# Dimension approximation of attractors of graph directed IFSs by self-similar sets

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**Abstract**

We show that for the attractor $(K_1, \ldots, K_q)$ of a graph directed iterated function system, for each $1 \leq j \leq q$ and $\varepsilon > 0$ there exists a self-similar set $K \subseteq K_j$ that satisfies the strong separation condition and $\dim_H K_j - \varepsilon < \dim_H K$. We show that we can further assume convenient conditions on the orthogonal parts and similarity ratios of the defining similarities of $K$. Using this property as a 'black box' we obtain results on a range of topics including on dimensions of projections, intersections, distance sets and sums and products of sets.

## 1 Introduction

The main goal of this paper is to develop a tool to deduce results about attractors of graph directed iterated function systems from results that are known for self-similar sets. We proceed by finding a self-similar subset of a given graph directed attractor such that the Hausdorff dimension of the self-similar set is arbitrary close to that of the graph directed attractor and the self-similar set has convenient properties such as the strong separation condition. Similar methods, approximating self-similar sets with well-behaved self-similar sets in the plane, were used by Peres and Shmerkin [15, Proposition 6, Theorem 2], by Orponen [14, Lemma 3.4] and in higher dimensions by Farkas [6, Proposition 1.8]. Fraser and Pollicott [8, Proposition 2.5] showed a result of similar nature for conformal systems on subshifts of finite type. After stating the approximation theorems we deduce many corollaries relating to the dimension of projections and smooth images, the distance set conjecture, the dimension of arithmetic products and dimension conservation.

The paper is organised as follows. In Section 1.1 we introduce notation and definitions. In Section 1.2 we state the main results of this paper, the approximation theorems. In Section 1.3 we deduce various corollaries that follow from the approximation theorems. In Section 1.4 we restate the approximation theorems for subshifts of finite type and note that the results of Section 1.3 remain true. Section 2 contains the proofs of the theorems.

### 1.1 Definitions and Notations

A *self-similar iterated function system* (SS-IFS) in $\mathbb{R}^d$ is a finite collection of maps $\{S_i\}_{i=1}^m$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that all the $S_i$ are contracting similarities. The *attractor* of the SS-IFS is the unique nonempty compact set $K \subseteq \mathbb{R}^d$ such that $K = \bigcup_{i=1}^m S_i(K)$. The attractor of an SS-IFS is called a *self-similar set*. We say that the SS-IFS $\{S_i\}_{i=1}^m$ satisfies the *strong separation condition* (SSC) if the sets $\{S_i(K)\}_{i=1}^m$ are disjoint.

Let $\{S_i\}_{i=1}^m$ be an SS-IFS. Then every $S_i$ can be uniquely decomposed as

$$S_i(x) = r_i T_i(x) + v_i \tag{1}$$

for all $x \in \mathbb{R}^d$, where $0 < r_i < 1$, $T_i$ is an orthogonal transformation and $v_i \in \mathbb{R}^d$ is a translation, for all indices $i$. The unique solution $s$ of the equation

$$\sum_{i=1}^{m} r_i^s = 1 \tag{2}$$

is called the *similarity dimension* of the SS-IFS. It is well-known that if an SS-IFS satisfies the SSC then $0 < \mathcal{H}^s(K) < \infty$. Let $\mathcal{T}$ denote the group generated by the orthogonal transformations $\{T_i\}_{i=1}^m$. We call $\mathcal{T}$ the *transformation group* of the SS-IFS.

We denote the set $\{1, 2, \ldots, m\}$ by $\mathcal{I}$. Let $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ i.e. $\mathbf{i}$ is a $k$-tuple of indices. Then we write $S_\mathbf{i} = S_{i_1} \circ \ldots \circ S_{i_k}$ and $K_\mathbf{i} = S_\mathbf{i}(K)$. When the similarities are decomposed as in (1) we write $r_\mathbf{i} = r_{i_1} \cdot \ldots \cdot r_{i_k}$ and $T_\mathbf{i} = T_{i_1} \circ \ldots \circ T_{i_k}$. For an overview of the theory of self-similar sets see, for example, [3, 4, 11, 12, 16].

Let $G(\mathcal{V}, \mathcal{E})$ be a directed graph, where $\mathcal{V} = \{1, 2, \ldots, q\}$ is the set of vertices and $\mathcal{E}$ is the finite set of directed edges such that for each $i \in \mathcal{V}$ there exists at least one $e \in \mathcal{E}$ starting from $i$. Let $\mathcal{E}_{i,j}$ denote the set of edges from vertex $i$ to vertex $j$ and $\mathcal{E}_{i,j}^k$ denote the set of sequences of $k$ edges $(e_1, \ldots, e_k)$ which form a directed path from vertex $i$ to vertex $j$. A *graph directed iterated function system* (GD-IFS) in $\mathbb{R}^d$ is a finite collection of maps $\{S_e : e \in \mathcal{E}\}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that all the $S_e$ are contracting similarities. The *attractor* of the GD-IFS is the unique $q$-tuple of nonempty compact sets $(K_1, \ldots, K_q)$ such that

$$K_i = \bigcup_{j=1}^{q} \bigcup_{e \in \mathcal{E}_{i,j}} S_e(K_j). \tag{3}$$

The attractor of a GD-IFS is called a *graph directed attractor*.

Let $\{S_e : e \in \mathcal{E}\}$ be a GD-IFS. Then every $S_e$ can be uniquely decomposed as

$$S_e(x) = r_e T_e(x) + v_e \tag{4}$$

for all $x \in \mathbb{R}^d$, where $0 < r_e < 1$, $T_e$ is an orthogonal transformation and $v_e \in \mathbb{R}^d$ is a translation, for all edges $e$.

Let $\mathbf{e} = (e_1, \ldots, e_k) \in \mathcal{E}_{i,j}^k$, then we write $S_\mathbf{e}$ for $S_{e_1} \circ \ldots \circ S_{e_k}$ and $K_\mathbf{e}$ for $S_\mathbf{e}(K_j) \subseteq K_i$. If the similarities are decomposed as in (4) then we write $r_\mathbf{e}$ for $r_{e_1} \cdot \ldots \cdot r_{e_k}$ and $T_\mathbf{e}$ for $T_{e_1} \circ \ldots \circ T_{e_k}$. If $\mathbf{e} = (e_1, \ldots, e_k) \in \mathcal{E}_{i,j}^k$ and $\mathbf{f} = (f_1, \ldots, f_n) \in \mathcal{E}_{j,l}^n$ for $i, j, l \in \mathcal{V}$ then we write $\mathbf{e} * \mathbf{f}$ for $(e_1, \ldots, e_k, f_1, \ldots, f_n) \in \mathcal{E}_{i,l}^{k+n}$.

The directed graph $G(\mathcal{V}, \mathcal{E})$ is called *strongly connected* if for every pair of vertices $i$ and $j$ there exist a directed path from $i$ to $j$ and a directed path from $j$ to $i$. We say that the GD-IFS $\{S_e : e \in \mathcal{E}\}$ is *strongly connected* if $G(\mathcal{V}, \mathcal{E})$ is strongly connected. For an overview of the theory of graph directed attractors see, for example, [4, 13, 17].

The set $\mathcal{C}_i := \bigcup_{k=1}^{\infty} \mathcal{E}_{i,i}^k$ is the set of directed cycles of $G(\mathcal{V}, \mathcal{E})$ that start and end in vertex $i$. Equipped with the $*$ operation $\mathcal{C}_i$ becomes a semigroup. Let $\mathcal{T}_{i,G}$ denote the group generated by the transformations

$$\{T_{e_1} \circ \ldots \circ T_{e_k} : (e_1, \ldots, e_k) \in \mathcal{C}_i\}$$

and we call $\mathcal{T}_{i,G}$ the *i-th transformation group* of the GD-IFS. It is easy to see that if the GD-IFS is strongly connected then $\mathcal{T}_{i,G}$ is conjugate to $\mathcal{T}_{j,G}$ for all $i, j \in \mathcal{V}$ and hence

$$|\mathcal{T}_{i,G}| = |\mathcal{T}_{j,G}|,$$

where $|.|$ denote the cardinality of a set.

