to an allocation $x'$ selected for $R$ by a priority mechanism $\varphi$ based on $\pi$. Then $C'(i^{in}, i^{out}) = C(i^{in}, i^{out})$ for each $i \in N$.

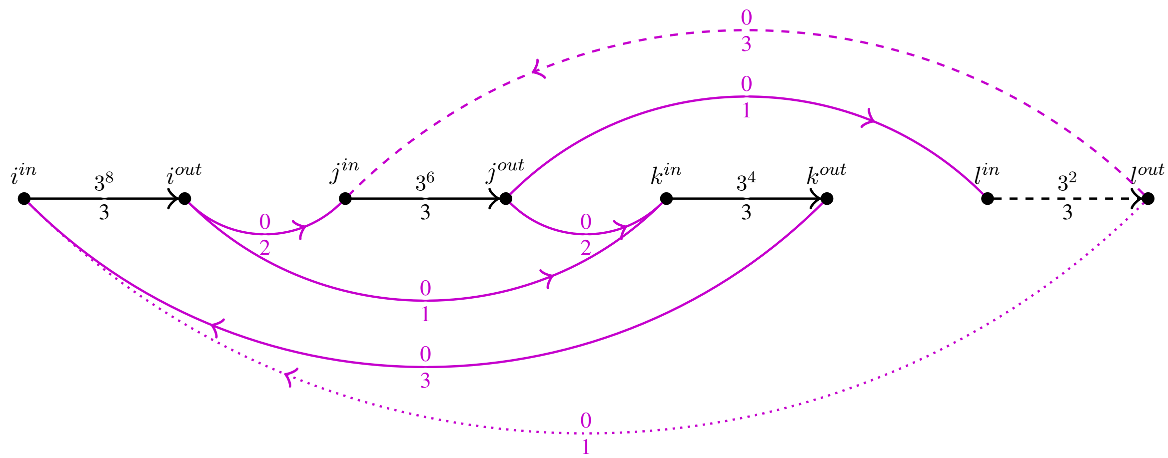


Figure 2: The circulation-based model for the instance in Example 4.

*Proof.* As in the statement of the proposition, let $C$ be a maximum weight circulation and let $C'$ be the circulation corresponding to an allocation $x'$ selected by a priority mechanism. Suppose, to obtain a contradiction, that $C'(j^{in}, j^{out}) \neq C(j^{in}, j^{out})$ for some $j \in N$. Let agent $i$ be the agent with the highest priority in $\pi$ where this holds. Suppose also, without loss of generality, that $\pi(k) = k$ for all $k \in N$. To reach the desired contradiction, we consider two cases.

*Case* (i): $C'(i^{in}, i^{out}) < C(i^{in}, i^{out})$. In this case, the maximum weight circulation $C$ assigns a higher allocation value to agent $i$ than the priority mechanism. We show by induction that this contradicts the rules of the priority mechanism. Suppose first that agent $i$ is the highest ranked agent according to the priority order $\pi$ and recall that the priority mechanism, by construction, restricts the set of maximal allocations to those that maximize the allocation value of $i$ (see condition (3) in Section 4). Thus there is no allocation that assigns agent $i$ a higher allocation value than the allocations in this chosen set, and, consequently, no circulation that assigns agent $i$ a higher value. Hence, agent $i$ cannot be the agent with the highest priority. Suppose now that agent $i$ is the second highest ranked agent according to the priority order $\pi$. Again, by condition (3) this agent restricts the set of allocations further. And so, the maximum weight circulation $C$ is still in the chosen set when agent $i$ restricts the set of allocations further, and it can, consequently, not have a higher allocation value for agent $i$ than $C'$. This argument can be repeated inductively to reach the conclusion that it cannot be the case that $C'(i^{in}, i^{out}) < C(i^{in}, i^{out})$.

