On 2-connected hypergraphs with no long cycles

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Abstract

We give an upper bound for the maximum number of edges in an n-vertex 2-connected r-uniform hypergraph with no Berge cycle of length k or greater, where $n \geq k \geq 4r \geq 12$. For n large with respect to r and k, this bound is sharp and is significantly stronger than the bound without restrictions on connectivity. It turned out that it is simpler to prove the bound for the broader class of Sperner families where the size of each set is at most r. For such families, our bound is sharp for all $n \geq k \geq r \geq 3$.

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1 Introduction

1.1 Basic definitions

The *upper rank* of a hypergraph $\mathcal{H}$ is the size of a largest edge. For brevity, instead of saying "a hypergraph of upper rank r" we will say "an r^--*graph*". When every edge has size r, i.e., $\mathcal{H}$ is r-uniform, we call $\mathcal{H}$ an "r-graph".

A hypergraph $\mathcal{H}$ is *Sperner* if no edge of $\mathcal{H}$ is contained in another edge. In particular, a Sperner hypergraph has no multiple edges, and all simple uniform hypergraphs are Sperner.

Definition 1.1. *A* **Berge cycle** *of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \dots, v_\ell\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo ℓ. The vertices $\{v_1, \dots, v_\ell\}$ are called* **base vertices** *of the Berge cycle.*

A **Berge path** *of length ℓ in a hypergraph is a set of $\ell + 1$ distinct vertices $\{v_1, \dots, v_{\ell+1}\}$ and ℓ distinct hyperedges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq \ell$. The vertices $\{v_1, \dots, v_{\ell+1}\}$ are called* **base vertices** *of the Berge path.*

Definition 1.2. *The* **incidence bigraph** *of a hypergraph $\mathcal{H}$ is the bipartite graph $I(\mathcal{H}) = (A, Y; E)$ such that $A = E(\mathcal{H})$, $Y = V(\mathcal{H})$ and for $a \in A$, $y \in Y$, $ay \in E(I(\mathcal{H}))$ if and only if $y \in a$ in $\mathcal{H}$.*

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A cycle C of length 2ℓ in $I(\mathcal{H})$ corresponds to a Berge cycle of length ℓ in $\mathcal{H}$ with the set of base vertices $C \cap Y$ and the set of edges $C \cap A$. Similarly, a path P of length $2\ell + 1$ (vertices) in $I(\mathcal{H})$ with endpoints in Y corresponds to a Berge path of length ℓ in $\mathcal{H}$ with the set of base vertices $P \cap Y$ and the set of edges $C \cap A$.

Definition 1.3. *A hypergraph $\mathcal{H}$ is called* **2-connected** *if its incidence bigraph is $I(\mathcal{H})$ is 2-connected.*

So $\mathcal{H}$ is 2-connected if it is connected and has no *cut vertex* $v \in V(\mathcal{H})$ (i.e., a partition of $V(\mathcal{H}) = \{v\} \cup V_1 \cup V_2$, $|V_i| \geq 1$, such that every edge is contained in either $\{v\} \cup V_1$ or $\{v\} \cup V_2$), nor does it have a *cut edge* (i.e., an edge $e \in \mathcal{H}$ and a partition of $V(\mathcal{H}) = V_1 \cup V_2$, $|V_i| \geq 1$, such that every edge $f \neq e$ is contained in either V_1 or in V_2).

Let $\mathcal{H}$ be a hypergraph and p be an integer. The *p-shadow*, $\partial_p \mathcal{H}$, is the collection of the p-sets that lie in some edge of $\mathcal{H}$. In particular, we will often consider the 2-shadow $\partial_2 \mathcal{H}$ of an r-uniform hypergraph $\mathcal{H}$. Each edge e of $\mathcal{H}$ yields in $\partial_2 \mathcal{H}$ a clique on $|e|$ vertices.

1.2 Graphs without long cycles

The classic Turán-type result on graphs without long cycles is:

Theorem 1.4 (Erdős and Gallai [1])**.** *Let $k \geq 3$ and let G be an n-vertex graph with more than $\frac{1}{2}(k-1)(n-1)$ edges. Then G contains a cycle of length at least k.*

This bound is sharp for infinitely many n: when $k - 2$ divides $n - 1$, the circumference of each connected n-vertex graph whose blocks (maximal connected subgraphs with no cut vertices) are cliques of order $k - 1$ is only $k - 1$.

There have been several alternate proofs and sharpenings of the Erdős-Gallai theorem including results by Woodall [18], Lewin [16], Faudree and Schelp [5, 6], and Kopylov [14]. See [10] for further details.

The strongest version was that of Kopylov who improved the Erdős–Gallai bound for 2-connected graphs. To state the theorem, we first introduce the family of extremal graphs.

Construction 1.5. *Fix $k \geq 4$, $n \geq k$, $\frac{k}{2} > a \geq 1$. Define the n-vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$ and the edge set of $H_{n,k,a}$ consists of all edges connecting A with B and all edges in $A \cup C$.*

Note that when $a \geq 2$, $H_{n,k,a}$ is 2-connected, has no cycle of length k or longer, and $e(H_{n,k,a}) = \binom{k-a}{2} + (n - k + a)a$.

Theorem 1.6 (Kopylov [14])**.** *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. If G is a 2-connected n-vertex graph with*

$$e(G) > \max\{e(H_{n,k,2}), e(H_{n,k,t})\},$$

then G has a cycle of length at least k.

Furthermore, Kopylov's proof yields that the only sharpness examples are the graphs $G_{n,k,t}$ and $G_{n,k,2}$. See [10] for details.

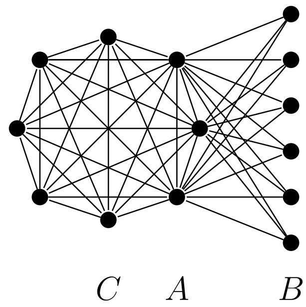

Figure 1: $H_{14,11,3}$

1.3 Hypergraphs without long Berge cycles

Recently, several interesting results were obtained for Berge paths and cycles. Notably, the results depend on the relationship between k and r.

Theorem 1.7 (Győri, Katona, and Lemons [11])**.** *Let $\mathcal{H}$ be an n-vertex r-graph with no Berge path of length k. If $r \geq k \geq 3$, then $e(\mathcal{H}) \leq \frac{(k-1)n}{r+1}$. If $k > r+1 > 3$, then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$.*

Later, the remaining case $k = r+1$ was resolved by Davoodi, Győri, Methuku, and Tompkins [2].

Furthermore, the bounds in Theorem 1.7 and in [2] are sharp for each k and r for infinitely many n.

Győri, Methuku, Salia, Tompkins, and Vizer [13] proved an asymptotic version of the Erdős–Gallai theorem for Berge paths in *connected* hypergraphs whenever r is fixed and n and k tend to infinity.

Theorem 1.8 (Győri, Methuku, Salia, Tompkins, and Vizer [13])**.** *Let r be given. Let $\mathcal{H}_{n,k}$ be a largest r-uniform connected n-vertex hypergraph with no Berge path of length k. Then*

$$\lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{e(\mathcal{H}_{n,k})}{k^{r-1}n}\right) = \frac{1}{2^{r-1}(r-1)!}.$$

For Berge cycles, the exact result for $k \geq r+3$ was obtained in [7]:

Theorem 1.9 (Füredi, Kostochka and Luo [7])**.** *Let $k \geq r+3 \geq 6$, and let $\mathcal{H}$ be an n-vertex r-graph with no Berge cycles of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$.*

This theorem is a hypergraph version of Theorem 1.4 for $k \geq r+3$. The case of $k \leq r-1$ was resolved by Kostochka and Luo [15].

Theorem 1.10 (Kostochka and Luo [15])**.** *Let $k \geq 4, r \geq k+1$, and let $\mathcal{H}$ be an n-vertex r-graph with no Berge cycles of length k or longer. Then $e(\mathcal{H}) \leq \frac{(k-1)(n-1)}{r}$.*

Recently, Ergemlidze, Győri, Methuku, Salia, Thompkins, and Zamora [4] extended the results to $k \in \{r+1, r+2\}$, and Győri, Lemons, Salia, and Zamora [12] extended the results to $k = r$.

Theorem 1.11 (Ergemlidze et al. [4])**.** *If $k \geq 4$ and $\mathcal{H}$ is an n-vertex r-graph with no Berge cycles of length k or longer, then $k = r+1$ and $e(\mathcal{H}) \leq n-1$, or $k = r+2$ and $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$.*

Theorem 1.12 (Győri et al. [12]). *If $r \geq 3$ and $\mathcal{H}$ is an n-vertex r-graph with no Berge cycles of length r or longer, then* $e(\mathcal{H}) \leq \max\{\lfloor \frac{n-1}{r} \rfloor (r-1), n-r+1\}$.

Theorems 1.9 and 1.10 are sharp for each k and r for infinitely many n. Furthermore, the present authors also proved in [8] exact bounds for all n when $k \geq r+4$.

For $r \geq k+1$, bounds for 2-connected hypergraphs stronger than for the general case were found in [15], although they are not known to be sharp.

Theorem 1.13 (Kostochka and Luo [15]). *Let $k \geq 4, r \geq k+1$, and let $\mathcal{H}$ be an n-vertex 2-connected, r-uniform hypergraph with no Berge cycle of length k or longer. Then* $e(\mathcal{H}) \leq \max\{k-1, \frac{k}{2r-k+2}(n-1)\}$.

In this paper, we find sharp bounds on the maximum number of edges in a 2-connected r-uniform hypergraph without Berge cycle of length k or longer in the case $k \geq 4r$ for $n > k2^r$. We do this by proving a more general sharp bound for Sperner r^--graphs.

2 Results

2.1 2-connected hypergraphs without long Berge cycles

Our goal is to prove a version of Kopylov's theorem for hypergraphs, i.e., to find the maximum number of edges in a 2-connected hypergraph with no Berge cycle of length k or greater.

Define

$$f(n,k,r,a) := \binom{k-a}{\min\{r, \lfloor \frac{k-a}{2} \rfloor\}} + (n-k+a)\binom{a}{\min\{r-1, \lfloor a/2 \rfloor\}}.$$

Also define

$$f^*(n,k,r,a) := \binom{k-a}{r} + (n-k+a)\binom{a}{r-1}.$$

Note that $f(n,k,r,a) = f^*(n,k,r,a)$ whenever $r \leq \lfloor (k-a)/2 \rfloor$ and $r-1 \leq \lfloor a/2 \rfloor$. Our main result is:

Theorem 2.1. *Let $n \geq k \geq r \geq 3$. If $\mathcal{H}$ is an n-vertex Sperner 2-connected r^--hypergraph with no Berge cycle of length k or longer, then* $e(\mathcal{H}) \leq \max\{f(n,k,r,\lfloor (k-1)/2 \rfloor), f(n,k,r,2)\}$.

This bound is sharp. To see this, we construct a series of hypergraphs (not necessarily uniform). The following can be viewed as a hypergraph version of Construction 1.5.

Construction 2.2. *For $n \geq k \geq r$, $1 \leq a \leq \lfloor (k-1)/2 \rfloor$, let $\mathcal{H}_{n,k,r,a}$ be the hypergraph with vertex set $A \cup B \cup C$ such that $|A| = k-2a$, $|B| = a, |C| = n-(k-a)$. The edge set of $\mathcal{H}_{n,k,r,a}$ is the family*

$$\{e \subseteq A \cup B : |e| = \min\{r, \lfloor (k-a)/2 \rfloor\}\} \cup \{c \cup e' : c \in C, e' \subseteq B, |e'| = \min\{r-1, \lfloor a/2 \rfloor\}\}.$$

For $a \geq 2$, $\mathcal{H}_{n,k,r,a}$ is 2-connected and contains no Berge cycle of length k or longer. We have that $|E(\mathcal{H}_{n,k,r,a})| = f(n,k,r,a)$, which is maximized when $a = \lfloor (k-1)/2 \rfloor$ or $a = 2$ by the convexity of f (as a function of a, see Claim 9.2 in the appendix). Furthermore, when $r \leq \lfloor (k-a)/2 \rfloor$ and $r-1 \leq \lfloor a/2 \rfloor$, $\mathcal{H}_{n,k,r,a}$ is r-uniform with $f^*(n,k,r,a)$ edges.

For integers $k \geq r$, let $n_{k,r}$ be the smallest positive integer n such that $f(n,k,r,\lfloor (k-1)/2 \rfloor) \geq f(n,k,r,2)$. Asymptotically $n_{k,r}$ is about $2^{r-1}k/r$. Then as a corollary of Theorem 2.1 we obtain the following result for r-graphs.

Theorem 2.3. *Let $n \geq n_{k,r} \geq k \geq 4r \geq 12$. If $\mathcal{H}$ is an n-vertex 2-connected r-graph with no Berge cycle of length k or longer, then $e(\mathcal{H}) \leq f(n,k,r,\lfloor (k-1)/2 \rfloor) = f^*(n,k,r,\lfloor (k-1)/2 \rfloor)$.*

For n large, this bound is almost $2^{r-1}/r$ stronger than the (exact) bound in Theorem 1.9 with no restriction on connectivity. Again we have sharpness example $\mathcal{H}_{n,k,r,\lfloor (k-1)/2 \rfloor}$.

2.2 Connected hypergraphs without long Berge path

We also obtain a result for connected graphs with no Berge path of length k.

Theorem 2.4. *Let $n \geq k \geq r \geq 3$. If $\mathcal{H}$ is an n-vertex Sperner connected r^--graph with no Berge path of length k, then $e(\mathcal{H}) \leq \max\{f(n,k,r,\lfloor (k-1)/2 \rfloor), f(n,k,r,1)\}$.*

For integers $k \geq r$, let $n'_{k,r}$ be the smallest positive integer n such that $f(n,k,r,\lfloor (k-1)/2 \rfloor) \geq f(n,k,r,1)$. Then we obtain the following result for r-uniform graphs with no Berge path of length k as a corollary of Corollary 2.4. This improves Theorem 1.8.

Theorem 2.5. *Let $n \geq n'_{k,r} \geq k \geq 4r \geq 12$. If $\mathcal{H}$ is an n-vertex connected r-graph with no Berge path of length k, then $e(\mathcal{H}) \leq f(n,k,r,\lfloor (k-1)/2 \rfloor) = f^*(n,k,r,\lfloor (k-1)/2 \rfloor)$.*

The family $\mathcal{H}_{n,k,r,\lfloor (k-1)/2 \rfloor}$ again shows sharpness of our bounds.

3 Proof outline

The basic idea of the proof is to consider instead of the family of r-graphs the larger family of Sperner r^--graphs. Then we can in some situations shrink some edges keeping the r^--graph Sperner.