To avoid the singular non-interesting case, when every $K_i$ is a single point we make the global assumption throughout the whole paper that there exists $i \in \mathcal{V}$ such that $K_i$ contains at least two points. This implies that if $\{S_e : e \in \mathcal{E}\}$ is strongly connected then every $K_j$ contains at least two points. Note that if $\{S_e : e \in \mathcal{E}\}$ is strongly connected then this assumption also implies that $\dim_H K_i = \dim_H K_j > 0$ for all $i, j \in \mathcal{V}$ even with no separation condition.

We use the following notation throughout the paper. We let $\dim_H K$ be the Hausdorff dimension of a set $K$. Let $\mathbb{O}_d$ be the group of all orthogonal transformations on $\mathbb{R}^d$ and $\mathbb{SO}_d$ be the group of special orthogonal transformations on $\mathbb{R}^d$ both of them equipped with the usual topology. If $\mathcal{T}$ is a set of orthogonal transformations then $\overline{\mathcal{T}}$ denotes the closure of $\mathcal{T}$. We denote the identity map of $\mathbb{R}^d$ by $Id_{\mathbb{R}^d}$. For a linear transformation $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ the Euclidean operator norm is

$$\|T\| = \sup_{x \in \mathbb{R}^d, \|x\|=1} \|Tx\|,$$

where $\|y\|$ denotes the Euclidean norm of $y \in \mathbb{R}^d$. Let $\mathcal{L}^d$ be the $d$-dimensional Lebesgue measure. We write $B(x, \gamma) \subseteq \mathbb{R}^d$ for the open ball of radius $\gamma$ centered at $x$.

## 1.2 The approximation theorems

In this section we state the main results of this paper, the approximation theorems. Given a graph directed attractor we find a self-similar subset that has arbitrarily close dimension and satisfies the strong separation condition. The first result shows that we can further require that the transformation group of the self-similar set is dense in that of the graph directed attractor.

**Theorem 1.1.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then for every $\varepsilon > 0$ there exists an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$ and the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$.*

*Remark* 1.2. Let $\{S_e : e \in \mathcal{E}\}$ and $\left\{\widehat{S}_e : e \in \widehat{\mathcal{E}}\right\}$ be two strongly connected GD-IFS in $\mathbb{R}^d$, with attractors $(K_1, \ldots, K_q)$ and $\left(\widehat{K}_1, \ldots, \widehat{K}_{\widehat{q}}\right)$, such that there exist $j \in \mathcal{V}$, $\widehat{j} \in \widehat{\mathcal{V}}$ and $\mathbf{e} \in \mathcal{C}_j$, $\mathbf{f} \in \widehat{\mathcal{C}}_{\widehat{j}}$ such that $\log r_{\mathbf{e}} / \log r_{\mathbf{f}} \notin \mathbb{Q}$. Then we can find SS-IFS $\{S_i\}_{i=1}^m$ and $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$

that satisfy the SSC, with attractors $K$ and $\widehat{K}$ such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, $\widehat{K} \subseteq \widehat{K}_j$, $\dim_H \widehat{K}_j - \varepsilon < \dim_H \widehat{K}$, the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$, the transformation group $\widehat{\mathcal{T}}$ of $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ is dense in $\widehat{\mathcal{T}}_{j,G}$ and $\log r_1 / \log \widehat{r}_1 \notin \mathbb{Q}$. See Remark 2.7.

In the next result instead of the dense subgroup condition we can require that the first level cylinder sets of the self-similar set are the same size and 'roughly homothetic', i.e. all the similarity ratios are the same and the orthogonal parts are $\varepsilon$-close.

**Theorem 1.3.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then for every $\varepsilon > 0$ and $O \in \overline{\mathcal{T}_{j,G}}$ there exist $r \in (0,1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, and $\|T_i - O\| < \varepsilon$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

If the transformation group is finite then we can even get that $T_i = O$ for every $i$.

**Corollary 1.4.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$ and assume that $\mathcal{T}_{j,G}$ is a finite group. Then for every $\varepsilon > 0$ and $O \in \mathcal{T}_{j,G}$ there exist $r \in (0,1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, and $T_i = O$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

Corollary 1.4 follows easily from Theorem 1.3 by letting

$$\varepsilon = \min\left\{\|T - O\| : T, O \in \mathcal{T}_{j,G}, T \neq O\right\} > 0.$$

One cannot hope to have in Theorem 1.3 that $T_i = O$ for every $i$ because in $\mathbb{R}^3$ there exist two rotations around lines that generate a free group over two elements and so $T_{\mathbf{i}}$ might all be different for every finite word $\mathbf{i}$. However, rotations on the plane commute, hence we can get all $T_i$ to be the same.

**Theorem 1.5.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^2$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then for every $\varepsilon > 0$ there exist $r \in (0,1)$, an orthogonal transformation $O \in \mathcal{T}_{j,G} \cap \mathbb{SO}_2$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G} \cap \mathbb{SO}_2$ and $T_i = O$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

## 1.3 Application of the approximation theorems

In this section we give applications of the dimension approximation results. The first application generalises a result of Hochman and Shmerkin [10, Corollary 1.7] on self-similar sets with SSC to graph directed attractors with no separation condition.

Let $0 < l \leq d$ be integers and let $G_{d,l}$ denote the *Grassmann manifold* of $l$-dimensional linear subspaces of $\mathbb{R}^d$ equipped with the usual topology (see for example [12, Section 3.9]).

**Theorem 1.6.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$, let $U$ be an open neighbourhood of $K_j$ and assume that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$. Then $\dim_H(g(K_j)) = \min\{\dim_H(K_j), l\}$ for every continuously differentiable map $g : U \longrightarrow \mathbb{R}^l$ such that $\operatorname{rank}(g'(x)) = l$ for some $x \in K_j$.*

Since $\text{rank}(g'(x)) = l$ it follows that there exists an open neighbourhood $V$ of $x$ such that $\text{rank}(g'(y)) = l$ for every $y \in V$. Because $g$ is a Lipschitz map $\dim_H (g(K_j)) \leq \min\{\dim_H(K_j), l\}$ is straightforward. Taking a small cylinder set inside $V$ we can further assume that $K_j \subseteq V$ (see Lemma 2.3). The opposite inequality follows by finding $K \subseteq K_j$ as in Theorem 1.1. Applying [10, Corollary 1.7] to $K$ finishes the proof of Theorem 1.6.

The following corollary applies to $g : U \longrightarrow \mathbb{R}^{d_2}$ where the dimension $d_2$ of the ambient space of the image can be greater than $l$.

**Corollary 1.7.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$, let $U$ be an open neighbourhood of $K_j$ and assume that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$. If $g : U \longrightarrow \mathbb{R}^{d_2}$ is a continuously differentiable map ($d_2 \in \mathbb{N}$) such that $\text{rank}(g'(x)) = l$ for every $x \in K_j$ and either of the following conditions is satisfied*

*(i) $g \in C^\infty$,*

*(ii) $\dim_H(K_j) \leq l$,*

*then $\dim_H (g(K_j)) = \min\{\dim_H(K_j), l\}$.*

Corollary 1.7 can be deduced from Theorem 1.6 as [6, Corollary 1.7] is deduced from [6, Theorem 1.6].

Another well-studied topic is Falconer's distance set conjecture. For a set $K$ we denote the *distance set of* $K$ by $D(K) = \{\|x - y\| : x, y \in K\}$. The conjecture is the following (see [2]):

**Conjecture 1.8.** *Let $K \subseteq \mathbb{R}^d$ be an analytic set. If $\dim_H K \geq \frac{d}{2}$ then $\dim_H D(K) = 1$, if $\dim_H K > \frac{d}{2}$ then $\mathcal{L}^1(D(K)) > 0$.*

Orponen [14, Theorem 1.2] showed that for a planar self-similar set $K$ if $\mathcal{H}^1(K) > 0$ then $\dim_H D(K) = 1$. Bárány [1, Corollary 2.8] extended this result by showing that if $K$ is a self-similar set in $\mathbb{R}^2$ and $\dim_H K \geq 1$ then $\dim_H D(K) = 1$. Bárány [1, Theorem 1.2] also showed that if $K \subseteq \mathbb{R}^3$, every $T_i = Id_{\mathbb{R}^3}$ in the SS-IFS of $K$ and $\dim_H K > 1$ then $\dim_H D(K) = 1$. Using our approximation theorems we deduce these results for graph directed attractors.