*Case* (ii): $C'(i^{in}, i^{out}) > C(i^{in}, i^{out})$. Note first that both $C$ and $C'$ are feasible circulations at profile $R$. Because agent $i$ is the agent with the highest priority in $\pi$ where $C'(i^{in}, i^{out}) \neq C(i^{in}, i^{out})$, by assumption, it follows that $C'(k^{in}, k^{out}) = C(k^{in}, k^{out})$ for all agents $k = 1, \ldots, i-1$. It will be demonstrated that agents $i+1, \ldots, n$ cannot make up for the loss $C$ suffered on arc $(i^{in}, i^{out})$ and thus $C$ cannot be of maximum weight since $C'$ is a feasible circulations at profile $R$. Recall first that the set $\mathbb{N}_0$ contains only positive integers, so the difference between $C'(i^{in}, i^{out})$ and $C(i^{in}, i^{out})$ is at least 1. By construction of the weights on the inner arcs, defined by condition (9), it then follows that:

$$[C'(i^{in}, i^{out}) - C(i^{in}, i^{out})] \cdot t_{max}^{2(n-i+1)} \geq t_{max}^{2(n-i+1)}. \tag{10}$$

Note next that, in the the extreme case, all agents with lower priorities than agent $i$ have flow value zero in $C'$ and a flow value of $t_{max}$ in $C$. This means that the weighted sum of the flow values at agents $i+1, \ldots, n$ at circulation $C$ is at most:

$$t_{max} \cdot \sum_{j=i+1}^{n} t_{max}^{2(n-j+1)}. \tag{11}$$

Now, the value of the sum (11) is strictly lower than the right hand side of inequality (10). Consequently, even in the the extreme case when all agents with lower priorities than agent $i$ have flow value zero in $C'$ and a flow value of $t_{max}$ in $C$, it holds that $w(C') > w(C)$. However, this contradicts that $C$ is a maximum weight circulation since $C'$ is a feasible circulation at graph $D_R$. □

**Appendix B.3: The Proof**

Let $\varphi$ be the priority mechanism based on $\pi$ where $\pi(i) = i$ for all $i \in N$. To obtain a contradiction, suppose that $\varphi$ can be manipulated by some agent $i \in N$ at a profile $R \in \tilde{\mathcal{R}}$. This means that there are two profiles $R \in \tilde{\mathcal{R}}$ and $R' = (R'_i, R_{-i}) \in \tilde{\mathcal{R}}$ such that for $x = \varphi(R)$ and $x' = \varphi(R')$ we have $x'_i P_i x_i$. Note that $R'_i \neq R_i$. Let $C^1$ and $C^2$ be the maximum weight circulations for the graphs $D_R$ and $D_{R'}$ induced by the profiles $R$ and $R' = (R'_i, R_{-i})$, respectively.

The next lemma shows that we may suppose that the set of acceptable agents reported by agent $i$ at preference relation $R'_i$ is a proper subset of the set of acceptable agents reported by agent $i$ at preference relation $R_i$.

**Lemma 3.** Without loss of generality, we may suppose $A_i(R'_i) \subseteq A_i(R_i)$.

*Proof.* We first show $U_i(R_i) \subseteq U_i(R'_i)$. To see this, suppose $j \in U_i(R_i)$ but $j \notin U_i(R'_i)$, i.e., that agent $j$ is unacceptable under $R_i$ but acceptable under $R'_i$. Since $x'_i P_i x_i$, it must then hold that $x'_{ij} = 0$ by definition of the preferences in $\tilde{\mathcal{R}}_i$. Hence, any regular arc of type $(j^{out}, i^{in})$ where $j \notin U_i(R'_i)$ in the graph $D_{R'}$ but $j \in U_i(R_i)$ in the graph $D_R$ will not be active in the solution $C^1$ at profile $R'$. Hence, $U_i(R_i) \subseteq U_i(R'_i) \cup \{j \in A_i(R'_i) : x'_{i(n+j)} = 0\}$. But then we may choose $R''_i$ such that $A_i(R''_i) = A_i(R'_i) \backslash \{j \in A_i(R'_i) : x'_{i(n+j)} = 0\}$ and $\bar{t}''_{ik} = \bar{t}'_{ik}$ for all $k \in A_i(R''_i)$, and $C^1$ remains a solution for $R'' = (R''_i, R_{-i}) \in \tilde{\mathcal{R}}$. But for $x'' = \varphi(R'')$ this implies $x''_i I_i x'_i$ and $x''_i P_i x_i$. Hence, $A_i(R''_i) \subseteq A_i(R_i)$ and $x''_i P_i x_i$. □