We start with a dense Sperner r^--graph $\mathcal{H}$. By definition, each edge e in $\mathcal{H}$ yields a clique of order $|e|$ in the 2-shadow of $\mathcal{H}$. If $\mathcal{H}$ contains a long Berge cycle C, then $\partial_2\mathcal{H}$ contains a cycle of the same length. However, the converse is not always true. So, our first goal is to reduce $\mathcal{H}$ to a smaller dense Sperner r^--graph $\mathcal{H}'$ for which we know that the existence of a long cycle in $\partial_2\mathcal{H}'$ implies the existence of a long cycle in $\mathcal{H}'$ itself.

Our second goal is to give an upper bound on the maximum size of a Sperner family of cliques of order at most r in the shadow $\partial_2\mathcal{H}'$ that does not have long cycles. This automatically yields a bound on $|\mathcal{H}'|$.

We systematically consider incidence graphs of r^--graphs instead of the r^--graphs themselves, because we find the language of 2-connected bipartite graphs convenient for our goals.

In Section 4, we prove two results for the maximum number of cliques in graphs without long cycles or paths which will later be applied to the 2-shadows of r^--graphs. Specifically, we give upper bounds for the size of Sperner families of cliques of size at most r in graphs with bounded circumference and graphs that do not contain long paths between every pair of vertices.

In Sections 5 and 6, we prove that our hypergraphs have such a dense subhypergraph that we may reduce to, working in the language of incidence bigraphs in Section 5 and the language of hypergraphs in Section 6. In Section 7, we combine the results from Sections 4-6 to prove Theorem 2.1. Finally, in Section 8 we prove Theorem 2.4 for Berge paths in connected hypergraphs.

4 Sperner cliques in graphs

A set family H is called *Sperner* if no element of H is contained in another element of H. In particular, every uniform family is Sperner.

The classic proof of LYM Inequality yields also the following result.

Theorem 4.1. *Let H be a set of h elements. Let $\mathcal{C}$ be a Sperner family of subsets of H such that $|C| \leq r$ for each $C \in \mathcal{C}$. Then $|\mathcal{C}| \leq \binom{h}{\min\{r, \lfloor h/2 \rfloor\}}$.*

4.1 Cliques in graphs with bounded circumference

In [17], Luo proved an upper bound for the maximum number of cliques in a 2-connected graph with bounded circumference.

Theorem 4.2 (Luo [17])**.** *Let n, k, r be positive integers with $n \geq k$. Let G be an n-vertex 2-connected graph with no cycle of length k or longer. Then the number of copies of K_r in G is at most*

$$\max\left\{\binom{k-2}{r} + (n-k+2)\binom{2}{r-1}, \binom{\lceil (k+1)/2 \rceil}{r} + (n - \lceil (k+1)/2 \rceil)\binom{\lfloor (k-1)/2 \rfloor}{r-1}\right\}.$$

We will prove a version of Theorem 4.2 for Sperner families of cliques.

Recall

$$f(n,k,r,a) := \binom{k-a}{\min\{r, \lfloor \frac{k-a}{2} \rfloor\}} + (n-k+a)\binom{a}{\min\{r-1, \lfloor a/2 \rfloor\}}.$$

For fixed positive integers $n \geq k \geq r$, $f(n,k,r,a)$ is convex over integers a in $[0, \lfloor (k-1)/2 \rfloor]$ (see the appendix for a proof). Thus the value of $f(n,k,r,a)$ is maximized at one of the endpoints of the domain.

For a graph G and a positive integer r, let $N_{\mathrm{Sp}}(G,r)$ denote the maximum size of a Sperner family $\mathcal{C}$ of subsets of $V(G)$ such that for each $C \in \mathcal{C}$, $G[C]$ is a clique of size at most r.

Theorem 4.3. *Let n, k, r be positive integers with $n \geq k$. Let G be an n-vertex 2-connected graph with no cycle of length k or longer. Then*

$$N_{\mathrm{Sp}}(G,r) \leq \max\{f(n,k,r,2), f(n,k,r,\lfloor (k-1)/2 \rfloor)\}.$$

To prove Theorem 4.3, we use a structural theorem by Kopylov for 2-connected graphs without long cycles.

Definition: For a positive integer α and a graph G, the α-*disintegration* of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha + 1$ or is empty. This resulting subgraph $H(G, \alpha)$ is called the $(\alpha + 1)$-*core* of G. It is well known (and easy) that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion. If $H(G, \alpha)$ is the empty graph, then we say that G is α-*disintegrable.*

Theorem 4.4 (Kopylov [14])**.** *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. Suppose that G is a 2-connected n-vertex graph with no cycle of length at least k.*

Then either

(4.4.1) *the t-core $H(G, t)$ is empty, the graph G is t-disintegrable; or*

(4.4.2) *$|H(G, t)| = s$ for some $t + 2 \leq s \leq k - 2$, and $H(G, t) = H(G, k - s)$, i.e., the rest of the vertices can be removed by a $(k - s)$-disintegration.*

Proof of Theorem 4.3. Set $t := \lfloor (k - 1)/2 \rfloor$. Let G be an n-vertex 2-connected graph with no cycle of length k or longer. Let $\mathcal{C}$ be a Sperner family of subsets of $V(G)$ that are cliques of size at most r with $|\mathcal{C}| = N_{\mathrm{Sp}}(G, r)$. Apply Theorem 4.4 to G. If (4.4.1) holds, then every vertex is deleted in the t-disintegration. At the time of its deletion, each vertex v has at most t neighbors and by Theorem 4.1, is contained in at most $\binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}}$ cliques of $\mathcal{C}$ (since each clique containing v has at most $r - 1$ other vertices). After $n - k + t$ steps in the disintegration process, the remaining $k - t$ vertices contain at most $\binom{k-t}{\min\{\lfloor((k-t)/2)\rfloor, r\}}$ elements of $\mathcal{C}$. Therefore $|\mathcal{C}| \leq N_{\mathrm{Sp}}(G, r) \leq f(n, k, r, t)$.

Now suppose (4.4.2) holds. Then we consecutively delete vertices of degree at most $k - s$ until we arrive at the core $H(G, t)$ of size s. As in the previous case, when deleting a vertex v of degree at most $k - s$, we remove at most $\binom{k-s}{\min\{(k-s)/2, r-1\}}$ cliques of $\mathcal{C}$ containing v. Since $H(G, t)$ contains at most $\binom{s}{\min\{s/2, r\}} = \binom{k-(k-s)}{\min\{(k-(k-s))/2\rfloor, r\}}$ cliques in $\mathcal{C}$, we obtain

$$|\mathcal{C}| = N_{\mathrm{Sp}}(G, r) \leq f(n, k, k - s) \leq \max\{f(n, k, r, 2), f(n, k, r, t)\}.$$

The last inequality holds by the convexity of f. □

4.2 k-path connected graphs

A graph G is ℓ-*hamiltonian* if for each linear forest L with ℓ edges (and no isolated vertex) on the vertex set $V(G)$ there is a hamiltonian cycle in $G \cup L$ that contains L.

A graph G is k-*path connected* if for each pair of vertices $x, y \in V(G)$, G contains an x, y-path with k or more vertices. In particular, every n-vertex 1-hamiltonian graph is n-path connected. The following theorem will be helpful for us.

Theorem 4.5 (Enomoto [3])**.** *Let G be a 3-connected graph on n vertices such that for every pair of vertices u, v such that $uv \notin E(G)$, $d(u) + d(v) \geq t$. Then G is k-path connected where $k = \min\{n, 2t - 1\}$.*

Define the function

$$h_{\mathrm{Sp}}(n,\ell,r,d) := \binom{n-d+\ell}{\min\{r, \lfloor \frac{n-d+\ell}{2} \rfloor\}} + (d-\ell)\binom{d}{\min\{r-1, \lfloor d/2 \rfloor\}}.$$

Note that $h_{\mathrm{Sp}}(n,\ell,r,d) = f(n, n+\ell, r, d)$. So Claim 9.2 implies (in the appendix) that for given positive n, r, and $\ell \geq 0$, the function $h_{\mathrm{Sp}}(n,\ell,r,d)$ is convex for $\ell \leq d \leq n$.

Theorem 4.6. *Let n, d, r, ℓ be integers with $0 \leq \ell < d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$. If G is an n-vertex graph with minimum degree $\delta(G) \geq d$, and G is not ℓ-hamiltonian, then*

$$N_{\mathrm{Sp}}(G,r) \leq \max\left\{ h_{\mathrm{Sp}}(n,\ell,r,d), h_{\mathrm{Sp}}(n,\ell,r,\lfloor \frac{n+\ell-1}{2} \rfloor) \right\}.$$

Proof. Let $\mathcal{C}$ be a Sperner family of cliques of size at most r in G. Suppose that $N_{\mathrm{Sp}}(G,K_r) > h_{\mathrm{Sp}}(n,\ell,r,\lfloor (n+\ell-1)/2 \rfloor)$. By a generalization of Pósa's theorem (Lemma 8 in [9]), there exists some $\ell < k < \lfloor (n+\ell-1)/2 \rfloor$ such that $V(G)$ contains a subset D of $k-\ell$ vertices with degree at most k (and so $k \geq \delta(G) \geq d$).

For each vertex $v \in D$, v is contained in at most $\binom{k}{\min\{k/2, r-1\}}$ cliques of $\mathcal{C}$, and $G - D$ contains at most $\binom{n-k+\ell}{\min\{\lfloor (n-k+\ell)/2 \rfloor, r\}}$ cliques of $\mathcal{C}$. Hence $|\mathcal{C}| \leq N_{\mathrm{Sp}}(G,r) \leq h_{\mathrm{Sp}}(n,\ell,r,k) \leq h_{\mathrm{Sp}}(n,\ell,r,d)$. □

Our new result is:

Theorem 4.7. *Let $n \geq 4$. Let G be an n-vertex 2-connected graph. If*

$$N_{\mathrm{Sp}}(G,r) > \frac{n-2}{k-3}\binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}, \tag{1}$$

then G is k-path connected.

Proof of Theorem 4.7. We use induction on n. If $n \leq k-1$, then by Theorem 4.1,

$$N_{\mathrm{Sp}}(G,r) \leq \binom{n}{\min\{r, \lfloor n/2 \rfloor} = \frac{n-2}{k-3}\left(\frac{k-3}{n-2}\binom{n}{\min\{r, \lfloor n/2 \rfloor\}}\right).$$

And for $n \leq k-1$,

$$\frac{k-3}{n-2}\binom{n}{\min\{r, \lfloor n/2 \rfloor\}} \leq \frac{k-3}{(k-1)-2}\binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} = \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}.$$

Hence (1) does not hold.

If $n = k$, consider any $x, y \in V(G)$ such that there is no hamiltonian x,y-path in G. If $xy \in E(G)$, then G is not 1-hamiltonian, then by Theorem 4.6 with $d = 2$ (since G is 2-connected),

$$N_{\mathrm{Sp}}(G,r) \leq \max\{h_{\mathrm{Sp}}(n,1,r,2), h_{\mathrm{Sp}}(n,1,r,\lfloor n/2 \rfloor)) = h_{\mathrm{Sp}}(n,1,r,2)$$

$$= \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 2 < \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}\frac{k-2}{k-3} = \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}\frac{n-2}{k-3},$$

and (1) again does not hold. If $xy \notin E(G)$, then the graph $G' := G \cup xy$ satisfies $N_{\rm Sp}(G', r) \geq N_{\rm Sp}(G, r)$, and G' is not 1-hamiltonian. So again we obtain $N_{\rm Sp}(G, r) \leq N_{\rm Sp}(G', r) \leq \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} \frac{n-2}{k-3}$.

Thus from now on we may assume $n \geq k + 1$.

Claim 4.8. *G is 3-connected.*

Proof. Suppose $\{v_1, v_2\}$ is a separating set. Let C_1 be the vertex set of a component of $G - \{v_1, v_2\}$ and $C_2 = V(G) - C_1$. For $i = 1, 2$, let G_i be obtained from $G - C_{3-i}$ by adding edge v_1v_2 if it is not in G. Let $n_i = |V(G_i)|$. By construction, each of G_1 and G_2 is 2-connected. Also,

$$n_1 + n_2 = n + 2 \quad \text{and} \quad N_{\rm Sp}(G, r) \leq N_{\rm Sp}(G_1, r) + N_{\rm Sp}(2, r). \tag{2}$$

By (2), some of G_i satisfies (1). By symmetry, suppose G_2 does. If $x, y \in V(G_2)$, then we are done by induction. Suppose neither of x and y is in $V(G_2)$. Then by induction, G_2 has a v_1, v_2-path P with at least k vertices. Also, the 2-connected graph G_1 has two disjoint paths P_1 and P_2 from $\{x, y\}$ to $\{v_1, v_2\}$. Then $P_1 \cup P \cup P_2$ forms a long x, y-path.

Finally, suppose $x \in V(G_2)$ and $y \notin V(G_2)$. Again by induction, G_2 has a v_1, x-path P with at least k vertices. Also, the 2-connected graph G_1 has a v_1, y-path P_1 that avoids v_2. Then $P \cup P_1$ is what we need. □

Claim 4.9. $\delta(G) \geq \frac{k+1}{2}$.

Proof. Suppose $v_1 \in V(G)$ and $d(v_1) \leq k/2$. Since G is 3-connected, we can choose a neighbor v_2 of v_1 so that $v_2 \notin \{x, y\}$. Let G' be obtained from G by contracting v_1 and v_2 into a new vertex that we again will call v_1. Since G was 3-connected, G' is 2-connected.

Let $\mathcal{S}_G$ be a maximum Sperner family of cliques of size at most r in G. We construct a family $\mathcal{S}'$ of cliques of size at most r in G' from $\mathcal{S}_G$ by
(a) deleting from $\mathcal{S}_G$ all cliques containing v_1; and
(b) replacing each clique $S \in \mathcal{S}_G$ with $v_2 \in S$ and $v_1 \notin S$ with the clique $S - v_2 + v_1$.

We claim that $\mathcal{S}'$ is Sperner. Indeed, suppose $S_1, S_2 \in \mathcal{S}'$ and $S_1 \subset S_2$. Since $\mathcal{S}_G$ was Sperner, $v_1 \in S_2 - S_1$. But then $S_2 - v_1 + v_2 \in \mathcal{S}_G$ and $S_1 \subset S_2 - v_1 + v_2$.

By construction and Theorem 4.1,

$$|\mathcal{S}_G| - |\mathcal{S}'| \leq \binom{d(v_1)}{\min\{r-1, \lfloor d(v_1)/2 \rfloor\}} \leq \binom{\lfloor k/2 \rfloor}{\min\{r, \lfloor k/4 \rfloor\}}.$$

But

$$\binom{\lfloor k/2 \rfloor}{\min\{r, \lfloor k/4 \rfloor\}} \leq \frac{1}{k-3} \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}},$$

and hence G' satisfies (1). So by the minimality of G, graph G' has a long x, y-path. But then G also does. □

Applying Theorem 4.5 completes the proof of our theorem. □

5 Constructing happy incidence bigraphs

5.1 Language of layered r^--bigraphs

A *layered bigraph* is a bigraph $G = (A, Y; E)$ in which parts A and Y are ordered.

An r^--*bigraph* is a layered bigraph $G = (A, Y; E)$ with $d(a) \leq r$ for each $a \in A$.