We define the *pinned distance set of* $K \subseteq \mathbb{R}^d$ to be $D_x(K) = \{\|x - y\| : y \in K\}$ for the pin $x \in \mathbb{R}^d$. Clearly $\dim_H D(K) \geq \dim_H D_x(K)$ for every set $K \subseteq \mathbb{R}^d$ with $x \in K$. For a fixed $x \in \mathbb{R}^d$ the map $D_x(y) = \|x - y\|$ is a locally Lipschitz map. Hence $\dim_H D_x(K) \leq \min\{\dim_H(K_j), 1\}$ for every set $K \subseteq \mathbb{R}^d$.

**Theorem 1.9.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^2$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then $\dim_H D_x(K_j) = \min\{\dim_H(K_j), 1\}$ for every $x \in \mathbb{R}^2$. In particular, if $\dim_H K_j \geq 1$ then $\dim_H D(K_j) = \dim_H D_x(K_j) = 1$ for every $x \in K_j$.*

*Proof.* Let $D_x(y) = \|x - y\|$ for $x, y \in \mathbb{R}^2$. We can find $K \subseteq K_j$ as in Theorem 1.1 for every $\varepsilon > 0$. Let $K_{\mathbf{i}}$ be a cylinder set such that $x \notin K_{\mathbf{i}}$. Then $\Lambda = K_{\mathbf{i}}$ is a self-similar set and $g(y) = D_x(y)$ satisfies the conditions of [1, Theorem 2.7] hence

$$\dim_H D(K_j) \geq \dim_H D_x(K) \geq \dim_H D_x(K_{\mathbf{i}}) = \min\{\dim_H(K_{\mathbf{i}}), 1\} = \min\{\dim_H(K), 1\}$$

and this completes the proof. ∎

**Theorem 1.10.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^3$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. If $|\mathcal{T}_{j,G}| < \infty$ and $\dim_H K_j > 1$ then $\dim_H D(K_j) = 1$.*

By Corollary 1.4 we can find an SS-IFS $\{S_i\}_{i=1}^m$ with $K \subseteq K_j$ such that $\dim_H K > 1$ and every $T_i = Id_{\mathbb{R}^3}$. Then the statement follows by applying [1, Theorem 1.2] for $K$.

**Theorem 1.11.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. If there exists $M \in G_{d,1}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,1}$ then $\dim_H D_x(K_j) = \min\{\dim_H(K_j), 1\}$. In particular, if $\dim_H K_j \geq 1$ then $\dim_H D(K_j) = \dim_H D_x(K_j) = 1$.*

Theorem 1.11 follows by applying Theorem 1.6 to $g(y) = D_x(y) = \|x - y\|$ for some arbitrarily chosen $x \in K_j$.

Bárány's paper [1] provides information about the dimension of the arithmetic products of self-similar sets in the line. In [1, Corollary 2.9] he shows that if $K \subseteq \mathbb{R}$ is a self-similar set then $\dim_H (K \cdot K) = \min\{2\dim_H(K), 1\}$ where $A \cdot B = \{a \cdot b : a \in A, b \in B\}$ for two sets $A$ and $B$. In particular, if $\dim_H K \geq \frac{1}{2}$ then $\dim_H (K \cdot K) = 1$. In [1, Theorem 1.3] he generalises this result to $K \cdot K \cdot K$ as he shows that if $\dim_H K > \frac{1}{3}$ then $\dim_H (K \cdot K \cdot K) = 1$.

**Theorem 1.12.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then $\dim_H (K_j \cdot K_j) = \min\{2\dim_H(K_j), 1\}$. In particular, if $\dim_H K_j \geq \frac{1}{2}$ then $\dim_H (K_j \cdot K_j) = 1$. If $\dim_H K_j > \frac{1}{3}$ then $\dim_H (K_j \cdot K_j \cdot K_j) = 1$.*

*Proof.* By [12, Theorem 8.10] and [4, Theorem 3.2] $\dim_H (K_j \times K_j) = 2\dim_H(K_j)$. Since multiplication is a locally Lipschitz map the upper bound $\dim_H (K_j \cdot K_j) \leq \min\{2\dim_H(K_j), 1\}$ follows trivially. By Corollary 1.4 for $\varepsilon > 0$ we can find an SS-IFS with attractor $K$ such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, and $T_i = Id_{\mathbb{R}}$ for every $i$. By choosing $\varepsilon$ small enough we may assume that if $\dim_H K_j > \frac{1}{3}$ then $\dim_H K > \frac{1}{3}$. Then we can apply [1, Corollary 2.9, Theorem 1.3] to $K$. Since $\varepsilon > 0$ is arbitrary the theorem follows. □

Peres and Shmerkin [15, Theorem 2] showed a similar result about arithmetic sums of self-similar sets. They proved that if there are two SS-IFS $\{S_i\}_{i=1}^m$ and $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ in $\mathbb{R}$ with attractors $K$ and $\widehat{K}$ such that $\log r_1 / \log \widehat{r}_1 \notin \mathbb{Q}$ then $\dim_H \left(K + \widehat{K}\right) = \min\left\{\dim_H(K) + \dim_H(\widehat{K}), 1\right\}$.

**Theorem 1.13.** *Let $\{S_e : e \in \mathcal{E}\}$ and $\left\{\widehat{S}_e : e \in \widehat{\mathcal{E}}\right\}$ be GD-IFS in $\mathbb{R}$ with attractors $(K_1, \ldots, K_q)$ and $\left(\widehat{K}_1, \ldots, \widehat{K}_{\widehat{q}}\right)$, assume there exist $j \in \mathcal{V}$, $\widehat{j} \in \widehat{\mathcal{V}}$ and $\mathbf{e} \in \mathcal{C}_j$, $\mathbf{f} \in \widehat{\mathcal{C}}_{\widehat{j}}$ such that $\log r_{\mathbf{e}} / \log r_{\mathbf{f}} \notin \mathbb{Q}$, then $\dim_H \left(K_j + \widehat{K}_{\widehat{j}}\right) = \min\left\{\dim_H(K_j) + \dim_H(\widehat{K}_{\widehat{j}}), 1\right\}$.*

*Proof.* By [12, Theorem 8.10] and [4, Theorem 3.2] $\dim_H \left(K_j \times \widehat{K}_{\widehat{j}}\right) = \dim_H(K_j) + \dim_H(\widehat{K}_{\widehat{j}})$. Since addition is a locally Lipschitz map the upper bound

$$\dim_H \left(K_j + \widehat{K}_{\widehat{j}}\right) \leq \min\left\{\dim_H(K_j) + \dim_H(\widehat{K}_{\widehat{j}}), 1\right\}$$

follows trivially. The opposite inequality follows from [15, Theorem 2] and Remark 1.2. □

The dimension approximation theorems have consequences in connection with Furstenberg's 'dimension conservation'. If $f : A \longrightarrow \mathbb{R}^{d_2}$ is a Lipschitz map where $A \subseteq \mathbb{R}^d$ we say that $f$ is *dimension conserving* if, for some $\delta \geq 0$,

$$\delta + \dim_H \left\{ y \in f(A) : \dim_H(f^{-1}(y)) \geq \delta \right\} \geq \dim_H A$$

with that convention that $\dim_H(\emptyset) = -\infty$ so that $\delta$ cannot be chosen too large. Furstenberg also introduces the definition of a 'homogeneous set' [9, Definition 1.4]. The main theorem of that paper [9, Theorem 6.2] states that the restriction of a linear map to a homogeneous compact set is dimension conserving. It is pointed out in the paper that if $K$ is a self-similar set, $\mathcal{T}$ has only one element and the SSC is satisfied then $K$ is homogeneous. It follows from the definition of homogeneous sets that $K_j$ is homogeneous even if $(K_1, \dots, K_q)$ is a graph directed attractor of a strongly connected GD-IFS, $\mathcal{T}_{j,G}$ is finite and the SSC is satisfied. Thus for such $K$ the restriction of any linear map to $K$ is dimension conserving.

Applying the dimension approximation results does not give exact dimension conservation. However, we can deduce 'almost dimension conservation'.