Recall now that, for any profile in $R \in \tilde{\mathcal{R}}$, each agent $k \in N$ reports a set of acceptable agents $A_k(R_k)$ together with an upper bound on how much time $\bar{t}_{kj}$ agent $k \in N$ at most would like to receive from each acceptable agent $j \in A_k(R_k)$. By Remark 1, the report $R_k$ is equivalent to the vector $\bar{t}_k$ where $\bar{t}_{kk} = t_k$ and $\bar{t}_{kj} = 0$ for all $j \in U_k(R_k)$. This together with the conclusion in Lemma 3 imply that there exists at least one agent $j$ that is acceptable for agent $i$ under $R_i$ where agent $i$ reports a strictly lower or higher time bound $\bar{t}'_{ij}$ at profile $R'$ than under profile $R$ (i.e., $\bar{t}'_{ij} < \bar{t}_{ij}$ or $\bar{t}'_{ij} > \bar{t}_{ij}$ ). In general, a manipulation $R'_i$ by agent $i$ can consist of both underreporting and overreporting $\bar{t}_{ij}$'s for acceptable agents. There are two possible cases for manipulations: one with overreporting and the other with only underreporting time bounds.

First, consider the case where there is overreporting. If there exists $j \in N \backslash \{i\}$ such that $x'_{ij} > \bar{t}_{ij}$, then by definition of $\tilde{\mathcal{R}}_i$, $\omega_i P_i x'_i$ and since $x$ is individually rational under $R$, we have $x_i P_i x'_i$, a contradiction. Otherwise $x'_{ij} \leq \bar{t}_{ij}$ for all $j \in N \backslash \{i\}$ and we can just replace $\bar{t}'_i$ with $\bar{t}''_i$ such that $\bar{t}''_{ij} = \min\{\bar{t}_{ij}, \bar{t}'_{ij}\}$ for all $j \in N \backslash \{i\}$. Let $R''_i$ denote $i$'s preference associated with $\bar{t}''_i$. Then $x'$ is still a maximizer for the profile $(R''_i, R_{-i})$ and therefore the manipulation only consists of underreporting upperbounds which are below $\bar{t}_i$.

Second, it remains to establish that agent $i$ cannot manipulate by underreporting time bounds for acceptable agents, i.e., $\bar{t}'_{ij} \leq \bar{t}_{ij}$ for all $j \in N \backslash \{i\}$. Below we are going to show that agent $i$ cannot gain by underreporting one time bound for an acceptable agent. This is enough to establish that agent $i$ never can gain by reporting a lower bound for several agents at the same time. Because any such misreport can be decomposed into a sequence of manipulations in which at each step only one upper bound $\bar{t}_{ij}$ is changed at the time and agent $i$ is never made better off at any step. Formally, let $k \in A_i(R_i)$ for which $\bar{t}'_{ik} < \bar{t}_{ik}$ and consider the misreport $\bar{t}^{(1)}_i$ where $\bar{t}^{(1)}_{ij} = \bar{t}_{ij}$ for all $j \neq k$ and $\bar{t}^{(1)}_{ik} = \bar{t}'_{ik}$. Let $x^{(1)}$ be the allocation chosen by the priority mechanism when $i$ reports $t^{(1)}$. Below we show that agent $i$ cannot gain by reporting $\bar{t}^{(1)}_i$ instead of $\bar{t}_i$. In particular, $\sum_{j \in A_i(R_i)} x_{ij} \geq \sum_{j \in A_i(R_i)} x^{(1)}_{ij}$. Thus, $x_i R_i x^{(1)}_i$. If there is another agent $\ell \neq k$ such that $\bar{t}^{(1)}_{i\ell} \neq \bar{t}'_{i\ell}$ then consider $\bar{t}^{(2)}$ where $\bar{t}^{(2)}_{ij} = \bar{t}^{(1)}_{ij}$ for all $j \neq \ell$ and $\bar{t}^{(2)}_{i\ell} = \bar{t}'_{i\ell}$. Suppose again that agent $i$ cannot gain by reporting $\bar{t}^{(2)}_i$ instead of $\bar{t}^{(1)}_i$. This means again that $\sum_{j \in A_i(R_i)} x^{(1)}_{ij} \geq \sum_{j \in A_i(R_i)} x^{(2)}_{ij}$. Thus, by transitivity $x_i R_i x^{(2)}$. This argument can be repeated inductively until the point that $\bar{t}^{(p)}_i = \bar{t}'_i$, and if in each step agent $i$ never gains by reporting $\bar{t}^{(j)}_i$ instead of $\bar{t}^{(j-1)}$ we have shown that agent $i$ cannot gain by reporting $\bar{t}'_i$ instead of $\bar{t}_i$. Hence, to complete the proof of Theorem 1, it is enough to show that agent $i$ cannot gain by misreporting $\bar{t}'_{ij}$ for one agent $j \in A_i(R_i)$.