A layered bigraph $G = (A, Y; E)$ is *Sperner* if the family $\{N(a) : a \in A\}$ is Sperner. By definition, if $N(a) = \{v, u\}$ in a Sperner bigraph, then the codegree of the pair vu is 1.

In particular, the incidence graph $G_{\mathcal{H}}$ of an r^--graph $\mathcal{H}$ is a Sperner r^--bigraph if and only if $\mathcal{H}$ is Sperner.

A vertex $a \in A$ of a layered bigraph $G = (A, Y; E)$ is *happy*, if the the codegree $d(x, y)$ of each pair $\{x, y\} \subseteq N(a)$ is at least $d(a) - 1$ (and *unhappy* otherwise). A layered bigraph $G = (A, Y; E)$ is *happy* if every vertex $a \in A$ is happy.

A vertex $y \in Y$ of degree 2 in is *special*, if each of the two neighbors is either unhappy or also has degree 2.

Vertices $x, y \in Y$ and $a \in A$ form a *special triple* if x and y are special (in particular they have degree 2), $N(a) = \{x, y\}$, and the other neighbors of x and y are unhappy.

Given a layered bigraph $G = (A, Y; E)$, let *the shadow* $\partial(G)$ be the graph F with vertex set Y such that $xy \in E(F)$ iff there is $a \in A$ with $\{x, y\} \subseteq N(a)$.

For each graph H, the *circumference*, $c(H)$, is the length of a longest cycle in H.

We first prove a simple corollary of Hall's Theorem.

Lemma 5.1 (Folklore). *Let $G = (A, B; E)$ be a bipartite graph with no isolated vertices such that for each $a \in A$ and every $b \in N(A)$, $d(a) \geq d(b)$. Then G has a matching covering A.*

Proof. Suppose that G has no matching covering A. By Hall's Theorem, there is $S \subseteq A$ with $|S| > |N(S)|$. Choose a minimum such S, say $S = \{a_1, \ldots, a_s\}$. By the minimality of S, G has a matching M covering $S' := S - a_s$, say $M = \{a_i b_i : 1 \leq i \leq s-1\}$. Since $|N(S)| \leq s - 1$, we have $N(S) = \{b_1, \ldots, b_{s-1}\}$. So,

$$d(a_1) + \ldots + d(a_{s-1}) + d(a_s) = e(S, N(S)) = d_S(b_1) + \ldots + d_S(b_{s-1}) \leq d(a_1) + \ldots d(a_{s-1}),$$

a contradiction. □

Lemma 5.2. *Let $r \geq 3$. If $G = (A, Y; E)$ is a happy Sperner r^--bigraph and $\partial(G)$ contains a cycle of length $\ell \geq r$, then G contains a cycle of length 2ℓ.*

Proof. Let $C = x_1, \ldots, x_\ell$ be a cycle of length $\ell \geq r$ in $\partial(G)$. Let F be the bipartite graph with parts $Q = E(C)$ and A such that a pair $(x_i x_{i+1}, a)$ is an edge in F if and only if $\{x_i x_{i+1}\} \subseteq N(a)$. If $\ell \geq r + 1$, then since each $a \in A$ has degree less than ℓ, a is adjacent to at most $d(a) - 1$ pairs $x_i x_{i+1}$. On the other hand, for each edge $(x_i x_{i+1}, a)$ in F, $d_F(\{x_i x_{i+1}\}) \geq d(a) - 1$ since G is happy. So by the previous lemma, F has a matching that covers $E(C)$, say with $x_i x_{i+1}$ matched to $f(x_i x_{i+1}) \in A$. Then we obtain the cycle $x_1, f(x_1x_2), x_2, f(x_2x_3), \dots, x_\ell, f(x_\ell x_1), x_1$ of length 2ℓ in G.

Now suppose $\ell = r$. If for every $a \in A$, $N_G(a) \neq \{x_1, \dots, x_r\}$, then $d_F(a) \le d(a) - 1$, and we are done as in the previous case. So suppose there exists an a such that $N_G(a) = \{x_1, \dots, x_r\}$. Then because G is Sperner, each $a' \in A - a$ is adjacent to at most $r - 1$ vertices in $\{x_1, \dots, x_r\}$, and hence $d_F(a') \le (r-1) - 1$. Consider the graph $F - a$. For $a' \in A - a$,

$$d_{F-a}(a') = d_F(a') \le \min\{r-2, d(a') - 1\}.$$

If some vertex $x_i x_{i+1}$ was adjacent to a in F, then $d_F(x_i x_{i+1}) \ge d(a) - 1 = r - 1$ and so $d_{F-a}(x_i x_{i+1}) \ge r-2$. Otherwise, for each $x_i x_{i+1}$ not adjacent to a in F, and each $a' \in N_F(x_i, x_{i+1})$, $d_{F-a}(x_i x_{i+1}) = d_F(x_i x_{i+1}) \ge d(a') - 1$, so we are finished as in the first case. □

The same proof also yields the following Lemma for paths of any length.

Lemma 5.3. *Let $G = (A, Y; E)$ be a happy r^--bigraph. If $\partial(G)$ contains a path with ℓ vertices, then G contains a path with $2\ell - 1$ vertices with endpoints in Y.*

We will often use the following known property of 2-connected graphs.

Lemma 5.4. *Let G be a 2-connected graph, $xy \in E(G)$ and $S \subset V(G)$ with $|S| \le |V(G)| - 2$.*
(1) *$G - xy$ is 2-connected iff $G - xy$ has a cycle containing x and y;*
(2) *the graph G/S obtained by gluing the vertices of S into one vertex s^* is 2-connected iff s^* is not a cut vertex of G/S.*

5.2 Unhappy r^--bigraphs

Definition 5.5. *Let $G = (A, Y; E)$ be a Sperner layered 2-connected r^--bigraph $G = (A, Y; E)$. A* **shrinking of** *G is one of the following operations:*
(1) *deleting an edge of G incident to an unhappy vertex,*
(2) *deleting a special vertex $y \in Y$ and all neighbors $b \in N(y)$ with $d(b) = 2$,*
(3) *deleting a special triple $x, y \in Y$ and $a \in A$, or*
(4) *gluing together all but one of the neighbors of some unhappy vertex $a \in X$.*

The goal of this subsection is to prove that unhappy Sperner layered 2-connected r^--bigraphs not admitting a shrinking have a special structure and high maximum average degree. The main result of the subsection is the following lemma.

Lemma 5.6. *Suppose $k \ge r \ge 3$ are integers. Let $G = (A, Y; E)$ be a Sperner layered 2-connected r^--bigraph with $c(G) < 2k$ that is not happy. Then either G admits a shrinking such that the resulting graph G' satisfies*
(S1) *G' is 2-connected;*
(S2) *$|E'| \le |E|$, $|Y'| \le |Y|$, and $|E'| + |Y'| < |E| + |Y|$;*
(S3) *G' is Sperner;*
(S4) *$|A| - |A'| \le |Y| - |Y'|$; and*
(S5) *$c(G') < 2k$,*
or for every unhappy vertex $a \in A$, there exists three vertices $y_1, y_2, y_3 \in N(a)$ and three subgraphs

B_1, B_2, B_3 *of* G *such that for* $i \in \{1,2,3\}$

(B1) $y_i \in V(B_i)$, $a \notin V(B_i)$, *and* y_i *is the only neighbor of* a *in* B_i;

(B2) B_i *is* 2*-connected and Sperner;*

(B3) *there exists a* $x_i \in Y$ *such that* $\{a, x_i\}$ *separates* B_i *from* $G - B_i$;

(B4) $G - (B_i - x_i) - a$ *is Sperner and* 2*-connected; and*

(B5) *for* $j \in \{1,2,3\} - \{i\}$, $|V(B_i) \cap V(B_j)| \leq 1$ *with equality if and only if* $x_i = x_j$.

Proof. Suppose, $G = (A, Y; E)$ is a Sperner layered 2-connected r^--bigraph with $c(G) < 2k$ that is not happy. Then it has an unhappy vertex $a \in A$. Let $N_G(a) = \{y_1, \ldots, y_t\}$. Since a is unhappy, $t \geq 3$. Assume that there are no G' satisfying the lemma. We derive a series of properties of such G.

A vertex $y_i \in N(a)$ is an *a-menace*, if there is a vertex $m(a, y_i) \in A - a$ such that $N(a) - y_i \subseteq N(m(a, y_i))$. Since G is Sperner,

$$G - ay_i \text{ is Sperner if and only if } y_i \text{ is not an } a\text{-menace.} \tag{3}$$

For brevity, we call pairs of vertices in Y of codegree 1 *thin* and of codegree at least 2 — *thick*.

Claim 5.7. $N(a)$ *contains a thin pair.*

Proof. Suppose that all pairs of $N(a)$ are thick pairs. For each $y_i \in N(a)$, the graph $G_i := G - ay_i$ trivially satisfies (S2), (S4), and (S5) in the definition of shrinking. We will show that G_i is also 2-connected, i.e., it satisfies (S1). Let $y_j, y_k \in N(a) - y_i$. Because every pair of $N(a)$ is thick, there exists distinct vertices $b_{ij}, b_{ik} \neq a$ such that $\{y_i, y_j\} \in N(b_{ij})$ and $\{y_i, y_k\} \in N(b_{ik})$. Applying Lemma 5.4 with the cycle $y_i b_{ij} y_j a y_k b_{ik} y_i$ certifies that G_i is 2-connected.

If for some $1 \leq i \leq t$, the graph G_i is Sperner, i.e., satisfies (S3), then we are done. Assume not. Because a is the only vertex with a changed neighborhood in G_i, for all i there exists a vertex b_i in G such that $\{y_1, \ldots, y_t\} - \{y_i\} \subset N(b_i)$. Furthermore, for $i \neq j$, $b_i \neq b_j$, otherwise some $N(b_i)$ contains $N(a)$, contradicting the fact that G is Sperner.

In particular, each pair in $N(a)$ belongs in the neighborhoods of a and $d(a) - 2$ additional vertices, contradicting that a is unhappy. □

Claim 5.8. *All distinct thick pairs in* $N(a)$ *are disjoint.*

Proof. Suppose not. First we show that there exist some thick pairs $\{y_{i^*}, y_{j^*}\}, \{y_{i^*}, y_{k^*}\}$ and a thin pair $\{y_{s^*}, y_{t^*}\}$ such that $s^*, t^* \neq i^*$. Let $\{y_i, y_j\}, \{y_i, y_k\}$ and $\{y_s, y_t\}$ be any intersecting thick pairs of $N(a)$ and a thin pair respectively where without loss of generality, $y_s \notin \{y_i, y_j\}$. If $y_t \neq y_i$ then we are done. If not then consider instead the pair $\{y_s, y_j\}$. If it is thin, then we take this pair instead of $\{y_s, y_t\}$. If it is thick, then we let $\{y_i, y_j\}, \{y_s, y_j\}$ be our intersecting thick pairs with y_j playing the role of y_{i^*} and $\{y_s, y_t\} = \{y_s, y_i\}$ be the thin pair.

Now consider the graph $G - ay_{i^*}$. As in the previous claim, it satisfies (S2), (S4), and (S5) as well as (S1) in the definition of shrinking where we define vertices $b_{i^*j^*}, b_{i^*k^*}$ similarly. Since no other vertex contains the pair $\{y_{s^*}, y_{t^*}\}$ in its neighborhood, $G - ay_{i^*}$ is Sperner. □

Claim 5.9. *The codegree of each pair in* $N(a)$ *is at most* 2.

Proof. Suppose there exist distinct vertices $b_1, b_2 \neq a$ both adjacent to y_1 and y_2. Since $\{y_1, y_2\}$ is a thick pair, $\{y_1, y_3\}$ and $\{y_2, y_3\}$ are thin by the previous claim. Let P be a shortest path in $G - a$ from y_3 to $\{y_1, y_2\}$. Note that if P contains b_1 or b_2, then by the minimality of $|P|$, either y_1 or y_2 follows directly after. Therefore we may assume by symmetry that $y_1 \in P$ and $b_2 \notin P$. Consider the graph $G - ay_1$. Trivially it satisfies (S2), (S4), and (S5). Because $\{y_2, y_3\}$ is thin, it also satisfies (S3). Finally, the cycle $y_3Py_1b_2y_2ay_3$ certifies that (S1) is satisfied. □

Claim 5.10. *If a proper subset S of $N(a)$ is a separating set in G, then S contains an a-menace.*

Proof. If the claim does not hold, choose a smallest separating subset $S = \{y_1, \dots, y_s\}$ of $N(a)$ not containing a-menaces. Since S is a proper subset of $N(a)$, $s < t$. Let D_1 and D_2 be components of $G - S$, where D_1 contains a. By the minimality of S,

$$\text{each } y_i \in S \text{ has a neighbor in } D_2. \tag{4}$$

Since G is 2-connected, there are two v_t, S-paths P_1 and P_2 sharing only v_t. By symmetry we may assume that P_1 avoids a. Let y_1 be the end of P_1 in S. By (4), there is a y_1, y_2-path P_3 all whose internal vertices are in D_2.

Consider $G' = G - ay_1$. Properties (S2), (S4) and (S5) in the claim of the lemma hold for G' by definition. Since y_1 is not an a-menace, by (3), G' is Sperner, i.e. (S3) holds. Cycle $y_2av_tP_1y_1P_3y_2$ together with Lemma 5.4 show that G' is 2-connected. Thus, G' satisfies the lemma. □

Claim 5.11. *$N(a)$ has no thick pairs.*

Proof. Suppose pair y_1y_2 is thick. By Claims 5.8 and 5.9, $d(y_1y_2) = 2$ and the common neighbor $b \in A - a$ of y_1 and y_2 has no other neighbors in $N(a)$. Let $N(b) = \{y_1, y_2, z_1, \dots, z_s\}$. Since G is Sperner, $s \geq 1$.

By Claim 5.8, neither of y_1 and y_2 is an a-menace. So, by Claim 5.10, $G - y_1 - y_2$ contains an a, b-path P_1. We may assume that v_t is the second and z_1 is the second to last vertices of P_1. Since $d(y_1y_2) = 2$, by Claim 5, $z_1 \notin N(a)$. So $v_t \neq z_1$.

Case 1: $d(y_1) = 2$. Then $d(y_1z_1) = 1$ and hence b is unhappy. So, since $d(y_1y_2) = 2$, by Claim 5.8, $d(y_2z_1) = 1$. Consider $G' = G - y_1$. As in the proof of Claim 5.10, (S2), (S4) and (S5) hold for G' by definition. Cycle $y_2ay_2P_1by_2$ together with Lemma 5.4 sertify that G' is 2-connected, i.e., (S1) holds. Only the neighborhoods of a and b in A' are distinct from those in A. So the fact that $d(y_2z_1) = d(y_2y_t) = 1$ shows that G' is Sperner. This proves Case 1.