**Theorem 1.14.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \dots, K_q)$, let $j \in \mathcal{V}$, let $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map ($d_2 \in \mathbb{N}$) and assume that $|\mathcal{T}_{j,G}| < \infty$. Then there exists $\delta \geq 0$ such that for every $\varepsilon > 0$*

$$\delta + \dim_H \left\{ y \in L(K_j) : \dim_H(L^{-1}(y) \cap K_j) \geq \delta - \varepsilon \right\} \geq \dim_H K_j.$$

*Proof.* By Corollary 1.4, for all $n \in \mathbb{N}$ we can find an SS-IFS that satisfies the SSC with attractor $K^{(n)}$ such that $K^{(n)} \subseteq K_j$, $\dim_H K_j - 1/n < \dim_H K^{(n)}$, and the orthogonal part of the similarities are $Id_{\mathbb{R}^d}$. Hence $L|_{K^{(n)}}$ is a dimension conserving map by [9, Theorem 6.2] and the fact that $K^{(n)}$ is homogeneous. Thus there exists $\delta_n \geq 0$ such that

$$\delta_n + \dim_H \left\{ y \in L(K^{(n)}) : \dim_H(L^{-1}(y) \cap K^{(n)}) \geq \delta_n \right\} \geq \dim_H K^{(n)} > \dim_H K_j - 1/n.$$

We can take a convergent subsequence $\delta_{n_k}$ of $\delta_n$ with limit $\delta$. Let $\varepsilon > 0$ be arbitrary. Then

$$\delta_{n_k} + \dim_H \left\{ y \in L(K_j) : \dim_H(L^{-1}(y) \cap K_j) \geq \delta - \varepsilon \right\} > \dim_H K_j - 1/n_k$$

whenever $\delta_{n_k} \geq \delta - \varepsilon$. Taking the limit on both sides we get the conclusion of the Theorem. □

When $G_{d,l}$ has a dense orbit under the action of $\mathcal{T}_{j,G}$ where $l \geq \dim_H(K_j)$ is the rank of the linear map, then we can prove dimension conservation.

**Theorem 1.15.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \dots, K_q)$, let $j \in \mathcal{V}$ and $U$ be an open neighbourhood of $K_j$. If $g : U \longrightarrow \mathbb{R}^{d_2}$ is a continuously differentiable map ($d_2 \in \mathbb{N}$) such that* $\operatorname{rank}(g'(x)) = l$ *for every $x \in K_j$ where $\dim_H(K_j) \leq l$ and there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$ then $g|_{K_j}$ is a dimension conserving map.*

This follows from Corollary 1.7 taking $\delta = 0$.

On the plane either $|\mathcal{T}_{j,G}| < \infty$ or $|\mathcal{T}_{j,G}| = \infty$ implies that $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{2,1}$ for every $M \in G_{2,1}$. Falconer and Jin [5, Theorem 4.8] showed a property in some sense stronger than 'almost dimension conservation' for the projections of a self-similar set with infinite transformation group $\mathcal{T}$ when $1 < \dim_H(K_j)$. We generalise their result to graph directed attractors with no separation condition. Let $\Pi_\theta$ denote the orthogonal projection map onto the line $\{(\lambda \cos\theta, \lambda \sin\theta) : \lambda \in \mathbb{R}\}$.

**Theorem 1.16.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^2$ with attractor $(K_1, \dots, K_q)$ and assume that $|\mathcal{T}_{j,G}| = \infty$. If $\dim_H(K_j) > 1$ then there exists $E \subsetneq [0, \pi)$ with $\dim_H E = 0$ such that for all $\theta \in [0, \pi) \setminus E$ and for all $\varepsilon > 0$*

$$\mathcal{L}^1 \left\{ y \in \Pi_\theta(K_j) : \dim_H(\Pi_\theta^{-1}(y) \cap K_j) \geq \dim_H(K_j) - 1 - \varepsilon \right\} > 0.$$

*Proof.* By Theorem 1.5 there exist SS-IFSs that satisfy the SSC with attractor $K^{(n)}$ such that $K^{(n)} \subseteq K_j$, $\dim_H K_j - 1/n < \dim_H K^{(n)}$, the transformation group $\mathcal{T}^{(n)}$ is dense in $\mathcal{T}_{j,G} \cap \mathbb{SO}_2$ and $T_i = O^{(n)}$, $r_i = r^{(n)}$ for every $i \in \mathcal{I}^{(n)}$. Then by [5, Theorem 4.6] whenever $\dim_H K_j - 1/n > 1$ there exists $E^{(n)} \subsetneq [0, \pi)$ with $\dim_H E^{(n)} = 0$ such that for all $\theta \in [0, \pi) \setminus E^{(n)}$ and for all $\varepsilon > 0$

$$\mathcal{L}^1 \left\{ y \in \Pi_\theta(K^{(n)}) : \dim_H(\Pi_\theta^{-1}(y) \cap K^{(n)}) \geq \dim_H(K^{(n)}) - 1 - \varepsilon/2 \right\} > 0.$$

Let $E = \bigcup_{n=1}^{\infty} E^{(n)}$. Then taking $1/n \leq \varepsilon/2$ and $\theta \in [0, \pi) \setminus E$ it follows that

$$\mathcal{L}^1 \left\{ y \in \Pi_\theta(K_j) : \dim_H(\Pi_\theta^{-1}(y) \cap K_j) \geq \dim_H(K_j) - 1 - \varepsilon \right\} > 0.$$

□

## 1.4 Approximation theorems for subshifts of finite type

Subshifts of finite type and graph directed attractors are the same in some sense (see [7, Proposition 2.5, Proposition 2.6]). Thus we can extend our results to subshifts of finite type. Let $\mathcal{J} = \{0, 1, \dots, M-1\}$ be a finite alphabet and $A$ be an $M \times M$ transition matrix indexed by $\mathcal{J} \times \mathcal{J}$ with entries in $\{0, 1\}$. We define the subshift of finite type corresponding to $A$ as

$$\Sigma_A = \left\{ \alpha = (\alpha_0, \alpha_1, \dots) \in \mathcal{J}^{\mathbb{N}} : A_{\alpha_i, \alpha_{i+1}} = 1 \,\text{for all}\, i = 0, 1, \dots \right\}.$$

We say $\Sigma_A$ is irreducible (or transitive) if the matrix $A$ is irreducible, which means that for all pairs $i, j \in \mathcal{J}$, there exists a positive integer $n$ such that $(A^n)_{i,j} > 0$. To each $i \in \mathcal{J}$ associate a contracting similarity map $S_i(x) = r_i \cdot T_i(x) + v_i$ on $\mathbb{R}^d$ where $r_i \in (0, 1)$, $T_i \in \mathbb{O}_d$ and $v_i \in \mathbb{R}^d$. For $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathcal{J}^{\mathbb{N}}$ and $k \in \mathbb{N}$ we write $\alpha|_k = (\alpha_0, \dots, \alpha_k) \in \mathcal{J}^k$ and for $\mathbf{i} = (i_0, \dots, i_{k-1}) \in \mathcal{J}^k$ we write

$$S_{\mathbf{i}} = S_{i_0} \circ \dots S_{i_{k-1}}.$$

Then $\Pi(\alpha) = \lim_{k \to \infty} S_{\alpha|_k}(0)$ exists for every $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathcal{J}^{\mathbb{N}}$. For a given subshift of finite type we study the set $F_A = \Pi(\Sigma_A) \subseteq \mathbb{R}^d$. For $j \in \mathcal{J}$ let $F_A^j = \Pi(\Sigma_A^j)$ where $\Sigma_A^j = \{(\alpha_0, \alpha_1, \dots) \in \Sigma_A : \alpha_0 = j\}$. Note that if $\Sigma_A$ is an irreducible then $\dim_H F_A = \dim_H F_A^j$ for every $j \in \mathcal{J}$.