It only remains to rule out that agent $i$ cannot gain by reporting a strictly lower time bound $\bar{t}_{ij}$. Translating this into the terminology of the circulation-based model, this can equivalently be expressed as the flow value $C(i^{in}, i^{out})$ at agent $i$ in a maximum weight circulation cannot be increased by reducing the capacity on a regular arc $(j^{out}, i^{in})$. Given this insight, a large part of the remaining proof will focus on a regular arc $(j^{out}, i^{in})$.

Recall now that $C^1$ denotes the maximum weight circulations for the true preferences $R$ induced by the graph $D_R$, and that $C^2$ denotes the maximum weight solution for the misrepresented preferences $R'$ induced by the graph $D_{R'}$. Furthermore, by the assumption that agent $i$ can manipulate the priority mechanism, it follows that $C^2$ has a larger flow value at agent $i$ than $C^1$ does, i.e., $C^2(i^{in}, i^{out}) > C^1(i^{in}, i^{out})$. By construction of the weights in condition (9), the circulation value of $C^2$ cannot be the same as the circulation value of $C^1$ if the flow value differs for at least one agent. Thus, the circulation value of $C^2$ must be strictly smaller than the circulation value of $C^1$, i.e., $w(C^2) < w(C^1)$. Note also that the circulation $C^2$ is a feasible circulation in $D_R$ since the flows remain below the capacity on each arc and it preserves flow conservation. However, the circulation $C^2$ is not optimal in the graph $D_R$ since the circulation value of $C^2$ is strictly smaller than the circulation value of $C^1$ and the circulation $C^1$ is optimal in $D_R$.

Consider next the function defined by the circulation $C^1 - C^2$ where $C^1(u, v) - C^2(u, v) \in \mathbb{Z}$

for each arc $(u, v)$ in the graph $D_R$. This function assigns a negative value to the arc $(u, v)$ if the flow through the arc is larger in circulation $C^2$ than in circulation $C^1$. For convenience, one can think of these "negative" arcs as arcs turned backwards, with the usual positive flow value on them. Since both $C^1$ and $C^2$ are circulations in the graph $D_R$, their difference also obeys flow conservation and as such, it can be decomposed into cycles.

A cycle decomposition is a collection of directed cycles in the graph so that the flow value on all edges of a specific cycle in the decomposition is the same, and the sum of flow values in all cycles containing an arc $(u, v)$ equals the flow value on $(u, v)$. The capacity or the weight on the edges plays no role in the decomposition. It is known that any feasible circulation has a cycle decomposition (Ford and Fulkerson, 1956). In the next paragraph, we will construct such a cycle decomposition of the circulation $C^1 - C^2$. For simplicity, we will decompose our circulation into cycles of flow value 1.