Case 2: $d(y_1) \geq 3$. Let $c \in N(y_1) - a - b$, where if possible we choose c to be adjacent to z_1. Since G is 2-connected, $G - y_1$ has a shortest path P_2 from c to $V(P_1) \cup \{y_2\}$. Let x be the end of P_2 in $V(P_1) \cup \{y_2\}$.

Case 2.1: $x \neq b$. Consider $G' = G - ay_1$. As above, (S2), (S4) and (S5) trivially hold for G'. Since only the neighborhood of a in A' is distinct from those in A and $d(y_2y_t) = 1$, G' is Sperner. We need now only to show that G' is 2-connected. If $x = y_2$, then cycle $cP_2y_2aP_1by_1c$ certifies this. If $x \in V(P_1) - b$, then our certificate is cycle $cy_1by_2aP_1(a, x)xP_2c$, where $P_1(a, x)$ denotes the subpath of P_1 from a to x.

Case 2.2: $x = b$. Note that because $x \neq z_1$, by the choice of c and the choice of P_2, $z_1 \notin N(c)$ for any $c \in N(y_1) - a - b$. In particular, $d(y_1z_1) = 1$, and so b is unhappy. The second to last vertex of P_2 is none of z_1, y_1, y_2, so we may assume it is z_2. Consider $G' = G - by_1$. Cycle $cP_2by_2ay_1c$ shows that G' is 2-connected. As above, (S2), (S4) and (S5) trivially hold for G'. Thus if G' is Sperner, then the claim is proved. If G' is not Sperner, then y_1 is a b-menace, and there is a vertex $g \in A - b$ such that $N(g) \supset \{y_2, z_1, z_2\}$. Since $z_1a \notin E$, $g \neq a$. But then instead of the path P_2, we can consider the path $P_2(c, z_2)z_2gz_1$, and will have Case 2.1. □

$$\textit{Let } G' \textit{ be obtained from } G \textit{ by gluing all vertices in } N(a) - y_t \textit{ into one vertex } y^*. \tag{5}$$

(S2) holds for G' trivially. When gluing the vertices, we lose edges only if some pair $y_i, y_j \in N(a)$ have a common neighbor. But because $\{y_i, y_j\}$ is thin, they have no common neighbors other than a. Hence $|E'| = |E| - (t-2)$ and $|Y'| = |Y| - (t-2)$ so (S4) holds. Property (S5) is less clear but still is true: If G' has a cycle C of length at least $2k$, then it must go through y^*. Furthermore, if C does not go through a, then either C is present in G with y^* replaced by some y_i, or it can be extended through a connecting some y_i and y_j. If C does through a, then it uses edges ay_t and ay^*; we can modify C in G to a cycle of the same length. Thus, (S5) also holds.

Since all pairs in $N(a)$ are thin, none of y_i is an a-menace. So by Claim 5.10 and Lemma 5.4, G' is 2-connected. Again, since all pairs in $N(a)$ are thin, $N_{G'}(a)$ is not contained in any other neighborhood. Hence, in order the lemma to fail, by symmetry there are $b_1, b_2 \in A - a$ such that $N_G(b_2) - y_2 \subset N_G(b_1)$ and $y_1b_1 \in E$. Note that b_1 and b_2 each contain exactly one vertex in $N(a)$ (y_1 and y_2 respectively), and there is $x \in N(b_1) \cap N(b_2)$ such that $x \notin N(a)$.

Claim 5.12. *$d(b_2) = 2$.*

Proof. Suppose $N(b_2) \supseteq \{y_2, x_1, x_2\}$. Then by the definition of b_1, $N(b_1) \supseteq \{y_1, x_1, x_2\}$. So by Claim 5.11 applied to b_1 and b_2, because the pair $\{x_1, x_2\}$ is thick, both b_1 and b_2 are happy . Since G is 2-connected, $G - a$ has a shortest path P from v_t to $Z = \{y_1, y_2, b_1, b_2, x_1, x_2\}$. Let z be the last vertex of P. By symmetry, we may assume $z \in \{y_2, b_2, x_2\}$. Consider $G' = G - ay_2$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $z = y_2$, then the cycle $aPy_2b_2x_2b_1y_1a$ shows that G' is 2-connected.

So suppose $z \in \{b_2, x_2\}$. Since b_2 is happy, there is another b_3 adjacent to y_2 and x_2. By definition, it is distinct from b_1 and a. So if $z = x_2$ and P does not pass through b_3, then we have cycle $aPx_2b_3y_2b_2x_2b_1y_1a$. Similarly, if $z = b_2$ and P does not pass through b_3, then we have cycle $aPb_2y_2b_3x_1b_1y_1a$. Finally, if P passes through b_3, then we have cycle $aP(a, b_3)b_3y_2b_2x_1b_1y_1a$. □

Claim 5.13. *$d(y_2) \geq 3$.*

Proof. Recall $x = N(b_1) \cap N(b_2)$. Assume $N(y_2) = \{a, b_2\}$. By Claim 5.10, $G - y_1 - y_2$ has an a, x-path P. We can choose a shortest such path. Let c be the second to last vertex in P.

Case 1: $c \neq b_1$. Consider $G' = G - b_2 - y_2$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. The cycle $aPxb_1y_1a$ shows that G' is 2-connected.

Case 2: $c = b_1$. Let z be the previous to c vertex of P. Since all pairs in $N(a)$ are thin, $z \neq v_t$. If b_1 is happy, then there exists a vertex $b_3 \neq b_1$ with $\{y_1, x\} \subseteq N(b_3)$. Then b_3 can play the role of b_1 in the definition of b_1 and b_2. In this case, we get Case 1 and are done. Thus, b_1 is unhappy. Hence all pairs in $N(b_1)$ are thin.

If $d(x) = 2$, consider $G' = G - b_2 - y_2 - x$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ and in $N(b_1)$ are thin, G' is Sperner. The cycle aPb_1y_1a shows that G' is 2-connected.

So suppose $b_4 \in N(x) - b_1 - b_2$. Since G is 2-connected, $G - x$ has an b_4, a-path P_1. If P_1 does not intersect $\{b_1, y_1\}$, then we have Case 1 with $P = aP_1b_4x$. So, suppose u is the first vertex in $\{b_1, y_1\}$ that is hit by P_1. Note that if P_1 meets $P - u$ before u, then we can modify it to avoid intersecting with $\{b_1, y_1\}$. Thus we assume below that this is not the case.

If $u = y_1$, consider $G' = G - ay_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. The cycle $aPb_1y_1P_1(y_1, b_4)xb_2y_2a$ shows that G' is 2-connected. Finally, if $u = b_1$, consider $G' = G - b_1x$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. The cycle $aPb_1P_1(b_1, b_4)xb_2y_2a$ shows that G' is 2-connected. □

Claim 5.14. *Set $\{x, y_1, y_2\}$ separates a from b_1.*

Proof. Suppose not. Then $G - \{x, y_1, y_2\}$ has an a, b_1-path P. Note that $b_2 \notin P$ since $N(b_2) = \{x, y_2\}$. Let the second vertex of P be v_t.

If b_1 is happy, then there is $b_3 \in A - b_1$ with $N(b_3) \supseteq \{y_1, x\}$. Consider $G' = G - ay_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. We need to show that G' is 2-connected. If $b_3 \in P$, then the cycle $aP(a, b_3)b_3y_1b_1xb_2y_2a$ certifies this. Otherwise, the cycle $aPb_1y_1b_3xb_2y_2a$ certifies this.

So, b_1 is unhappy, and all pairs in $N(b_1)$ are thin. If $d(y_1) = 2$, consider $G' = G - y_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ and in $N(a)$ are thin, G' is Sperner. The cycle $aPb_1xb_2y_2a$ shows that G' is 2-connected.

Thus, $d(y_1) \geq 3$. Let $c \in N(y_1) - a - b_1$. Let P_1 be a shortest path in $G - y_1$ from c to $V(P) \cup \{x, y_2\}$. Let z be the last vertex of P_1. If $z \in V(P) - b_1$, consider $G' = G - ay_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. The cycle $aP(a, z)zP_1cy_1b_1xb_2y_2a$ certifies that G' is 2-connected.

If $z \in \{b_1, x, y_2\}$, consider $G' = G - b_1y_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. Let P_2 denote the path $ay_2b_2xb_1$. Then the cycle $ay_1cP_1zP_2(z, b_1)b_1Pa$ certifies that G' is 2-connected. □

Claim 5.15. *Set $\{x, a\}$ separates y_2 from $N(a) - y_2$.*

Proof. Suppose not. Let P be a shortest a, x-path in $G - y_1 - y_2$. By Claim 5.14, P does not go through b_1. Let the second vertex of P be v_t. Let P_1 be a shortest path in $G - a - x$ from y_2 to $(N(a) - y_2) \cup V(P)$. Let z be the last vertex of P_1. If $b_1 \in V(P)$, then we can take $z = y_1$. Consider $G' = G - ay_2$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $z \in N(a) - v_t$ then the cycle $y_2P_1zaPxb_2y_2$ certifies that G' is 2-connected. Otherwise, the cycle $y_2P_1zP(z, a)ay_1b_1xb_2y_2$ does it. □

Let C_2 be the vertex set of the component of $G - a - x$ containing y_2 and let $G_2 = G[C_2 \cup \{a, x\}]$. By Claim 5.15, $C_2 \cap N(a) = \{y_2\}$. If x has no neighbors in $C_2 - b_2$, then by Claim 5.14, y_2 would be a cut vertex, a contradiction. Thus, in view of b_2, no vertex in $G_2 - a$ separates x from y_2. Since no vertex in $G_2 - a$ may separate $\{y_2, x\}$ from any other vertex, we conclude

$$G_2 - a \text{ is 2-connected and the unique neighbor of } a \text{ in } C_2 \text{ is } y_2. \qquad (6)$$

Claim 5.16. *Set $\{x, a\}$ separates y_1 from $N(a) - y_1$.*

Proof. Suppose not. If $d(b_1) = 2$, then by symmetry of b_1 and b_2 and the previous claim, we are done. So $d(b_1) \geq 3$. Let $x' \in N(b_1) - y_1 - x$. Let P be a shortest a, x-path in $G - y_1 - y_2$. By Claim 5.15, P does not go through b_2. Let the second vertex of P be v_t.

Let P_1 be a shortest path in $G - a - x$ from $\{y_1, b_1\}$ to $V(P) \cup (N(a) - y_1 - y_2)$. Let z_1 be the first vertex of P_1 and z_2 — the last. If $z_1 = y_1$, consider $G' = G - ay_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $z_2 \in N(a) - v_t$ then the cycle $y_1P_1z_2ay_2b_2xb_1y_1$ certifies that G' is 2-connected. Otherwise, the cycle $y_1P_1z_2P(z_2, a)ay_2b_2xb_1y_1$ does it.

So suppose $z_1 = b_1$.

Case 1: b_1 is unhappy. If $z_2 \in V(P)$, then we consider $G' = G - xb_1$. As above, (S2),(S4) and (S5) hold for G'. Since b_1 is unhappy, all pairs in $N(b_1)$ are thin, and hence G' is Sperner. The cycle $b_1P_1z_2P(z_2, x)xb_2y_2ay_1b_1$ certifies that G' is 2-connected. So below we assume $z_2 = y_3$ and $t \geq 4$.

If $d(y_1) = 2$, then we consider $G' = G - y_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ and $N(b_1)$ are thin, G' is Sperner. The cycle $b_1P_1y_3ay_2b_2xb_1$ certifies that G' is 2-connected.

Thus there is $b_0 \in N(y_1) - a - b_1$. If $G - b_1 - y_1$ has a path from b_0 to $N(a) - y_1$, then we would have the case $z_1 = b_1$ above. Hence there is no such path. But then $G - V(P) - N(a)$ has a b_0, b_1-path P_2. In this case, we consider $G' = G - y_1b_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. The cycle $y_1b_0P_2b_1xb_2y_2ay_1$ certifies that G' is 2-connected.

Case 2: b_1 is happy. Then there is another common neighbor b_1' of x and y_1. Again, consider $G' = G - ay_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $b_1' \notin P_1$ and $z_2 \in N(a) - v_t$ then the cycle $b_1P_1z_2ay_2b_2xb_1'y_1b_1$ certifies that G' is 2-connected. If $b_1' \notin P_1$ and $z_2 \in V(P)$ then the cycle $b_1P_1z_2P(z_2, a)ay_2b_2xb_1'y_1b_1$ does it. If $b_1' \in P_1$, then we switch the roles of b_1 and b_1': consider the path $P_1' = P_1(b_1', z_2)$. □

Claim 5.17. *Vertex a has only one neighbor (namely, y_1) in the component C_1 of $G - x - a$ containing y_1 and b_1.*

Proof. Otherwise, $\{x, a\}$ would not separate y_1 from $N(a) - y_1$, a contradiction to Claim 5.16. □

Similarly to the definition of G_2, let C_1 be the vertex set of the component of $G - a - x$ containing y_1 and let $G_1 = G[C_1 \cup \{a, x\}]$. By Claim 5.17, $C_1 \cap N(a) = \{y_1\}$.

Claim 5.18. *$G_1 - a$ is 2-connected.*

Proof. Case 1: $G - a - b_1$ has an x, y_1-path P. Then $P + b_1$ forms a cycle in $G_1 - a$ containing x and y_1. Since G is 2-connected and $\{y_1, x\}$ is a separating set in G_1, this finishes the case.

Case 2: $d(b_1) = 2$. Then y_1 can play the role of y_2, and we are done by (6).

Case 3: Vertex b_1 separates y_1 from x in $G_1 - a$, and b_1 has a neighbor $y' \notin \{x, y_1\}$. If b_1 were happy, there would be $b' \neq b_1$ adjacent to x and y_1 and we would have Case 1. So, b_1 is unhappy. Let P_1 be a shortest path from y' to $\{a, x\}$ in $G - b_1$. and z be the last vertex on P_1.

Suppose first that $z = a$. Then by Claim 5.17, the second to last vertex of P_1 is y_1. Consider $G' = G - y_1 b_1$. As above, (S2),(S4) and (S5) hold for G'. Since b_1 is unhappy, all pairs in $N(b_1)$ are thin. Thus G' is Sperner. The cycle $y' P_1 a y_2 b_2 x b_1 y'$ certifies that G' is 2-connected.