If $\Sigma_A$ is an irreducible subshift of finite type then there exists a strongly connected GD-IFS with attractor $(F_A^1, \dots, F_A^{M-1})$, see [7, Proposition 2.6]. For the completeness we include the construction. Let $\mathcal{J}$ be the set of vertices. We draw a directed edge $e = e_{i,j}$ from $i$ to $j$ if $A_{i,j} = 1$, let $S_e = S_i$ and let $\mathcal{E} = \{e_{i,j} : i, j \in \mathcal{J}, A_{i,j} = 1\}$. If $A$ is irreducible then the graph $G(\mathcal{J}, \mathcal{E})$ is strongly connected. We have that

$$F_A^i = \bigcup_{j\in\mathcal{J}, A_{i,j}=1} S_i(F_A^j) = \bigcup_{j\in\mathcal{J}} \bigcup_{e\in\mathcal{E}_{i,j}} S_e(F_A^j).$$

Then the set of directed cycles in $G(\mathcal{J}, \mathcal{E})$ is

$$\mathcal{C}_j = \bigcup_{k=1}^{\infty} \mathcal{E}_{j,j}^k = \bigcup_{k=1}^{\infty} \Big\{ (\alpha_0, \dots, \alpha_{k-1}) \in \mathcal{J}^k : A_{j,\alpha_0} = 1, A_{\alpha_{k-1},j} = 1$$

$$\text{and}\, A_{\alpha_i,\alpha_{i+1}} = 1 \,\text{for all}\, i = 0, 1, \dots k-2 \Big\}.$$

Hence we define the *$j$-th transformation group $\mathcal{T}_{j,A}^G$ of $\Sigma_A$* to be the group generated by the semigroup

$$\left\{ T_\alpha : k \in \mathbb{N}, \alpha = (\alpha_0, \dots, \alpha_{k-1}) \in \mathcal{J}^k, A_{j,\alpha_0} = 1 \,\text{and}\, A_{\alpha_i,\alpha_{i+1}} = 1 \,\text{for all}\, i = 0, 1, \dots k-2 \right\}.$$

Note that if $A_{j,\alpha_0} = 1$ then $T_{\alpha_0} = T_j$. Now we are ready to reformulate Theorem 1.1 for subshifts of finite type.

**Theorem 1.17.** *Let $\Sigma_A$ be an irreducible subshift of finite type and $j \in \mathcal{J}$. Then for every $\varepsilon > 0$ there exists an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq F_A^j \subseteq F_A$, $\dim_H F_A - \varepsilon < \dim_H K$ and the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,A}^G$.*

We can also state the subshift of finite type analogue of Theorem 1.3.

**Theorem 1.18.** *Let $\Sigma_A$ be an irreducible subshift of finite type and $j \in \mathcal{J}$. Then for every $\varepsilon > 0$ and $O \in \overline{\mathcal{T}_{j,A}^G}$ there exist $r \in (0, 1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq F_A^j \subseteq F_A$, $\dim_H F_A - \varepsilon < \dim_H K$, and $\|T_i - O\| < \varepsilon$, $r_i = r$ for every $i \in \{1, \dots, m\}$.*

As in the case of Corollary 1.4 for graph directed attractors we can conclude the following for subshifts of finite type.

**Corollary 1.19.** *Let $\Sigma_A$ be an irreducible subshift of finite type, let $j \in \mathcal{J}$ and assume that $\mathcal{T}_{j,A}^G$ is a finite group. Then for every $\varepsilon > 0$ and $O \in \mathcal{T}_{j,A}^G$ there exist $r \in (0, 1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq F_A^j \subseteq F_A$, $\dim_H F_A - \varepsilon < \dim_H K$, and $T_i = O$, $r_i = r$ for every $i \in \{1, \dots, m\}$.*

In the plane we can formulate the subshift of finite type analogue of Theorem 1.5.

**Theorem 1.20.** *Let $\Sigma_A$ be an irreducible subshift of finite type in $\mathbb{R}^2$ and let $j \in \mathcal{J}$. Then for every $\varepsilon > 0$ there exist $r \in (0, 1)$, an orthogonal transformation $O \in \mathcal{T}_{j,A}^G \cap \mathbb{SO}_2$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor $K$ such that $K \subseteq F_A^j \subseteq F_A$, $\dim_H F_A - \varepsilon < \dim_H K$, the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,A}^G \cap \mathbb{SO}_2$ and $T_i = O$, $r_i = r$ for every $i \in \{1, \dots, m\}$.*

As a consequence we can restate every result of Section 1.3 for subshifts of finite type and the proofs proceed similarly. We omit the restatement of those results as they are very similar to those in Section 1.3. However, we note that Fraser and Pollicott [8, Theorem 2.10] proved the subshift of finite type version of Theorem 1.6 for systems satisfying the 'strong separation property'. In [8, Theorem 2.7] they also proved the subshift of finite type version of Theorem 1.9 for the even more general case of conformal systems rather then similarities.

# 2 Proof of the approximation theorems

We prove our main approximation theorems in this section

**Proposition 2.1.** *If $T \in \mathbb{O}_d$ then for all $N \in \mathbb{N}$ there exists $k \in \mathbb{N}$, $k \geq N$, such that the group generated by $T^k$ is dense in the group generated by $T$.*

See [6, Poposition 2.2].

**Lemma 2.2.** *Let $S_1, \ldots, S_k : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ $(k \geq 2)$ be contracting similarities such that $S_1$ and $S_2$ have no common fixed point. Then there exist $F_1, \ldots, F_k : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $F_1 = S_1$, $F_2 = S_2$, for each $i \in \{3, \ldots, k\}$ either $F_i = S_1^{k_i} \circ S_i$ or $F_i = S_2^{k_i} \circ S_i$ for some $k_i \in \mathbb{N}$, and $F_i$ and $F_j$ have no common fixed point for all $i, j \in \{1, \ldots, k\}$, $i \neq j$.*

See [6, Lemma 7.2].

**Lemma 2.3.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then*

$$\bigcup \{K_{\mathbf{e}} : \mathbf{e} \in \mathcal{C}_j, \operatorname{diam}(K_{\mathbf{e}}) < \gamma\}$$

*is dense in $K_j$ for every $\gamma > 0$.*

*Proof.* Fix $\gamma > 0$. Let $x \in K_j$ and $\gamma > \delta > 0$ arbitrary and we find a point $y \in \bigcup \{K_{\mathbf{e}} : \mathbf{e} \in \mathcal{C}_j, \operatorname{diam}(K_{\mathbf{e}}) < \gamma\}$ which is $\delta$-close to $x$. For every $n$ we have that $\bigcup_{i=1}^{q} \bigcup_{\mathbf{f} \in \mathcal{E}_{j,i}^n} K_{\mathbf{f}}$ is a cover of $K_j$. For $n$ large enough $\operatorname{diam}(K_{\mathbf{f}}) < \delta < \gamma$ for every $\mathbf{f} \in \bigcup_{i=1}^{q} \mathcal{E}_{j,i}^n$. Let $i \in \mathcal{V}$ and $\mathbf{f}_1 \in \mathcal{E}_{j,i}^n$ be such that $x \in K_{\mathbf{f}_1}$. It follows that $\|x - y\| < \delta$ for every $y \in K_{\mathbf{f}_1}$. Since $\{S_e : e \in \mathcal{E}\}$ is strongly connected $\bigcup_{k=1}^{\infty} \mathcal{E}_{i,j}^k \neq \emptyset$, so let $\mathbf{f}_2 \in \bigcup_{1=N}^{\infty} \mathcal{E}_{i,j}^k$. Then $\mathbf{e} = \mathbf{f}_1 * \mathbf{f}_2 \in \mathcal{C}_j$ and $K_{\mathbf{e}} \subseteq K_{\mathbf{f}}$, thus $\operatorname{diam}(K_{\mathbf{e}}) < \gamma$ while $\|x - y\| < \delta$ for every $y \in K_{\mathbf{e}}$. □

**Lemma 2.4.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$. If $K_j$ contains at least two points for some $j \in \mathcal{V}$ then there exist $\mathbf{e}, \mathbf{f} \in \mathcal{C}_j$ such that $S_{\mathbf{e}}$ and $S_{\mathbf{f}}$ have no common fixed point.*

*Proof.* Assume that $S_{\mathbf{e}}$ have the same fixed point $x$ for every $\mathbf{e} \in \mathcal{C}_j$. Then

$$\bigcup \{K_{\mathbf{e}} : \mathbf{e} \in \mathcal{C}_j, \operatorname{diam}(K_{\mathbf{e}}) < \gamma\} \subseteq B(x, \gamma)$$

for all $\gamma > 0$ and it follows from Lemma 2.3 that $K_j = \{x\}$ which is a contradiction. □

**Lemma 2.5.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ and $j \in \mathcal{V}$. Then there exists a finite set of cycles $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\} \subseteq \mathcal{C}_j$ such that $T_{\mathbf{e}_1}, \ldots, T_{\mathbf{e}_k}$ generate $\mathcal{T}_{i,G}$.*