Note first that a cycle decomposition of $C^1 - C^2$ need not be unique for the profiles $R$ and $R'$. To obtain one such decomposition, we use a simple inductive algorithm that produces a cycle decomposition of $C^1 - C^2$ in a finite number of iterations. This algorithm uses the flow value of $C^1 - C^2$ on each arc $(u, v)$ in the graph $D_R$ but will not use any information about the arc capacities or weights (arc weights are only considered below). First, identify any directed cycle, say $\mathcal{C}$, based on the circulation $C^1 - C^2$ and take its forward or backward arc with a lowest absolute flow value on it. Suppose that the lowest absolute flow value at some agent in the cycle $\mathcal{C}$ is $q$, then $q$ feasible cycles of type $\mathcal{C}$ can be identified. These cycles represent the first $q$ cycles in the decomposition of $C^1 - C^2$. Then, reduce the flow value on each arc included in the cycle $\mathcal{C}$ by $q$. This will give an "updated" circulation, based on the "original" circulation $C^1 - C^2$. Notice that the updated circulation is indeed a circulation, preserving flow conservation at each vertex, but compared to $C^1 - C^2$, it is guaranteed to have at least one more arc with zero flow value. We proceed in this manner until the whole circulation $C^1 - C^2$ is decomposed into cycles. Note also that since $\mathbb{N}_0$ is restricted to a set of positive bounded integers, this procedure ends in a finite number of iterations. Moreover, the absolute flow value on an arc monotonically (but not strictly monotonically) decreases in each inductive step, until it reaches 0.

Note that the cycles in the decomposition are not necessarily arc-disjoint from each other (i.e., several distinct cycles in the decomposition can pass through the same arc), but due to the inductive argument above, each arc in the cycle decomposition is either a forward arc or a backward arc, depending on the sign of $C^1(u, v) - C^2(u, v)$. More precisely, forward arcs are positive, while backward arcs are negative. Thus, it cannot be the case that one cycle in the decomposition uses an arc with positive value, while another cycle uses the same arc with negative value.

Consider now the cycle decomposition of the circulation $C^1 - C^2$ as constructed above. We now turn towards arc weights: the total weight of a cycle in the decomposition is defined as the sum of weights on each arc in the cycle. Based on the sign of the total weight of a cycle, we distinguish positive, zero and negative weight cycles in our decomposition. A positive weight cycle is called an *augmenting cycle*. Note that all augmenting cycles pass through $(j^{out}, i^{in})$, because any augmenting cycle which does not pass through $(j^{out}, i^{in})$ would increase the circulation value of $C^2$ in $D_{R'}$, which is impossible since $C^2$ is optimal in the graph $D_{R'}$.

**Lemma 4.** Suppose that $C^1 - C^2$ is decomposed into cycles using the inductive decomposition algorithm from above. Then:

(i) there exists an augmenting cycle,

(ii) a cycle of weight zero consists exclusively of arcs of weight zero,

(iii) there are no negative weight cycles.

*Proof.* The proof of Part (i) follows directly since $w(C^1) > w(C^2)$ and $w(C^1)$ equals $w(C^2)$ plus the weight of each cycle in the cycle decomposition of $C^1 - C^2$. Part (ii) follows by construction of the weights, i.e., a cycle of weight zero consists exclusively of arcs of weight zero (obviously, no combination of the weights on inner arcs with coefficients in the open interval between 0 and $t_{max}$ can add up to zero).

Part (iii) is proved by contradiction. Suppose that there is a cycle $\mathcal{C}$ of negative total weight in the cycle decomposition of $C^1 - C^2$. Let the reverse of $\mathcal{C}$ be denoted by $\overleftarrow{\mathcal{C}}$. The reverse $\overleftarrow{\mathcal{C}}$ has positive total weight and preserves the sign of $C^2 - C^1$ on each of its arcs by construction of the inductive decomposition algorithm. Moreover, we will show that, $\overleftarrow{\mathcal{C}}$ can be added to $C^1$ without violating flow conservation or any capacity constraint in $D_R$. Thus, $C^1 + \overleftarrow{\mathcal{C}}$ is a circulation of larger weight than $C^1$. Let now $(u, v)$ be an arbitrary arc in the reverse cycle $\overleftarrow{\mathcal{C}}$. It will be demonstrated that:

$$0 \leq C^1(u, v) + \overleftarrow{\mathcal{C}}(u, v) \leq c(u, v). \tag{12}$$

Condition (12) implies that $C^1$ cannot be a maximum weight circulation in the graph $D_R$ which contradicts our assumption. We need to consider two cases. Suppose first that $\overleftarrow{\mathcal{C}}(u, v) \geq 0$. Then:

$$C^1(u, v) + \overleftarrow{\mathcal{C}}(u, v) \leq C^1(u, v) + [C^2(u, v) - C^1(u, v)] = C^2(u, v) \leq c(u, v).$$

Note also that because $C^1(u, v)$ and $\overleftarrow{\mathcal{C}}(u, v)$ are non-negative at the arc $(u, v)$, it follows directly that $C^1(u, v) + \overleftarrow{\mathcal{C}}(u, v) \geq 0$. Hence, condition (12) holds when $\overleftarrow{\mathcal{C}}(u, v) \geq 0$. Suppose next that $\overleftarrow{\mathcal{C}}(u, v) < 0$. In this case:

$$C^1(u, v) + \overleftarrow{\mathcal{C}}(u, v) < C^1(u, v) \leq c(u, v).$$

Furthermore:

$$C^1(u, v) + \overleftarrow{\mathcal{C}}(u, v) \geq C^1(u, v) + [C^2(u, v) - C^1(u, v)] = C^2(u, v) \geq 0.$$

Hence, condition (12) also holds when $\overleftarrow{\mathcal{C}}(u, v) < 0$. □

Lemma 4 thus demonstrated that all cycles in the cycle decomposition of $C^1 - C^2$, which pass through an inner arc, are augmenting cycles. However, we do not know whether these cycles use the arc $(j^{out}, i^{in})$ as a forward arc or as a backward arc. The following lemma sheds light on this.

**Lemma 5.** Suppose that $C^1 - C^2$ is decomposed into cycles using the inductive decomposition algorithm from the above, and let $(j^{out}, i^{in})$ be an arbitrary arc in some cycle in the cycle decomposition of $C^1 - C^2$. Then $(j^{out}, i^{in})$ is a forward arc.

*Proof.* Note first that $C^2(j^{out}, i^{in})$ is bounded from above by the decreased capacity of $(j^{out}, i^{in})$ in $D_{R'}$. If $C^1(j^{out}, i^{in}) \leq C^2(j^{out}, i^{in})$, then $C^1$ is feasible in the graph $D_{R'}$ and has a larger weight than

$C^2$, which contradicts the optimality of $C^2$ in the graph $D_{R'}$. Thus, $C^1(j^{out}, i^{in}) - C^2(j^{out}, i^{in}) > 0$, which implies that $(j^{out}, i^{in})$ is a forward arc in all cycles in the decomposition of $C^1 - C^2$. $\square$

Finally, consider the flow value $C^1(i^{in}, i^{out}) - C^2(i^{in}, i^{out})$. To prove Theorem 1, we only need to establish that $C^1(i^{in}, i^{out}) - C^2(i^{in}, i^{out}) \geq 0$ because this contradicts the assumption that $x_i' P_i x_i$. For this condition to be false, the arc $(i^{in}, i^{out})$ must be a backward arc in at least one cycle in the cycle decomposition of $C^1 - C^2$. However, as concluded in the above, being a backward arc in one cycle also implies being a backward arc in all cycles. From Lemma 4 we know that all cycles that passes through $(i^{in}, i^{out})$ are augmenting cycles. Lemma 5 then states that the augmenting cycles use $(j^{out}, i^{in})$ as a forward arc, and they must, consequently, leave $i^{in}$ either as a forward arc, along the only outgoing arc $(i^{in}, i^{out})$, or as a backward arc, along any of the regular arcs running to $i^{in}$. Neither of these two cases allows $(i^{in}, i^{out})$ to be a backward arc. This concludes the proof and shows that agent $i$ cannot manipulate the priority mechanism $\varphi$ at any profile $R \in \tilde{\mathcal{R}}$.

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