Suppose now that $z = x$. Since Case 1 does not hold, $y_1 \notin P_1$. Consider $G' = G - x b_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. The cycle $y' P_1 x b_2 y_2 a y_1 b_1 y'$ certifies that G' is 2-connected. □

Claim 5.19. *$G - C_1$ and $G - C_2$ are 2-connected Sperner r^--graphs.*

Proof. Let P be a shortest y_3, x-path in $G - a$. By Claim 5.15 and 5.16, P avoids $C_1 \cup C_2$. For $i = 1, 2$, the cycle $y_3 P x b_{3-i} y_{3-i} a y_3$ certifies that $G - C_i$ is 2-connected. Since the degrees of the vertices in $G - C_1$ and $G - C_2$ are dominated by those in G, $G - C_1$ and $G - C_2$ are r^--graphs. Since a is the only vertex in $A \cap V(G - C_i)$ whose degree decreased w.r.t. G and all pairs in $N(a)$ are thin, $G - C_1$ and $G - C_2$ are Sperner. □

Now set $B_1 = G_1 - a$, $B_2 = G_2 - a$, and $x_1 = x_2 = x$. Note that the choice of y_t in (5) was arbitrary. So we may repeat the proof instead taking G'' to be the graph obtained by gluing $N(a) - y_1$ into a single vertex y^{**}. If G'' satisfies (S1) - (S5), then we are done. Otherwise we find some vertices $y'_1, y'_2 \in N(a) - y_1$ which play the role of y_1 and y_2. We may assume that $y'_1 \notin \{y_1, y_2\}$ and it is coupled with some vertex x' which plays the role of x.

Again, repeating the previous proofs for Claims 5.12-5.19 with y'_1 and y'_2, we obtain that either G admits a shrinking, or we can define G'_1 similarly to play the role of G_1 (defined after Claim 5.17) for y'_1. Let $B_3 = G'_1 - a$, $y_3 = y'_1$, and $x_3 = x'$. We now show that (B1) - (B5) hold.

(B1) and (B3) are trivial. Since G was Sperner each vertex of $A \cap V(B_i)$ has the sane neighborhood in B_i, B_i is also Sperner. Hence together with (6) and Claim (5.18), we get (B2). Claim 5.19 proves (B4). Claims 5.15 and 5.16 imply that $V(B_1) \cap V(B_2) = \{x\}$, and $y'_1 (= y_3)$ is contained in a component of $G - \{a, x\}$ not containing y_1 and y_2. In particular, B_3 is disjoint from B_1 and B_2 except possibly at x' if $x' = x$. This proves (B5) and thus the Lemma 5.6. □

5.3 Consequences of Lemma 5.6

This technical lemma implies the following more applicable fact.

Lemma 5.20. *Suppose $k \geq 5$, $r \geq 3$ are integers with $k \geq r$. Set $t = \lfloor (k-1)/2 \rfloor$. Let $G = (A, Y; E)$ be a Sperner layered 2-connected r^--bigraph with $c(G) < 2k$ that is not happy. Then either G admits a shrinking such that the resulting graph satisfies (S1) - (S5), or there exists an unhappy vertex $a^* \in A$ and some block B^* satisfying the hypothesis of Lemma 2.4 such that B^* is happy and $|A \cap B^*| \leq \binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}}(|Y \cap B^*| - 2)$.*

Proof. Suppose G does not admit any shrinking. By Lemma 5.6, for each unhappy vertex a we obtain some $\{y_i, x_i, B_i\}$ for $i \in \{1, 2, 3\}$ satisfying (B1) - (B5).

Claim 5.21. *For each unhappy a, at most one B_i has a (x_i, y_i)-path of length k or longer.*

Proof. Suppose without loss of generality that for $i \in \{1, 2\}$, there exists a (y_i, x_i)-path P_i in B_i of length at least k. Recall that $y_1, y_2 \in N(a)$. Let P_3 be a (x_1, x_2)-path internally disjoint from

$V(B_1) \cup V(B_2)$ (where P_3 may be a singleton). Then $P_1 \cup P_3 \cup P_2 \cup a$ is a cycle of length at least $2k-1$, i.e., length at least $2k$. $\square$

Among all vertices in A that are not happy, choose a and a corresponding 2-connected graph B_1 from Lemma 2.4 so that (a) B_1 does not have a (x_i, y_i)-path of length k or longer, and (b) subject to (a), $|V(B_1)|$ is minimized.

Suppose first that B_1 contains an unhappy vertex a'. By Lemma 2.4, there exists $\{x'_i, y'_i, B'_i\}$ for $i \in \{1, 2, 3\}$ satisfying (B1)-(B5) with a'.

Claim 5.22. *At most one $j \in \{1, 2, 3\}$ satisfies $V(B'_j) \not\subseteq V(B_1)$.*

Proof. Suppose without loss of generality $V(B'_2) \not\subseteq V(B_1)$ and $V(B'_3) \not\subseteq V(B_1)$. Then since $\{x_1, a\}$ separates B_1 from $G - (B_1 - x) - a$, and B'_2 and B'_3 are 2-connected, $\{x_1, a\} \subseteq V(B'_2)$ and $\{x_1, a\} \subseteq V(B'_3)$. But this violates (B5). $\square$

Therefore we may assume $V(B'_1), V(B'_2) \subseteq V(B_1)$. By Claim 5.21, we can also assume that $V(B'_1)$ has no (x'_1, y'_1)-path of length k or longer. Furthermore, since $a' \in V(B_1) - V(B'_1)$, $|V(B'_1)| < |V(B_1)|$. But this contradicts the choice of a and B_1. Thus B_1 cannot have any unhappy vertices, i.e., B_1 is happy.

Consider the shadow $\partial(B_1)$ of B_1. By Lemma 5.3, $\partial(B_1)$ is not $\lceil (k+1)/2 \rceil$-path connected, otherwise B_1 would contain an (x_1, y_1)-path of length at least $2\lceil (k+1)/2 \rceil - 1 \geq k$, a contradiction.

Let $\alpha = \lceil (k-1)/2 \rceil$, $\beta = \lfloor (k-1)/2 \rfloor$.

Claim 5.23. $\frac{1}{\alpha-2}\binom{\alpha}{\min\{r, \lfloor \alpha/2 \rfloor\}} \leq \binom{\beta}{\min\{r-1, \lfloor \beta/2 \rfloor\}}$.

Proof. First suppose $\alpha = \beta$, i.e., k is odd. Then the case $\min\{r, \lfloor \alpha/2 \rfloor\} = \alpha/2$ is trivial. Otherwise $\frac{1}{\alpha-2}\binom{\alpha}{r} = \frac{1}{\alpha-2}\frac{\alpha-r+1}{r}\binom{\beta}{r-1} \leq \binom{\beta}{r-1}$. So assume $\alpha = \beta + 1$. If $\min\{r, \lfloor \alpha/2 \rfloor\} = r$ (so $\min\{r-1, \lfloor \beta/2 \rfloor = r-1\}$), then we have $\frac{1}{\alpha-2}\binom{\alpha}{r} = \frac{1}{\beta-1}\frac{\beta+1}{r}\binom{\beta}{r-1} \leq \binom{\beta}{r-1}$. Otherwise if $\lfloor \alpha/2 \rfloor < r$, then $\lfloor \beta/2 \rfloor \leq r-1$, and $\frac{1}{\alpha-2}\binom{\alpha}{\lfloor \alpha/2 \rfloor} = \frac{1}{\beta-1}\binom{\beta+1}{\lfloor (\beta+1)/2 \rfloor} = \frac{1}{\beta-1}\frac{\beta+1}{\lceil (\beta+1)/2 \rceil}\binom{\beta}{\lfloor (\beta+1)/2 \rfloor - 1} \leq \binom{\beta}{\lfloor \beta/2 \rfloor}$. $\square$

Therefore because $\partial(B_1)$ is not $(\alpha+1)$-path connected, by Theorem 4.7 and the previous claim,

$$|A \cap B_1| \leq N_{\mathrm{Sp}}(\partial(B_1), r) \leq \frac{|Y \cap B_1| - 2}{\alpha - 2}\binom{\alpha}{\min\{r, \lfloor \alpha/2 \rfloor\}} \leq (|Y \cap B_1| - 2)\binom{\beta}{\min\{r-1, \lfloor \beta/2 \rfloor\}}.$$

$\square$

6 Constructing happy r^--graphs

In this section, we translate Lemma 5.6 into the language of r^--graphs. We also refine it.

6.1 Unhappy r^--graphs

A Sperner r^--graph $\mathcal{H}$ is *happy* if its layered incidence bigraph $I(\mathcal{H})$ is happy, and is *unhappy* otherwise. The happy and unhappy vertices in $I(\mathcal{H})$ correspond to happy and unhappy edges in $\mathcal{H}$.

For an unhappy edge e in an unhappy r^--graph $\mathcal{H}$ and a vertex $v \in e$, let $F(\mathcal{H}, e, v)$ denote the r^--graph obtained from $\mathcal{H}$ by replacing e with $e - v$.

A vertex v of degree 2 in an unhappy r^--graph $\mathcal{H}$ is *special* if each of the two incident edges, say e_1 and e_2, is either unhappy or a graph edge (i.e., contains exactly two vertices). If v is special and incident with e_1 and e_2, then $F(\mathcal{H}, v, e_1, e_2)$ is the r^--graph obtained from $\mathcal{H}$ by deleting v and for $i = 1, 2$ deleting e_i if $|e_i| = 2$ and replacing e_i with $e_i - v$ otherwise.

A graph edge vu in an unhappy r^--graph $\mathcal{H}$ is *special* if both v and u are special, and both adjacent to vu edges are unhappy. If vu is special and adjacent to e_1 and e_2, then $F(\mathcal{H}, vu)$ is the r^--graph obtained from $\mathcal{H}$ by deleting v and u, replacing e_1 with $e_1 - v$, and replacing e_2 with $e_2 - u$.

A 2-*block* in a 2-connected $\mathcal{H}$ is a 2-connected $\mathcal{H}' \subset \mathcal{H}$ such that only two vertices of $\mathcal{H}'$ have neighbors outside of $\mathcal{H}'$. These two vertices will be called *outer vertices of* $\mathcal{H}'$.

A 2-block $\mathcal{H}'$ with outer vertices x and y in an unhappy Sperner r^--graph $\mathcal{H}$ is *special* if $\mathcal{H}'$ is happy and there is exactly one edge, say a, in $G - E(\mathcal{H}')$ containing y, and this edge does not contain x.

Given a special 2-block $\mathcal{H}'$ with outer vertices x and y in an unhappy Sperner r^--graph $\mathcal{H}$, the r^--graph $F(\mathcal{H}, \mathcal{H}', x, y)$ is obtained from $\mathcal{H}$ by deleting all vertices of $\mathcal{H}' - x - y$ together with the edges containing them and adding edge $\{x, y\}$ if it is not in $\mathcal{H}$.

Translating from the language of incidence bipartite graphs to hypergraphs, we obtain the following versions of Lemmas 5.2 and 5.3 about Berge cycles and Berge paths.

Lemma 6.1. *Let $r \geq 3$. Let $\mathcal{H}$ be a happy r^--graph. If the 2-shadow $\partial_2\mathcal{H}$ contains a cycle of length $\ell \geq r+1$, then $\mathcal{H}$ contains a Berge cycle of length ℓ on the same base vertices. Furthermore, if $\partial_2\mathcal{H}$ contains a path, then $\mathcal{H}$ contains a Berge path with the same base vertices.*

For simplicity, for an r^--graph $\mathcal{H}$, denote $\sum |E(\mathcal{H})| := \sum_{e \in E(\mathcal{H})} |e|$. For example, if $\mathcal{H}$ is r-uniform, then $\sum |E(\mathcal{H})| = r|E(\mathcal{H})|$. We also obtain the following as a corollary of Lemma 5.20.

Lemma 6.2. *Suppose $k \geq r \geq 3$ are integers, and set $t = \lfloor (k-1)/2 \rfloor$. Let $\mathcal{H}$ be a Sperner 2-connected r^--graph with $c(\mathcal{H}) < k$ that is not happy. Then we can obtain a Sperner 2-connected r^--graph $\mathcal{H}'$ such that*

(i) $\sum |E(\mathcal{H}')| \leq \sum |E(\mathcal{H})|$, $|V(\mathcal{H}')| \leq |V(\mathcal{H})|$, *and* $\sum |E(\mathcal{H}')| + |V(\mathcal{H})| < \sum |E(\mathcal{H})| + |V(\mathcal{H}')|$;

(ii) $|E(\mathcal{H})| - |E(\mathcal{H}')| \leq \binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}}(|V(\mathcal{H})| - |V(\mathcal{H}')|)$; *and*

(iii) $c(\mathcal{H}') < k$

using one of the following transformations:

(T1) *for an unhappy edge e and $v \in e$, replacing H with $F(H, e, v)$;*

(T2) *for a special vertex v with incident edges e_1 and e_2, replace H with $F(H, v, e_1, e_2)$;*

(T3) *for a special edge vu, replace H with $F(H, vu)$;*

(T4) *glue together all but one vertices of an unhappy edge;*

(T5) *for a special 2-block H' with outer vertices say x, y, replace H with $F(H, H', x, y)$.*

Furthermore, if (T5) *is not applied, then instead of* (ii)*, we obtain* $|E(\mathcal{H})| - |E(\mathcal{H}')| \le (|V(\mathcal{H})| - |V(\mathcal{H}')|)$.

6.2 A refinement of Lemma 6.2

Suppose we start from a Sperner 2-connected unhappy r^--graph $\mathcal{H}$ with at least k vertices and $c(\mathcal{H}) < k$. Lemma 6.2 provides that we can obtain from $\mathcal{H}$ a happy Sperner 2-connected r^--graph in several steps using the following rule at each step:

$$\textit{if possible, apply (T1); if not then try (T2), then (T3) and so on.} \tag{7}$$

We may think that we have started from $\mathcal{H} = \mathcal{H}_0$ and after Step i obtain $\mathcal{H}_i$ from $\mathcal{H}_{i-1}$ using one of (T1)–(T5).

Claims 2.7–2.8 in the proof of Lemma 6.2 yield that following (7), at each Step i,

if (T1) *is not applied on Step* $i+1$*, then in each unhappy edge* a *of* $\mathcal{H}_i$*, thick pairs form a matching,* (8)

and

if neither (T1) *nor* (T2) *is applied on Step* $i+1$*, then all pairs of vertices in each unhappy edge* a *of* $\mathcal{H}_i$ *are thin.* (9)

Claim 6.3. *If* (T2) *was applied on Step* i*, then* (T1) *cannot be applied on Step* $i+1$.

Proof. Suppose $\mathcal{H}_i = F(\mathcal{H}_{i-1}, v, e_1, e_2)$ and $\mathcal{H}_{i+1} = F(\mathcal{H}_i, e_0, w)$.

Case 1: Edge e_0 is neither $e_1 - v$ nor $e_2 - v$. We want to show that in this case, e_0 is unhappy in $\mathcal{H}_{i-1}$ and $\mathcal{H}' = F(\mathcal{H}_{i-1}, e_0, w)$ is a Sperner 2-connected r^--graph satisfying (i)–(iii) with $\mathcal{H}_{i-1}$ in place of $\mathcal{H}$. That would contradict Rule (7).

To prove the first part (that e_0 is unhappy in $\mathcal{H}_{i-1}$), recall that e_0 is unhappy in $\mathcal{H}_i$. But the codegree in $\mathcal{H}_i$ of each pair in $V(\mathcal{H}_i)$ is the same as in $\mathcal{H}_{i-1}$.