*Proof.* For every $i \in \mathcal{V}$, $i \neq j$ let us fix a directed path $a_i$ from $j$ to $i$ and a directed path $b_i$ from $i$ to $j$. For $i = j$ let $a_j$ and $b_j$ be the empty path. We claim that the finite set of cycles

$$\bigcup_{i,l \in \mathcal{V}} \{a_i * e * b_l : e \in \mathcal{E}_{i,l}\} \bigcup \{a_i * b_i : i \in \mathcal{V}\} \subseteq \mathcal{C}_j$$

satisfies the lemma. Let $\mathbf{e} = (e_1, \ldots, e_n) \in \mathcal{C}_j$ be an arbitrary cycle that visits the vertices $i_0, i_1, \ldots, i_n$ respectively. Then with that convention that $T_\emptyset = Id_{\mathbb{R}^d}$ we have that

$$\begin{aligned} T_{\mathbf{e}} &= T_{e_1} \circ \cdots \circ T_{e_n} \\ &= T_{b_{i_0}}^{-1} \circ T_{e_1} \circ T_{b_{i_1}} \circ \cdots \circ T_{b_{i_{n-1}}}^{-1} \circ T_{e_n} \circ T_{b_{i_n}} \\ &= T_{a_{i_0} * b_{i_0}}^{-1} \circ T_{a_{i_0} * e_1 * b_{i_1}} \circ \cdots \circ T_{a_{i_{n-1}} * b_{i_{n-1}}}^{-1} \circ T_{a_{i_{n-1}} * e_n * b_{i_n}} \end{aligned}$$

which completes the proof. □

The proof of the following lemma is based on the idea of the beginning of the proof of Peres and Shmerkin [15, Theorem 2].

**Lemma 2.6.** *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$ and let $\mathbf{e} \in \mathcal{C}_j$. Then for every $\varepsilon > 0$ there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^n$ that satisfies the SSC, with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_{\mathbf{e}}$ and $\dim_H K_j - \varepsilon < \dim_H \widehat{K}$.*

*Proof.* Let $t = \dim_H K_j = \dim_H K_{\mathbf{e}}$. Since $K_j$ has at least two points it follows that $K_j$ has infinitely many points and so $\mathcal{H}^0(K_j) = \infty$. On the other hand, since $\{S_e : e \in \mathcal{E}\}$ is strongly connected $\mathcal{H}^t(K_j) < \infty$ by [4, Thm 3.2]. Thus $t > 0$ and hence without the loss of generality we can assume that $t > \varepsilon > 0$. Since $\mathcal{H}^{t-\frac{\varepsilon}{2}}(K_{\mathbf{e}}) = \infty$ we can find $\delta > 0$ such that for any $3\delta$-cover $\mathcal{U}$ of $K_{\mathbf{e}}$ we have that $\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^{t-\frac{\varepsilon}{2}} > 1$. Let $r_{\min} = \min\{r_e : e \in \mathcal{E}\} < 1$ and let

$$\mathcal{J} = \left\{ \mathbf{f} \in \bigcup_{i \in \mathcal{V}} \bigcup_{k=1}^{\infty} \mathcal{E}_{j,i}^k : \exists \mathbf{g} \in \bigcup_{i \in \mathcal{V}} \bigcup_{k=1}^{\infty} \mathcal{E}_{j,i}^k, \mathbf{f} = \mathbf{e} * \mathbf{g},\, r_{\min}\delta \leq \operatorname{diam}(K_{\mathbf{f}}) < \delta \right\}.$$

Then $\{K_{\mathbf{f}} : \mathbf{f} \in \mathcal{J}\}$ is a cover of $K_{\mathbf{e}}$. Let $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{J}$ be such that $K_{\mathbf{f}_1}, \ldots, K_{\mathbf{f}_n}$ is a maximal pairwise disjoint sub-collection of $\{K_{\mathbf{f}} : \mathbf{f} \in \mathcal{J}\}$. Let $U_i$ be the $\delta$-neighbourhood of $K_{\mathbf{f}_i}$ for $i \in \{1, \ldots, n\}$. By the maximality $\{U_i : i \in \{1, \ldots, n\}\}$ is a $3\delta$-cover of $K_{\mathbf{e}}$. Hence by the choice of $\delta$

$$\sum_{i=1}^n (3\delta)^{t-\frac{\varepsilon}{2}} \geq \sum_{i=1}^n (\operatorname{diam}(U_i))^{t-\frac{\varepsilon}{2}} > 1.$$

It follows that

$$n \geq (3\delta)^{-\left(t-\frac{\varepsilon}{2}\right)}. \tag{5}$$

Assume that the paths $\mathbf{f}_1, \ldots, \mathbf{f}_n$ end in vertices $i_1, \ldots, i_n$ respectively. For every $i \in \mathcal{V}$ let us fix a directed path $b_i$ from $i$ to $j$. Let $c_{\min} = \min_{i \in \mathcal{V}} \left\{ r_{b_i} \frac{\operatorname{diam}(K_j)}{\operatorname{diam}(K_i)} \right\}$. Then

$$\operatorname{diam}(K_{\mathbf{f}_l * b_{i_l}}) \geq c_{\min} \cdot \operatorname{diam}(K_{\mathbf{f}_l}) \geq c_{\min} \cdot r_{\min} \cdot \delta \tag{6}$$

and $\mathbf{f}_l * b_{i_l} \in \mathcal{C}_j$ for every $l \in \{1, \ldots, n\}$.

Let $\widehat{K}$ be the attractor of the SS-IFS $\left\{S_{\mathbf{f}_l * b_{i_l}}\right\}_{l=1}^{n}$. Then $\widehat{K} \subseteq K_{\mathbf{e}}$, the SS-IFS $\left\{S_{\mathbf{f}_l * b_{i_l}}\right\}_{l=1}^{n}$ satisfies the SSC and

$$\dim_H \widehat{K} \geq \frac{\log(\frac{1}{n})}{\log(\frac{c_{\min} \cdot r_{\min} \cdot \delta}{\operatorname{diam}(K_j)})} \geq \frac{-\left(t - \frac{\varepsilon}{2}\right) \cdot \log(3) - \left(t - \frac{\varepsilon}{2}\right) \cdot \log(\delta)}{\log(\operatorname{diam}(K_j)) - \log(c_{\min}) - \log(r_{\min}) - \log(\delta)}$$

by (5), (6) and because the similarity dimension of $\left\{S_{\mathbf{f}_l * b_{i_l}}\right\}_{l=1}^{n}$ is $\dim_H \widehat{K}$, see (2). So, by choosing $\delta$ small enough, $\dim_H \widehat{K} > t - \varepsilon$. Hence the SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^{n} = \left\{S_{\mathbf{f}_l * b_{i_l}}\right\}_{l=1}^{n}$ satisfies the lemma. □

Similar ideas to the proof of Theorem 1.1 were used by Farkas [6, Proposition 1.8] in the case of self-similar sets.

*Proof of Theorem* 1.1. According to Lemma 2.4 and Lemma 2.5 there exist $\mathbf{e}_1, \dots, \mathbf{e}_k \in \mathcal{C}_j$ such that $S_{\mathbf{e}_1}$ and $S_{\mathbf{e}_2}$ have no common fixed point and $T_{\mathbf{e}_1}, \dots, T_{\mathbf{e}_k}$ generate $\mathcal{T}_{j,G}$ (note that $\mathbf{e}_1 \in \mathcal{C}_j$ can be chosen arbitrarily, see Remark 2.7). Hence by Lemma 2.2 there exist $\mathbf{f}_1, \dots, \mathbf{f}_k \in \mathcal{C}_j$ such that $S_{\mathbf{f}_i}$ and $S_{\mathbf{f}_l}$ have no common fixed point for every $i, l \in \{1, \dots, k\}$ where $i \neq l$, $\mathbf{f}_1 = \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_2$ and $T_{\mathbf{f}_1}, \dots, T_{\mathbf{f}_k}$ generate $\mathcal{T}_{j,G}$. Let $x_i$ be the unique fixed point of $S_{\mathbf{f}_i}$ for every $i \in \{1, \dots, k\}$. Let $d_{\min} = \min\left\{\|x_i - x_l\| : i, l \in \{1, \dots, k\}, i \neq l\right\} > 0$, $r_{\max} = \max\left\{r_{\mathbf{f}_i} : i \in \{1, \dots, k\}\right\} < 1$ and $N \in \mathbb{N}$ such that $r_{\max}^N \cdot \operatorname{diam}(K_j) < d_{\min}/2$. Then $S_{\mathbf{f}_i}^{k_i}(K_j) \cap S_{\mathbf{f}_l}^{k_l}(K_j) = \emptyset$ for all $i, l \in \{1, \dots, k\}$, $i \neq l$, $k_i, k_l \in \mathbb{N}$, $k_i, k_l \geq N$. By Proposition 2.1 for all $i \in \{1, \dots, k\}$ we can find $k_i \in \mathbb{N}$, $k_i \geq N$ such that the group generated by $T_{\mathbf{f}_i}^{k_i}$ is dense in the group generated by $T_{\mathbf{f}_i}$. It follows that the group generated by $T_{\mathbf{f}_1}^{k_1}, \dots, T_{\mathbf{f}_k}^{k_k}$ is dense in $\mathcal{T}_{j,G}$ and $S_{\mathbf{f}_i}^{k_i}(K) \cap S_{\mathbf{f}_l}^{k_l}(K) = \emptyset$ for all $i, l \in \mathcal{I}$, $i \neq l$. Let $S_i = S_{\mathbf{f}_i}^{k_i}$ for all $i \in \{1, \dots, k\}$.

Let $F = \bigcup_{i=1}^{k} S_{\mathbf{f}_i}^{k_i}(K_j)$. If $K_j = F$ then $\{S_i\}_{i=1}^{k}$ satisfies the SSC with attractor $K = K_j$ and the proof is complete. So we can assume that $F \subsetneq K_j$. By Lemma 2.3 we can find $\mathbf{e} \in \mathcal{C}_j$ such that $K_{\mathbf{e}} \cap F = \emptyset$. It follows from Lemma 2.6 that there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^{n}$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_{\mathbf{e}}$ and $\dim_H K_j - \varepsilon < \dim_H \widehat{K}$. Let $m = k + n$, $S_{k+l} = \widehat{S}_i$ for all $l \in \{1, \dots, n\}$ and $K$ be the attractor of the SS-IFS $\{S_i\}_{i=1}^{m}$. Then the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^{m}$ is dense in $\mathcal{T}_{j,G}$, $\widehat{K} \subseteq K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H \widehat{K} \leq \dim_H K$ and $\{S_i\}_{i=1}^{m}$ satisfies the SSC. □

*Remark* 2.7. As we noted in the proof of Theorem 1.1 $\mathbf{e}_1 \in \mathcal{C}_j$ can be chosen arbitrarily because if there are two maps with different fixed points then at least one of them have a different fixed point from $S_{\mathbf{e}_1}$, and if you add any further map to a generator set of $\mathcal{T}_{j,G}$ it will remain a generator set of $\mathcal{T}_{j,G}$. The similarity ratio of $\widehat{S}_1$ is $r_1 = r_{\mathbf{f}_1}^{k_1} = r_{\mathbf{e}_1}^{k_1}$. Thus $\log r_1 = k_1 \log r_{\mathbf{e}_1}$.

For $k_1, \dots, k_m \in \mathbb{N}$ such that $\sum_{l=1}^{m} k_l = k$ let

$$N(k_1, \dots, k_m) = \left|\left\{(i_1, \dots, i_k) \in \mathcal{I}^k : |\{j : 1 \leq j \leq k, i_j = l\}| = k_l \text{ for every } 1 \leq l \leq m\right\}\right|, \tag{7}$$

i.e. the number of words in $\mathcal{I}^k$ such that the symbol $l$ appears in the word exactly $k_l$ times for every $1 \leq l \leq m$.

**Lemma 2.8.** *Let $(p_1, \dots, p_m)$ be a probability vector. Then there exists $c > 0$ such that for each $k \in \mathbb{N}$ there exist $k_1, \dots, k_m \in \mathbb{N}$ such that $\sum_{l=1}^{m} k_l = k$ and*

$$N(k_1, \dots, k_m) \geq c \cdot k^{-m/2} \cdot p_1^{-k_1} \cdot \dots \cdot p_m^{-k_m}.$$

*Proof.* Choose $(i_1, \dots, i_k) \in \mathcal{I}^k$ at random such that $P(i_j = l) = p_l$ independently for each $j$. Then

$$P\left(|\{j : 1 \leq j \leq k, i_j = l\}| = k_l \text{ for every } 1 \leq l \leq m\right) = N(k_1, \dots, k_m) p_1^{k_1} \cdot \dots \cdot p_m^{k_m}. \tag{8}$$

Let

$$N_{k,l} = N_{k,l}(\mathbf{i}) = |\{j : 1 \leq j \leq k, i_j = l\}| = \sum_{j=1}^{k} \mathbf{1}_{i_j = l}$$

for $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}^k$. We have that $\mathrm{E}(N_{k,l}) = kp_l$ and $\mathrm{E}\left((N_{k,l} - kp_l)^2\right) = kp_l(1 - p_l)$, hence by Chebyshev's inequality

$$P\left(|N_{k,l} - kp_l| \geq \sqrt{2k}\right) \leq \frac{kp_l(1 - p_l)}{2k} \leq p_l/2.$$

Thus

$$P\left(|N_{k,l} - kp_l| < \sqrt{2k} \text{ for every } 1 \leq l \leq m\right) \geq 1 - \sum_{l=1}^{m} p_l/2 = 1/2.$$

Then by (8)

$$\sum_{\substack{kp_l - \sqrt{2k} < k_l < kp_l + \sqrt{2k} \\ \text{for every } l = 1, \dots, m}} N(k_1, \dots, k_m) p_1^{k_1} \cdot \dots \cdot p_m^{k_m}$$
$$= P\left(|N_{k,l} - kp_l| < \sqrt{2k} \text{ for every } 1 \leq l \leq m\right) \geq 1/2.$$

There are less than $\left(2\sqrt{2k} + 1\right)^m \leq \left(4\sqrt{k}\right)^m$ terms in the sum, so there exist $k_1, \dots, k_m \in \mathbb{N}$ such that $\sum_{l=1}^{m} k_l = k$

$$N(k_1, \dots, k_m) p_1^{k_1} \cdot \dots \cdot p_m^{k_m} \geq 2^{-1} \left(4\sqrt{k}\right)^{-m} = 2^{-2m-1} \cdot k^{-m/2}$$

which completes the proof. □

*Note* 2.9. Let $(k_1, \dots, k_m) \in \mathbb{N}^m$ be the closest (or one of the closest) point to $(kp_1, \dots, kp_m) \in \mathbb{R}^m$ such that $(k_1, \dots, k_m)$ is on the hyperplane $x_1 + \dots + x_m = k$. Using Stirling's formula one can show that

$$N(k_1, \dots, k_m) \geq c \cdot k^{(1-m)/2} \cdot p_1^{-k_1} \cdot \dots \cdot p_m^{-k_m}$$

for some $c > 0$ independent of $k$. However, the conclusion of Lemma 2.8 is enough for us so we have given a more elementary proof for that.

The main idea of the proof of Theorem 1.3 is based on the proof by Peres and Shmerkin [15, Proposition 6].

*Proof of Theorem* 1.3. From Theorem 1.1 there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_j$, $\dim_H K_j - \frac{\varepsilon}{2} < \dim_H \widehat{K}$ and the transformation group $\mathcal{T}$ of $\left\{\widehat{S}_i\right\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$. Let $s > 0$ be the unique solution of $\sum_{i=1}^m r_i^s = 1$ where $r_i$ are the similarity ratios of the maps $\widehat{S}_i$. Since $\left\{\widehat{S}_i\right\}_{i=1}^m$ satisfies the SSC it follows that $\dim_H \widehat{K} = s > \dim_H K_j - \frac{\varepsilon}{2}$, see (2).