To prove the second part, we use the fact that $\mathcal{H}'$ can be obtained from $\mathcal{H}_{i+1}$ by adding back vertex v and for $j = 1, 2$ constructing e_j either by adding v to $e_j - v \in \mathcal{H}_{i+1}$ when $|e_j| \ge 3$ or adding edge e_j when $|e_j| = 2$. Since the incidence graph $I(\mathcal{H}_{i+1})$ is 2-connected and this operation corresponds to adding a vertex of degree 2 or an ear to $I(\mathcal{H}_{i+1})$, $I(\mathcal{H}')$ also is 2-connected. Since $\mathcal{H}_{i+1}$ is Sperner, and $\mathcal{H}'$ differs from it only e_1, e_2 and v, H' is also Sperner: new edges are not contained in any old edge because of v, and no old edge can be contained in e_j, since otherwise it would be contained in $e_j - v$ in $\mathcal{H}_{i+1}$. Properties (i)–(iii) are trivial.

Case 2: $e_0 = e_1 - v$. In this case, we know that e_1 is unhappy in $\mathcal{H}_{i-1}$ and want to show that $\mathcal{H}' = F(\mathcal{H}_{i-1}, e_1, w)$ is a Sperner 2-connected r^--graph satisfying (i)–(iii) with $\mathcal{H}_{i-1}$ in place of $\mathcal{H}$. Now $\mathcal{H}'$ can be obtained from $\mathcal{H}_{i+1}$ by adding back vertex v, adding v to $e_0 - w$ and constructing e_2 either by adding v to $e_2 - v \in \mathcal{H}_{i+1}$ when $|e_2| \ge 3$ or adding edge e_2 when $|e_2| = 2$. The rest is as in Case 1. □

Practically the same proof yields the following similar claim.

Claim 6.4. *If* (T3) *was applied on Step i, then* (T1) *cannot be applied on Step $i+1$.* □

The proof of the next claim is somewhat different.

Claim 6.5. *If* (T4) *was applied on Step i, then* (T1) *cannot be applied on Step $i+1$.*

Proof. Suppose $\mathcal{H}_{i-1}$ has an unhappy edge $a = \{y_1, \dots, y_t\}$ such that $\mathcal{H}_i$ is obtained from H_{i-1} by gluing $\{y_1, \dots, y_{t-1}\}$ into a new vertex y^*, and $\mathcal{H}_{i+1} = F(\mathcal{H}_i, e, w)$. By (9),

all pairs of vertices in each unhappy edge of $\mathcal{H}_{i-1}$ are thin. In particular, the size of each edge in $\mathcal{H}_i$ apart from the edge y^*y_t is the same as in $\mathcal{H}_{i-1}$. (10)

Case 1: $w \neq y^*$. By (10), in $\mathcal{H}_{i-1}$, $|e \cap a| \leq 1$. So, since e is unhappy in $\mathcal{H}_i$, it is also unhappy in $\mathcal{H}_{i-1}$. We want to show that $\mathcal{H}' = F(\mathcal{H}_{i-1}, e, w)$ is a Sperner 2-connected r^--graph satisfying (i)–(iii). Since each pair in e is thin, $\mathcal{H}'$ is Sperner. Properties (i)–(iii) are evident, so we need to check that $\mathcal{H}'$ is 2-connected.

By construction, $\mathcal{H}'$ can be obtained from the 2-connected $\mathcal{H}_{i+1}$ by blowing up vertex y^* into vertices $y_1, \dots, y_{t-1}$ (each of a positive degree) and replacing edge y^*y_t with a. In terms of the incidence graphs, in the 2-connected $I(\mathcal{H}_{i+1})$, we split y^* into $t-1$ vertices of degree at least 1, delete vertex y^* and add vertex a adjacent to $y_1, \dots, y_t$. It is easy to check that the new graph is 2-connected.

Case 2: $w = y^*$. By (10), there is a unique $v_1 \in a - y_t$ such that $e' = e - y^* + v_1 \in \mathcal{H}_{i-1}$. Since e is unhappy in $\mathcal{H}_i$, it has a pair xy of codegree at most $|e|-2$. If $y^* \notin \{x, y\}$, then the codegree of xy in $\mathcal{H}_{i-1}$ also is at most $|e|-2$. And if $y^* = y$, then the codegree of y_1x in $\mathcal{H}_{i-1}$ is at most $|e|-2$. Thus e' is unhappy in $\mathcal{H}_{i-1}$. The rest is as in Case 1. □

6.3 Stopping at $k-1$ vertices

Lemma 6.6. *Suppose $r \geq 3$ and $k \geq r$ are integers. Let $\mathcal{H}$ be a Sperner 2-connected r^--graph with $c(\mathcal{H}) < k$ and at least k vertices that is not happy. Suppose $\mathcal{H} = \mathcal{H}_0, \dots, \mathcal{H}_i, \mathcal{H}_{i+1}$ is a sequence of r^--graphs obtained by iteratively applying Lemma 6.2 following Rule* (7) *to $\mathcal{H}$ until $\mathcal{H}_{i+1}$ is happy. If* (T5) *was never applied and $|V(\mathcal{H}_{i+1})| = k-1$, then $|E(\mathcal{H}_{i+1})| \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2$.*

Proof. Since (T1) does not change the number of vertices and $\mathcal{H}_0$ has at least k vertices, one of (T2), (T3), or (T4) was applied. Moreover, by Claims 6.3–6.5, one of (T2), (T3), or (T4) was applied to $\mathcal{H}_i$ to obtain the happy r^--graph $\mathcal{H}_{i+1}$. For short, denote $\mathcal{H}' = \mathcal{H}_{i+1}$.

If $\mathcal{H}'$ has a vertex of degree at most 3, then the number of edges in $\mathcal{H}'$ is at most $\binom{k-3}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + \binom{3}{\min\{r-1, 1\}}$, and we are done. Hence

$$\delta(\mathcal{H}') \geq 3. \tag{11}$$

In the following, for any r^--graph $\mathcal{A}$ and any vertex $v \in V(\mathcal{A})$, we use $\mathcal{A} - v$ to denote the r^--graph obtained by removing vertex v and shrinking any edge e that contains v to the edge $e - v$, unless $|e| = 2$, in which case we simply delete e in $\mathcal{A} - v$. Note that $\mathcal{A} - v$ need not be Sperner, even if $\mathcal{A}$ is Sperner.

Case 1: (T4) was the last applied operation. Let $a = \{y_1, \ldots, y_t\}$ be the unhappy edge such that $\mathcal{H}'$ is obtained from H_i by gluing $\{y_1, \ldots, y_{t-1}\}$ into a new vertex y^*. Since $\mathcal{H}'$ is happy, $\mathcal{H}_i - a$ is happy. The r^--graph $F(\mathcal{H}_i, a, y_t)$ satisfies (i)-(iii) and is Sperner by (10). So if $F(\mathcal{H}', a, y_t)$ is 2-connected, then we would have applied (T1) to $\mathcal{H}_i$ instead of (T4), a contradiction to Rule (7). Therefore

the incidence graph $I(\mathcal{H}_i - a)$ has a vertex x_t separating y_t from $\{y_1, \ldots, y_{t-1}\}$. (12)

If x_t corresponds to an edge b in $\mathcal{H}_i - a$, then some pair of its vertices is thin. So, since $\mathcal{H}_i - a$ is happy, $|b| = 2$. Then instead of x_t, we can choose as a vertex x'_t separating y_t from $\{y_1, \ldots, y_{t-1}\}$ the neighbor of x_t that is farther from y_t. Thus we may assume that x_t corresponds to a vertex in $\mathcal{H}_i - a$.

If $x_t \notin \{y_1, \ldots, y_{t-1}\}$, then y_t and y^* are also separated by x_t in $\mathcal{H}' - y^*y_t$. Since there are at least 2 components in $\mathcal{H}' - y^*y_t - x_t$, the largest block of $\mathcal{H}' - y^*y_t$ has at most $|V(\mathcal{H}') - 1| = k - 2$ vertices.

We have that

$$|E(\mathcal{H}')| = |E(\mathcal{H}' - y^*y_t)| + 1 \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 1 + 1 = \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2.$$

If $x_t \in \{y_1, \ldots, y_{t-1}\}$, then let $\mathcal{C}$ be a component of $(\mathcal{H}_i - a) - x_t$ which does not contain y_t. Then $\mathcal{C}$ contains a vertex $y \notin \{y_1, \ldots, y_{t-1}\}$, otherwise every edge of $\mathcal{C} + x_t$ in $\mathcal{H}_i$ would be a subset of the edge a, contradicting that $\mathcal{H}_i$ is Sperner. Thus in $\mathcal{H}' - y^*y_t$, y and y_t are in different blocks. Hence we again get $|E(\mathcal{H}')| \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2$.

Case 2: $\mathcal{H}_{i+1} = F(\mathcal{H}_i, v, e_1, e_2)$ for some special vertex v. By (8), if $|e_1| \geq 4$, then some pair in $e_1 - v$ is thin, and hence $e_1 - v$ is unhappy in $\mathcal{H}_{i+1}$, a contradiction the happiness of $\mathcal{H}_{i+1}$. Thus $|e_1|, |e_2| \leq 3$. Since $\mathcal{H}_i$ was unhappy, we may assume that $|e_1| = 3$, say $e_1 = \{v, v', v''\}$. By (8), either vv' or vv'' is a thin pair in $\mathcal{H}_i$. Suppose vv'' is thin. Consider $\mathcal{H}'' = F(\mathcal{H}_i, e_1, v')$. Since vv'' is thin, $\mathcal{H}''$ is Sperner. If $\mathcal{H}''$ is 2-connected, we get a contradiction to Rule (7). Thus the incidence graph $I(\mathcal{H}'')$ has a cut vertex x separating v' from $\{v, v''\}$. We claim that

we can choose x corresponding to a vertex in $\mathcal{H}''$ distinct from v. (We allow $x = v''$.) (13)

Indeed, if v separates v' from v'' in $I(\mathcal{H}'')$, then vertex e_1 in the incidence graph $I(\mathcal{H}_i)$ separates v' from v'', a contradiction to the 2-connectedness of $\mathcal{H}_i$. If x corresponds to an edge in $I(\mathcal{H}_i)$, then again x contains thin pairs. If $|x| \geq 3$. Then x is unhappy. By the choice $\mathcal{H}_{i+1}$, the only unhappy edge in $\mathcal{H}''$ could be e_2. Recall that in this case, $|e_2| = 3$, say $x = e_2 = \{v, w, w'\}$. But in this case, one of v, w and w' also separates v' from v'', and we know that it is not v. Recall that vv'' is a thin pair, and so $v'' \notin \{w, w'\}$. Otherwise if $|x| = 2$, then both of its vertices are cut vertices. This proves (13).

Recall that $|V(\mathcal{H}'')| = |V(\mathcal{H}_i)| = k$ and $e(\mathcal{H}'') = e(\mathcal{H}_i) \leq e(\mathcal{H}_{i+1}) + 1$. Suppose first that each component of $\mathcal{H}'' - x$ has at least 3 vertices. Since $\mathcal{H}'' - x$ has $k-1$ vertices and at least 2 connected components, $k \geq 7$, and the largest component of $\mathcal{H}'' - x$ has at most $k - 4$ vertices. Therefore we obtain

$$e(\mathcal{H}_{i+1}) \le e(\mathcal{H}'') \le \binom{k-3}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + \binom{4}{2} \le \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2.$$

Now suppose that some component $\mathcal{C}$ of $\mathcal{H}'' - x$ contains at most 2 vertices. By (11), $|\mathcal{C}| = 2$ and each of the two vertices in $\mathcal{C}$ either has degree in $\mathcal{H}''$ less than in $\mathcal{H}_{i+1}$ or is v. But the only vertex having degree in $\mathcal{H}''$ less than in $\mathcal{H}_{i+1}$ is v', and the vertices v and v' are in distinct components of $\mathcal{H}'' - x$.

Case 3: $\mathcal{H}_{i+1} = F(\mathcal{H}_i, vu)$ for some special edge vu. Let e_1 be the unhappy edge incident to v and e_2 be the unhappy edge incident to u. By (9), all pairs in e_1 and e_2 are thin. So since $\mathcal{H}_{i+1}$ is happy, $|e_1| = |e_2| = 3$. Let $e_1 = \{v, v', v''\}$ and $e_2 = \{u, u', u''\}$, where possibly $v' = u'$. As in Case 2, consider $\mathcal{H}'' = F(\mathcal{H}_i, e_1, v')$. Since vv'' is thin, $\mathcal{H}''$ is Sperner. If $\mathcal{H}''$ is 2-connected, we get a contradiction to Rule (7). Thus the incidence graph $I(\mathcal{H}'')$ has a cut vertex x separating v' from $\{v, v''\}$.

Similarly to the proof of (13), we derive

we can choose x corresponding to a vertex in $\mathcal{H}''$ distinct from v and u. (We allow $x = v''$.) (14)

Furthermore, $x \notin \{u', u''\}$. Now $|V(\mathcal{H}'')| = |V(\mathcal{H}_i)| = k + 1$ and $e(\mathcal{H}'') = e(\mathcal{H}_i) = e(\mathcal{H}_{i+1}) + 1$.

Note that there cannot be any isolated vertices in $\mathcal{H}'' - x$ since by (11), $\delta(\mathcal{H}'') \ge 3$. Also, as in the previous case, there cannot be a component of $\mathcal{H}'' - x$ with exactly 2 vertices. So we may assume that each component of $\mathcal{H}'' - x$ has at least 3 vertices.

Let $\mathcal{C}$ be the component of $\mathcal{H}'' - x$ that contains v. Then $\mathcal{C}$ must also contain u and at least two of the vertices in $\{v'', u', u''\}$. Therefore $|\mathcal{C}| \ge 4$. In particular, since $\mathcal{H}'' - x$ contains exactly k vertices and at least 2 connected components, $k \ge |\mathcal{C}| + 3 \ge 7$.

As in Case 2, if the largest component of $\mathcal{H}'' - x$ has at most $k - 4$ vertices (so $k \ge 8$ since $|\mathcal{C}| \ge 4$), then

$$e(\mathcal{H}_{i+1}) \le e(\mathcal{H}'') \le \binom{k-3}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + \binom{5}{2} \le \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2,$$

a contradiction.

Now suppose a component $\mathcal{C}'$ of $\mathcal{H}'' - x$ has $k - 3$ or $k - 2$ vertices. If $\mathcal{C}'$ contains v, (i.e., $\mathcal{C}' = \mathcal{C}$), then since $\mathcal{C}$ contains u as well, and u and v are incident to exactly 3 edges (vu, e_1, and e_2),

$$e(\mathcal{H}''[\mathcal{C} + x]) \le \binom{|\mathcal{C}'| - 2 + 1}{\min\{r, \lfloor (|\mathcal{C}'| - 2 + 1)/2 \rfloor\}} + 3.$$

For $|\mathcal{C}'| = k - 3$ we get

$$e(\mathcal{H}'') \le \binom{k-4}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + 3 + \binom{4}{2} \le \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2,$$

and for $|\mathcal{C}'| = k-2$ we get

$$e(\mathcal{H}'') \leq \binom{k-3}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + 3 + \binom{3}{2} \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2.$$

So $\mathcal{C}' \neq \mathcal{C}$. But since $|\mathcal{C}| \geq 4$, we have $|V(\mathcal{H}'')| \geq |\mathcal{C}'| + |\mathcal{C}| + 1 \geq 4 + (k-3) + 1 = k+2$, a contradiction. □

7 Proof of Theorem 2.1

Proof. Apply Lemma 6.2 repeatedly to $\mathcal{H}$ following Rule (7) to obtain an r^--hypergraph $\mathcal{H}'$ that is happy. By Lemma 6.1, $\partial_2\mathcal{H}'$ has no cycle of length k or longer.

Let n_S and m_S be the number of vertices and r^--edges respectively that were deleted going from $\mathcal{H}$ to $\mathcal{H}'$ by applying operations (T1)-(T4), and let n_B and m_B be the number of vertices and r^--edges respectively that were deleted from applying operation (T5). So $n = |V(\mathcal{H}')| + n_S + n_B$ and $|E(\mathcal{H})| \leq N_{\mathrm{Sp}}(\partial_2\mathcal{H}', r) + m_S + m_B$. If $|V(\mathcal{H}')| \geq k$, then by Theorem 4.3 (applied to $\partial_2\mathcal{H}'$) and Lemma 6.2, we have

$$\begin{aligned} |E(\mathcal{H}')| &\leq N_{\mathrm{Sp}}(\partial_2\mathcal{H}', r) + m_S + m_S \\ &\leq \max\{f(|V(\mathcal{H}')|, k, r, 2), f(|V(\mathcal{H}')|, k, r, t)\} + n_S + \binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}} n_B \end{aligned} \tag{15}$$

First suppose that $n_B = 0$, i.e., (T5) was never applied. Examining the coefficient of n_S we see $1 \leq \min\{2, \binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}}\}$. So in the case $|V(\mathcal{H}')| \geq k$, from (15), we get $|E(\mathcal{H}')| \leq \max\{f(n,r,k,2), f(n,r,k,t)\}$, as desired. Otherwise, if $|V(\mathcal{H}')| \leq k-1$, then either

$$|E(\mathcal{H}')| \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2 = f(k-1, k, r, 2)$$

by Lemma 6.6, or $|V(\mathcal{H}')| \leq k-2$ and

$$|E(\mathcal{H}')| \leq \binom{|V(\mathcal{H}')|}{\min\{r, \lfloor |V(\mathcal{H}')|/2 \rfloor\}} \leq f(|V(\mathcal{H}')|, k, r, 2).$$

Either way we obtain $|E(\mathcal{H})| \leq f(n,k,r,2)$.

So we may assume that at least one application of (T5) was required to obtain $\mathcal{H}'$.

Denote $H' := \partial_2\mathcal{H}'$ and let Q be the t-core of H' (that is, the resulting graph from applying t-disintegration to H'). If H' is t-disintegrable, i.e., Q is empty, then $N_{\mathrm{Sp}}(H', r) \leq f(|V(H')|, k, r, t)$ and so by (15), we get $|E(\mathcal{H})| \leq f(n,k,r,t)$. So we may assume that Q is non-empty. In particular, since $\delta(Q) \geq t+1$, $|V(Q)| \geq t+2$.

Claim 7.1. *The graph Q is 1-hamiltonian.*

Proof. First note that $|V(Q)| \leq k-1$: the case for $|V(H')| \leq k-1$ is trivial, and if $|V(H')| \geq k$, then by applying Kopylov's Theorem (Theorem 4.4), we obtain $|V(Q)| \leq k-2$.

Next, we claim that Q is 3-connected. If not, then there exists a cut set $\{x, y\} \subset V(Q)$ and at least two components in $H' - \{x, y\}$. Since $\delta(Q) \geq t+1$, for each of these components C, $|C \cup \{x, y\}| \geq t+2$. Hence $|V(Q)| \geq 2(t+2) - 2 \geq k$, a contradiction to $|V(Q)| \leq k-1$.

Therefore Q is 3-connected. By Enomoto's Theorem (Theorem 4.5), Q is s-path connected where $s = \min\{|V(Q)|, 2(t+1)\} = |V(Q)|$. I.e., Q is 1-hamiltonian. □

Let $q := |V(Q)|$. Let $\mathcal{B}$ be a special (in particular, happy) block that was removed in some application of (T5), and set $B = \partial_2 \mathcal{B}$. Let x_B and a_B be the vertex-edge cut pair corresponding to $\mathcal{B}$, where some vertex $y_B \in V(\mathcal{B}) \setminus V(\mathcal{H}')$ is contained in a_B.

Claim 7.2. *Suppose H' is s-path connected. There does not exist a (x_B, y_B)-path of length at least $k-s+1$ in B.*

Proof. Since $\mathcal{H}$ is 2-connected, its incidence bigraph contains two shortest disjoint paths P_1, P_2 from $\{x_B, a_B\}$ to $V(\mathcal{H}')$ (where possibly $|V(P_1)$ or $V(P_2) = 1)$. Note that these paths are internally disjoint from $V(\mathcal{H}') \cup V(\mathcal{B})$. In $\mathcal{H}$, P_1 and P_2 yield Berge paths $\mathcal{P}_1$ and $a \cup \mathcal{P}_2$ from x_B to $V(\mathcal{H}')$ and y_B to $V(\mathcal{H}')$ respectively. Say P_i has endpoint $v_i \in V(\mathcal{H}')$.

Now suppose there exists a path of length at least $k-s+1$ from x_B to y_B. This yields a Berge path $\mathcal{P}_3$ from x_B to y_B with at least $k-s+1$ base vertices such that all edges of $\mathcal{P}_3$ are contained in $V(\mathcal{B})$. Similarly, we find a Berge path $\mathcal{P}_4$ from v_1 to v_2 with at least s base vertices such that all edges of $\mathcal{P}_4$ are contained in $V(\mathcal{H}')$.

Then $\mathcal{P}_1 \cup \mathcal{P}_3 \cup a \cup \mathcal{P}_2 \cup \mathcal{P}_4$ is a Berge cycle of length at least $(k-s+1)+s-1 = k$, a contradiction.

□

Claim 7.3. *If H' contains a subgraph S that is s-path connected, then H' is also s-path connected.*

Proof. Let $\{x, y\} \subset V(H')$. We will show that there exists an (x, y)-path in H' with at least s vertices. Let P_x, P_y be two disjoint shortest paths from $\{x, y\}$ to $V(S)$, say with endpoints v_x and v_y respectively (where possibly one or both paths are singletons). Such paths exist because H' is 2-connected. Let P_S be a (v_x, v_y)-path in S of length at least S. Then $P_x \cup P_S \cup P_y$ has length at least s. □

Therefore the previous claim shows that H' is q-path connected. Applying Claim 7.2 and Theorem 4.7, we get

$$e(\mathcal{B}) \leq N_{\mathrm{Sp}}(B, r) \leq \frac{|V(B)| - 2}{k-q-2} \binom{k-q}{\min\{r, \lfloor (k-q)/2 \rfloor\}}. \tag{16}$$

Summing up over all blocks deleted via big cuts, we obtain

$$m_B \leq n_B \left(\frac{1}{k-q-2} \binom{k-q}{\min\{r, \lfloor (k-q)/2 \rfloor\}} \right) \tag{17}$$

Claim 7.4. *For each integer $s \geq 3$, $\frac{1}{s-2}\binom{s}{\min\{r, \lfloor s/2 \rfloor\}} \leq \binom{s}{\min\{r-1, \lfloor s/2 \rfloor\}}$.*

Proof. The case for $\min\{r, \lfloor s/2 \rfloor\} = \lfloor s/2 \rfloor$ is trivial. So we may assume $s \geq 2r+2$. We have $\frac{1}{s-2}\binom{s}{r} = \frac{1}{s-2}\frac{s-r+1}{r}\binom{s}{r-1} \leq \binom{s}{r-1}$. □

So first suppose that $|V(\mathcal{H}')| \geq k$. By Kopylov's theorem, $t+2 \leq q \leq k-2$, and $V(H') - V(Q)$ can be removed via $(k-s)$-disintegration. Therefore

$$e(\mathcal{H}') \leq \binom{q}{\min\{r, \lfloor q/2 \rfloor\}} + (|V(\mathcal{H}')| - q)\binom{k-q}{\min\{r-1, \lfloor (k-q)/2 \rfloor\}},$$

and hence by (17) and the previous claim,

$$e(\mathcal{H}) = e(\mathcal{H}') + m_B + m_S \leq$$

$$\leq \binom{q}{\min\{r, \lfloor q/2 \rfloor\}} + (|V(\mathcal{H}')| - q)\binom{k-q}{\min\{r-1, \lfloor (k-q)/2 \rfloor\}} + n_B \left(\frac{1}{k-q-2}\binom{k-q}{\min\{r, \lfloor (k-q)/2 \rfloor\}}\right) + n_S$$

$$\leq \binom{q}{\min\{r, \lfloor q/2 \rfloor\}} + (n-q)\binom{k-q}{\min\{r-1, \lfloor (k-q)/2 \rfloor\}} \leq \max\{f(n,k,r,t), f(n,k,r,2)\},$$

where the last inequality follows from the convexity of the function f. So from now on we may assume $|V(H')| \leq k-1$.

Claim 7.5. *Let S be a 1-hamiltonian subgraph of H' with $s := |V(S)|$ and $t+2 \leq s \leq k-2$. Let S' be the result of $(k-s)$-disintegration applied to H'. Then S' is also 1-hamiltonian.*

Proof. We will show a stronger statement: S' is $(k - |V(S')|)$-hamiltonian. Suppose not. Set $s' := |V(S')|$. Applying Theorem 4.6 with $d = k-s$ (so $d \leq 2t+2-(t+2) = t$) and $\ell = k - s'$, we get

$$N_{\mathrm{Sp}}(S', r) \leq \max\{h_{\mathrm{Sp}}(s', k-s', r, k-s), h_{\mathrm{Sp}}(s', k-s', r, \lfloor s'/2 \rfloor,)\}.$$

If $h_{\mathrm{Sp}}(q', k-s', r, k-s) \geq h_{\mathrm{Sp}}(s', k-s', r, \lfloor s'/2 \rfloor)$, then

$$\begin{aligned} N_{\mathrm{Sp}}(S', r) &\leq h_{\mathrm{Sp}}(s', k-s', r, k-s) \\ &= \binom{s}{\min\{r, \lfloor s/2 \rfloor\}} + (s'-s)\binom{k-s}{\min\{r-1, \lfloor (k-s)/2 \rfloor\}} \\ &= f(s', k, r, k-s), \end{aligned}$$

Recall that since S is 1-hamiltonian, H' is s-path connected. Hence for each $\mathcal{B}$ deleted in an application of (T5), $\partial_2 \mathcal{B}$ is not $(k-s+1)$-path connected.

It follows that

$$e(\mathcal{H}) \leq N_{\mathrm{Sp}}(H', r) + m_B + m_S$$

$$\leq f(s', k, r, k-s) + (|V(H')| - s' + n_B)\binom{k-s}{\min\{r-1, \lfloor (k-s)/2 \rfloor\}} + n_S \leq f(n, k, r, k-s).$$

So by the convexity of the function f, we are done.

Next suppose $h_{\mathrm{Sp}}(s', k-s', r, k-s) \leq h_{\mathrm{Sp}}(s', k-s', r, \lfloor s'/2 \rfloor)$. For simplicity, let $a := \lfloor s'/2 \rfloor$. We have that $2 \leq a \leq \lfloor (k-1)/2 \rfloor = t$.

$$
\begin{aligned}
N_{\mathrm{Sp}}(S',r) &\le h_{\mathrm{Sp}}(s',k-s',r,a)\\
&= \binom{s'-(a-k+s')}{\min\{r,\lfloor (s'-(a-k+s'))/2\rfloor\}} + (a-k+s')\binom{a}{\min\{r-1,\lfloor a/2\rfloor\}}\\
&= \binom{k-a}{\min\{r,\lfloor (k-a)/2\rfloor\}} + (s'-(k-a))\binom{a}{\min\{r-1,\lfloor a/2\rfloor\}}\\
&\le f(s',k,r,a) \le f(s',k,r,t).
\end{aligned}
$$

Therefore

$$
e(\mathcal{H}) \le f(s',k,r,t) + (|V(H')| - s' + n_B)\binom{k-s}{\min\{r-1,\lfloor (k-s)/2\rfloor\}} + n_S \le f(n,k,r,t).
$$

□

Starting from the 1-hamiltonian subgraph Q of H', we obtain a sequence of graphs $Q = Q_0 \subset Q_1 \subset \ldots \subset Q_q$ such that Q_i is the resulting 1-hamiltonian subgraph obtained from $(k-|V(Q_{i-1})|)$-disintegration applied to H'. The sequence ends when either the graph Q_{q+1} resulting from the $(k-|V(Q_q)|)$-disintegration of H' is exactly Q_q, or $|V(Q_q)| = k-1$. In the former case, we have that $|V(Q_{q+1})| = |V(Q_q)| =: q'$. Then

$$
\begin{aligned}
e(\mathcal{H}) &\le N_{\mathrm{Sp}}(H',r) + m_B + m_S\\
&\le f(q',k,r,k-q') + (|V(H')| - q' + n_B)\binom{k-q'}{\min\{r-1,\lfloor (k-q')/2\rfloor\}} + n_S \le f(n,k,r,k-q').
\end{aligned}
$$

Finally suppose that $|V(Q_q)| = k-1$. Then H' is $(k-1)$-path connected. Because $\mathcal{H}'$ is 2-connected, we can complete a Berge path in $\mathcal{H}'$ with at least $k-1$ vertices to a Berge cycle of length at least k. This proves the theorem.

□

8 Proof of Theorem 2.4 for paths

Proof. Let $\mathcal{H}$ be a counterexample of Theorem 2.4 with minimum $\sum_{e\in E(\mathcal{H})}|e|$ on at least $k+1$ vertices. If $\mathcal{H}$ contains a Berge cycle of length $k+1$ or longer, then removing any edge from this Berge cycle yields a Berge path with at least $k+1$ base vertices, a contradiction. If $\mathcal{H}$ contains a Berge cycle of length exactly k, then we use the following Lemma which contradicts that $n := |V(\mathcal{H})| \ge k+1$.

Lemma 8.1 (Győri, Katona, and Lemons [11])**.** *Let $\mathcal{H}$ be a connected hypergraph with no Berge path of length k. If there is a Berge cycle of length k on the vertices $v_1,\ldots,v_k$ then these vertices constitute a component of $\mathcal{H}$.*

Therefore $\mathcal{H}$ contains no Berge cycle of length k or longer. If $\mathcal{H}$ is 2-connected, then by Theorem 2.1, $e(\mathcal{H}) \le \max\{f(n,k,r,2), f(n,k,r,\lfloor (k-1)/2\rfloor)\}$, and we are done.