Since $\overline{\mathcal{T}}$ is compact we can take a finite open $\frac{\varepsilon}{2}$-cover $\{U_i\}_{i=1}^n$ of $\overline{\mathcal{T}}$ and for every $i \in \{1, \ldots, n\}$ we take $O_i \in U_i \cap \mathcal{T}$. For every $i \in \{1, \ldots, n\}$ we fix a finite word $\mathbf{j}_i \in \bigcup_{k=1}^\infty \mathcal{I}^k$ such that $\left\|T_{\mathbf{j}_i} - O_i^{-1} \circ O\right\| < \frac{\varepsilon}{2}$ (we can find such a $\mathbf{j}_i$ due to the compactness of $\overline{\mathcal{T}}$, see for example [6, Lemma 2.1]). Then for every $T \in U_i$ it follows that

$$\left\|T \circ T_{\mathbf{j}_i} - O\right\| \leq \left\|T \circ T_{\mathbf{j}_i} - O_i \circ T_{\mathbf{j}_i} + O_i \circ T_{\mathbf{j}_i} - O\right\| \leq \|T - O_i\| + \left\|T_{\mathbf{j}_i} - O_i^{-1} \circ O\right\| < \varepsilon. \tag{9}$$

Let $k \in \mathbb{N}$, $p_l = r_l^s$ for every $1 \leq l \leq m$. Then by Lemma 2.8 there exist $k_1, \ldots, k_m \in \mathbb{N}$ such that $\sum_{l=1}^m k_l = k$ and

$$N(k_1, \ldots, k_m) \geq c \cdot k^{(1-m)/2} \cdot p_1^{-k_1} \cdot \cdots \cdot p_m^{-k_m}$$

for some $c > 0$ independent of $k$. Then for every

$$\mathbf{i} \in \mathcal{J}_0 := \left\{(i_1, \ldots, i_k) \in \mathcal{I}^k : |\{j : 1 \leq j \leq k, i_j = l\}| = k_l \text{ for every } 1 \leq l \leq m\right\}$$

it follows that

$$\rho := r_{\mathbf{i}} = \prod_{l=1}^m r_l^{k_l}$$

and

$$|\mathcal{J}_0| = N(k_1, \ldots, k_m) \geq c \cdot k^{-m/2} \prod_{l=1}^m r_i^{-sk_l}.$$

Since $\{U_i\}_{i=1}^n$ is a finite $\frac{\varepsilon}{2}$-cover of $\overline{\mathcal{T}}$ we can find $U_i$ such that for at least $n^{-1}|\mathcal{J}_0|$ words $\mathbf{i} \in \mathcal{J}_0$ we have that $T_{\mathbf{i}} \in U_i$. Let $\mathcal{J} = \{\mathbf{i} \in \mathcal{J}_0 : T_{\mathbf{i}} \in U_i\}$, then $\|T_{\mathbf{i}} - O_i\| < \frac{\varepsilon}{2}$, $r_{\mathbf{i}} = \rho$ for every $\mathbf{i} \in \mathcal{J}$ and

$$|\mathcal{J}| \geq n^{-1} N(k_1, \ldots, k_m) \geq n^{-1} c (\sqrt{k})^{-m} \prod_{l=1}^m r_i^{-sk_l}.$$

Let $K$ be the attractor of the SS-IFS $\left\{\widehat{S}_{\mathbf{i}} \circ \widehat{S}_{\mathbf{j}_i} : \mathbf{i} \in \mathcal{J}\right\}$. Then $r := r_{\mathbf{i}} r_{\mathbf{j}_i} = \rho r_{\mathbf{j}_i}$, $\left\|T_{\mathbf{i}} \circ T_{\mathbf{j}_i} - O\right\| < \varepsilon$ by (9) for every $\mathbf{i} \in \mathcal{J}$ and by (2)

$$\begin{aligned}\dim_H K = \frac{\log|\mathcal{J}|}{-\log \rho r_{\mathbf{j}_i}} &\geq \frac{-\log n + \log c - m \log(\sqrt{k}) - s\left(\sum_{l=1}^m k_l \log r_l\right)}{-\left(\sum_{l=1}^m k_l \log r_l\right) - \log r_{\mathbf{j}_i}} \\ &\geq s - \frac{\varepsilon}{2} > \dim_H K_j - \varepsilon\end{aligned}$$

if $k$ is large enough. ☐

**Lemma 2.10.** *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}^d$ that satisfies the SSC with attractor $K$ and let $\varepsilon > 0$. Let $\widehat{K}$ be the attractor of the SS-IFS $\left\{S_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ where either $f_{\mathbf{i}} = S_1$ or $f_{\mathbf{i}} = Id_{\mathbb{R}^d}$. For $k$ large enough $\dim_H K - \varepsilon < \dim_H \widehat{K} \leq \dim_H K$.*

*Proof.* It follows that $\dim_H \widehat{K} \leq \dim_H K$ because $\widehat{K} \subseteq K$. Since the SSC is satisfied $\dim_H K = s$ where $\sum_{i=1}^m r_i^s = 1$, see (2). Thus $\sum_{i=1}^m r_i^{s-\varepsilon} > 1$. Let $k \in \mathbb{N}$ be such that $\left(\sum_{i=1}^m r_i^{s-\varepsilon}\right)^k > 1/r_1^{s-\varepsilon}$. Let $p_{\mathbf{i}} = r_1$ if $f_{\mathbf{i}} = S_1$ and $p_{\mathbf{i}} = 1$ if $f_{\mathbf{i}} = Id_{\mathbb{R}^d}$. Then

$$\sum_{\mathbf{i} \in \mathcal{I}^k} r_{\mathbf{i}}^{s-\varepsilon} p_{\mathbf{i}}^{s-\varepsilon} \geq \sum_{\mathbf{i} \in \mathcal{I}^k} r_{\mathbf{i}}^{s-\varepsilon} r_1^{s-\varepsilon} = \left(\sum_{i=1}^m r_i^{s-\varepsilon}\right)^k r_1^{s-\varepsilon} > 1$$

and hence $\dim_H K - \varepsilon = s - \varepsilon < \dim_H \widehat{K}$ because $\left\{S_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ satisfies the SSC. □

*Proof of Theorem* 1.5. If $|\mathcal{T}_{j,G}| < \infty$ then Theorem 1.5 follows from Corollary 1.4 because $\mathcal{T}_{j,G} \cap \mathbb{SO}_2$ is a finite cyclic group since $\mathbb{SO}_2$ is commutative, so we assume that $|\mathcal{T}_{j,G}| = \infty$. By Theorem 1.1 there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_j$, $\dim_H K_j - \frac{\varepsilon}{2} < \dim_H \widehat{K}$ and the transformation group $\mathcal{T}$ of $\left\{\widehat{S}_i\right\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$. It is easy to see that $|\mathcal{T}| = \infty$ implies that $\mathcal{T}$ contains a rotation of infinite order. If $\mathcal{T}$ contains reflections, without loss of generality say $T_1$ is a reflection, we iterate the SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ a large number of times and compose the orientation reversing maps with $\widehat{S}_1$. The new SS-IFS looks like $\left\{\widehat{S}_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ where $f_{\mathbf{i}} = \widehat{S}_1$ if $T_{\mathbf{i}} \notin \mathbb{SO}_2$ and $f_{\mathbf{i}} = Id_{\mathbb{R}^2}$ if $T_{\mathbf{i}} \in \mathbb{SO}_2$. Since $\mathcal{T}$ contains a rotation of infinite order it follows that the transformation group of $\left\{\widehat{S}_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ contains a rotation of infinite order. It follows from Lemma 2.10 that if we choose $k$ large enough then the Hausdorff dimension of the attractor of $\left\{\widehat{S}_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ approximates $\dim_H \widehat{K}$. Hence we can assume that there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_j$, $\dim_H K_j - \frac{\varepsilon}{2} < \dim_H \widehat{K}$ and $\mathcal{T} \subseteq \mathbb{SO}_2$ contains a rotation of infinite order.

The rest of the proof is very similar to the proof of Theorem 1.3. There is no need for the $\frac{\varepsilon}{2}$-cover $\{U_l\}_{l=1}^p$ of $\overline{\mathcal{T}}$. Instead we fix $\mathbf{j} \in \mathcal{I}^l$ for some $l \in \mathbb{N}$ such that $T_{\mathbf{j}}$ is of infinite order. From here on we proceed as in the the proof of Theorem 1.3 with minor differences. Since $\mathbb{SO}_2$ is commutative it follows that for all the $N(k_1, \ldots, k_m)$ words $\mathbf{i} \in \mathcal{J}_0$ we have that $T_{\mathbf{i}} = T$ for some $T \in \mathcal{T}$. Then either $T$ or $T \circ T_{\mathbf{j}}$ is of infinite order. Hence proceeding as in the proof of Theorem 1.3 we can show that either $\left\{\widehat{S}_{\mathbf{i}} : \mathbf{i} \in \mathcal{J}\right\}$ or $\left\{\widehat{S}_{\mathbf{i}} \circ \widehat{S}_{\mathbf{j}_l} : \mathbf{i} \in \mathcal{J}\right\}$ satisfies the theorem. □

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