Now suppose $\mathcal{H}$ is not 2-connected. Then the incidence bigraph $I_{\mathcal{H}}$ of $\mathcal{H}$ contains a set of cut vertices. If a cut vertex x of $I_{\mathcal{H}}$ corresponds to an edge in $\mathcal{H}$, then we say x is a cut edge of $\mathcal{H}$. Otherwise, we say x is a cut vertex of $\mathcal{H}$.

Suppose $\mathcal{H}$ has an cut-edge e. We claim that for each component $\mathcal{C}$ of $\mathcal{H} \setminus e$,

$$|V(\mathcal{C}) \cap e| \leq 1. \tag{18}$$

Indeed suppose that some component $\mathcal{C}$ of $\mathcal{H} \setminus e$ contains at least 2 vertices in e. Let $\mathcal{H}'$ be the r^--graph obtained by shrinking e to remove all but one vertex in $\mathcal{C}$ from e. Then $\mathcal{H}'$ is still connected and Sperner (since e is a cut edge of $\mathcal{H}$). Furthermore, after this operation, the length of a longest path cannot increase. This contradicts the choice of $\mathcal{H}$.

Now suppose $\mathcal{H}$ contains a cut edge e. By (18), e intersects every component of $\mathcal{H} \setminus e$ in at most one vertex. Let $\mathcal{H}'$ be the r^--graph obtained by contracting two vertices of e into a single vertex (and then deleting e if it now contains only one vertex). The new r^--graph $\mathcal{H}'$ is Sperner, contains no Berge P_k, and is connected. If $|V(\mathcal{H}')| \geq k+2$, we obtain that $\mathcal{H}'$ contradicts the choice of $\mathcal{H}$ (note that $e(\mathcal{H}') \geq e(\mathcal{H}) - 1 \geq \max\{f(n,k,r,1), f(n,k,r,\lfloor (k-1)/2 \rfloor)\} - 1 \geq \max\{f(n-1,k,r,1), f(n-1,k,r,\lfloor (k-1)/2 \rfloor)\}$).

Iterating this process, we may assume that $\mathcal{H}$ contains no cut edges unless $n = k+1$.

Case 1: $\mathcal{H}$ does not have a cut edge.

Any block $\mathcal{B}$ of $\mathcal{H}$ is a subhypergraph of $\mathcal{H}$. In particular, $\mathcal{B}$ is a Sperner 2-connected r^--graph. Let $\mathcal{B}_1, \ldots, \mathcal{B}_p$ be the blocks of $\mathcal{H}$. For each i, let s_i be the length of a longest Berge cycle in $\mathcal{B}_i$. Without loss of generality, we may assume $s_1 \geq \ldots \geq s_p$.

Claim 8.2. *For all $i \geq 2$, $s_1 + s_i \leq k+1$.*

In particular, $s_i \leq (k+1)/2$ for all $i \geq 2$.

Proof. Suppose $s_1 + s_i \geq k+2$. Let C_1 be a Berge cycle of $\mathcal{B}_1$ of length s_1 and let C_i be a Berge cycle of $\mathcal{B}_i$ of length s_i. Let P be a shortest Berge path from $V(\mathcal{B}_1)$ to $V(\mathcal{B}_i)$. Note that P contains at most one edge from each Berge cycle. Then removing an edge from each Berge cycle, we obtain together with P a Berge path whose base vertices cover $V(C_1) \cup V(C_i)$. Since $|V(C_1) \cap V(C_i)| \leq 1$, this path has at least $s_1 + s_i - 1 \geq k+1$ base vertices. □

For each block $\mathcal{B}_i$, let $n_i := |V(\mathcal{B}_i)|$. If $n_i = s_i$, then

$$e(\mathcal{B}_i) \leq \binom{s_i}{\min\{r, \lfloor s_i/2 \rfloor\}} \leq (n_i - 1)\binom{s_i - 1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}}.$$

If $n_i \geq s_i + 1$, then we apply Theorem 2.1 to $\mathcal{B}_i$ with cycle length $s_i + 1$. We obtain

$$e(\mathcal{B}_i) \leq \max\{f(n_i, s_i+1, r, 2), f(n_i, s_i+1, \lfloor s_i/2 \rfloor\}.$$

Furthermore,

$$\begin{aligned} f(n_i, s_i+1, r, 2) &= \binom{s_i-1}{\min\{r, \lfloor (s_i-1)/2 \rfloor\}} + 2(n_i - s_i + 1) \\ &\le (s_i-1)\binom{s_i-2}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}} + (n_i - s_i)\binom{s_i-2}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}} \\ &= (n_i-1)\binom{s_i-2}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}}. \end{aligned}$$

And $f(n_i, s_i+1, r, \lfloor s_i/2 \rfloor) \le (n_i-1)\binom{s_i-1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}}$.

In all cases we get

$$e(\mathcal{B}_i) \le (n_i-1)\binom{s_i-1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}}. \tag{19}$$

For $\mathcal{B}_1$, if $n_1 = s_1$ then $e(\mathcal{B}_1) \le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}}$ and so by (19),

$$e(\mathcal{H}) \le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + \sum_{i=2}^{p}(n_i-1)\binom{s_i-1}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}}. \tag{20}$$

If $s_1 \ge \lceil (k+1)/2 \rceil$, then from (20) we obtain

$$\begin{aligned} e(\mathcal{H}) &\le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + \sum_{i=2}^{p}(n_i-1)\binom{k-s_1}{\min\{r-1, \lfloor (k-s_1)/2 \rfloor\}} \le f(n, k, r, k-s_1) \\ &\le \max\{f(n, k, r, 1), f(n, k, r, \lfloor (k-1)/2 \rfloor\}). \end{aligned}$$

Otherwise,

$$e(\mathcal{H}) \le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + \sum_{i=2}^{p}(n_i-1)\binom{s_1-1}{\min\{r-1, \lfloor (s_1-1)/2 \rfloor\}} \le f(n, k, r, \lfloor (k-1)/2 \rfloor).$$

If $n_1 \ge s_1 + 1$, then we get

$$e(\mathcal{B}_1) \le \max\{f(n_1, s_1+1, r, 2), f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor\}).$$

If $f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor) \ge f(n_1, s_1+1, r, 2)$, then together with (19), we get

$$e(\mathcal{H}) \le f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor) + \sum_{i=2}^{p}(n_i-1)\binom{\lfloor \frac{k-1}{2} \rfloor}{\min\{r-1, \lfloor \frac{k-1}{4} \rfloor\}} \le f(n, k, r, \lfloor (k-1)/2 \rfloor).$$

If $f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor) < f(n_1, s_1+1, r, 2)$, then

$$f(n_1, s_1+1, r, 2) = \binom{s_1-1}{\min\{r, \lfloor (s_1-1)/2 \rfloor\}} + 2(n_1 - s_1 + 1) \le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + 2(n_1 - s_1).$$

Thus we obtain

$$e(\mathcal{H}) \le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + 2(n_1 - s_1) + \sum_{i=2}^{p} (n_i - 1)\binom{s_i - 1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}},$$

and we are done as in the the case for (20).

Case 2: $n = k+1$ and $\mathcal{H}$ contains a cut edge.

Let e be a cut edge of $\mathcal{H}$. By (18), each component $\mathcal{C}$ of $\mathcal{H} \setminus e$ contains only at most one vertex of e. If $|e| \ge 3$, then $e(\mathcal{H} \setminus e) \le \binom{k+1-2}{\min\{r, \lfloor (k+1-2)/2 \rfloor\}}$. Hence $e(\mathcal{H}) \le \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 1 < f(n, k, r, 1)$.

So we may assume $|e| = 2$. Suppose first that $\mathcal{H} \setminus e$ contains a component $\mathcal{C}$ with $2 \le |V(\mathcal{C})| \le k-1$. Then

$$e(\mathcal{H}) \le 1 + \binom{|V(\mathcal{C})|}{\min\{r, \lfloor |V(\mathcal{C})|/2 \rfloor\}} + \binom{(k+1) - |V(\mathcal{C})|}{\min\{r, \lfloor ((k+1) - |V(\mathcal{C})|)/2 \rfloor\}} \le 1 + \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 1$$
$$= f(n, r, k, 1).$$

Thus $\mathcal{H} \setminus e$ must consist of one component of size k and one of size 1. The same also holds for every other cut edge e' of $\mathcal{H}$. This together with (18) implies that if $\mathcal{H}$ has two cut edges e, e', then e' is a cut edge of $\mathcal{H} \setminus e$, and vice versa. Therefore $e(\mathcal{H}) \le \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 2 = f(n, k, r, 1)$.

So we may assume that e is the only cut edge of $\mathcal{H}$. Let $\mathcal{C}$ be the component of $\mathcal{H}$ of size k. This component cannot contain a Berge cycle of length k, otherwise with e we would obtain Berge path with of length k.

If $\mathcal{C}$ is 2-connected, then by Theorem 2.1,

$$e(\mathcal{H}) = e(\mathcal{C}) + 1 \le \max\{f(k, k, r, 2), f(k, k, r, \lfloor (k-1)/2 \rfloor)\} < f(n, k, r, 1).$$

Otherwise $\mathcal{C}$ has a cut vertex v and a block $\mathcal{B}$ with $2 \le |V(\mathcal{B})| \le k-1$. Therefore

$$e(\mathcal{C}) \le \binom{|V(\mathcal{B})|}{\min\{r, \lfloor |V(\mathcal{B})|/2 \rfloor\}} + \binom{k - |V(\mathcal{B})| + 1}{\min\{r, \lfloor (k - |V(\mathcal{B})| + 1)/2 \rfloor\}} \le \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 1,$$

so we get $e(\mathcal{H}) = e(\mathcal{C}) + 1 \le f(n, k, r, 1)$. This proves the theorem. □

9 Concluding remarks

1. As it is mentioned in Theorem 2.3, if $k \ge 4r$ and n is asymptotically larger than $\frac{2^{r-1}}{r}k$, then our bound is also exact for r-graphs: a sharpness example is $\mathcal{H}_{n,k,r,\lfloor (k-1)/2 \rfloor}$. We think that for smaller n, our bound for r-graphs is not exact. It would be interesting and challenging to find exact bounds for the number of edges in n-vertex 2-connected r-graphs with no cycles of length k or longer for $k > r$ and $k \le n < \frac{2^{r-1}}{r}k$.

2. When r is large, $k \ge 4r$ and n is polynomial in k, then $\mathcal{H}_{n,k,r,2}$ has not much more than $\binom{k-2}{r}$ edges. Also $\mathcal{H}_{n,k,r,2}$ is not uniform whenever $r \ge 4$. The following construction of 2-connected

r-uniform hypergraphs also has more than $\binom{k-2}{r}$ edges in this case, although fewer edges than $\mathcal{H}_{n,k,r,2}$ has (and it works only for n such that $n-k+2$ is divisible by $r-1$).

Construction 9.1. *Fix $k \geq 4r \geq 12$, $s \geq 1$, $n = k-2+s(r-1)$. Define the n-vertex r-graph $F_{n,k,r,s}$ as follows. The vertex set of $F_{n,k,r,s}$ is partitioned into $s+1$ sets $A_1, \ldots, A_s, C$ such that $|C| = k-2$ and $|A_i| = r-1$ for all $i \in [s]$. We fix two special vertices $c_1, c_2 \in C$. The edge set of $F_{n,k,r,s}$ consists of all edges contained in C and of the $2(r-1)$ edges of the form $A_i \cup \{c_j\}$ for $i \in [s]$ and $j \in [2]$.*

We do not currently know of any *uniform* hypergraphs with more edges and no Berge cycles of length k or longer.

3. Note that here we use r^--graphs to prove a bound for r-graphs when $k > r$ and in [15] we used r^+-graphs (i.e. hypergraphs with the lower rank at least r) in the case $k < r$.

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Appendix about convexity

Claim 9.2. *For fixed positive integers n, k, and r, the function*

$$f(n,k,r,a) = \binom{k-a}{\min\{r, \lfloor \frac{k-a}{2} \rfloor\}} + (n-k+a)\binom{a}{\min\{r-1, \lfloor a/2 \rfloor\}}$$

is convex over integers $\max\{0, k-n\} \leq a \leq k$.

In particular, if we consider $f(n,k,r,a)$ over a domain of integers, say $\{c, \ldots, d\}$ where $c, d \in \mathbb{Z}$, $\max\{0, k-n\} \leq c \leq d \leq k$ then $f(n,r,k,a)$ attains its maximum at either $a = c$ or $a = d$.

Proof. Since we only consider integer values of a, we may view $f(n,k,r,a)$ as a sequence of numbers. We say a sequence of real numbers $(f_i)_{i=u}^{v}$ is *convex* if $f_{i-1} + f_{i+1} \geq 2f_i$ for all $u < i < v$.

Fact 9.3. *Let $u < v < w$ be integers. Suppose $(f_i)_{i=u}^{v+1}$ and $(g_i)_{i=v}^{w}$ are convex sequences of real numbers such that $f_v = g_v$ and $f_{v+1} = g_{v+1}$. Then the sequence $(h_i)_{i=u}^{w}$ where*

$$h_i := \begin{cases} f_i, & u \leq i \leq v+1 \\ g_i, & v \leq i \leq w \end{cases}$$

is convex.

Indeed for any $u < i < w$, either $(h_{i-1}, h_i, h_{i+1}) = (f_{i-1}, f_i, f_{i+1})$ or $(h_{i-1}, h_i, h_{i+1}) = (g_{i-1}, g_i, g_{i+1})$.

The following two facts are easy to check.

Fact 9.4. *The sequence $(x_i)_{i=0}^{\infty}$ where $x_i := \binom{i}{\lfloor i/2 \rfloor}$ is convex.*

Fact 9.5. *For any fixed positive integer r, the sequence $(y_i)_{i=0}^{\infty}$ where $y_i = \binom{i}{r}$ is convex.*

By Facts 9.3–9.5, function $g_1(a) := \binom{k-a}{\min\{\lfloor \frac{k-a}{2} \rfloor, r\}}$ is convex for integers $0 \leq a \leq k$, and $g_2(a) := \binom{a}{\min\{r-1, \lfloor a/2 \rfloor\}}$ is convex for integers $a \geq 0$. Here we use that $g_1(a) = \binom{k-a}{r}$ when $a \leq k - 2r$ and $g_1(a) = \binom{k-a}{\lfloor (k-a)/2 \rfloor}$ when $a \geq k - 2r - 1$. One can show similar cut-offs for $g_2(a)$.

Note that $g_2(a)$ is non-decreasing. We also show that $(n-k+a) \cdot g_2(a)$ is convex for integers $a \geq \max\{0, k-n\}$:

$$\begin{aligned} & (n-k+(a-1)) \cdot g_2(a-1) + (n-k+(a+1)) \cdot g_2(a+1) \\ = \ & (n-k+a) \cdot (g_2(a-1) + g_2(a+1)) - g_2(a-1) + g_2(a+1) \\ \geq \ & (n-k+a) \cdot (2g_2(a)) + 0 \\ = \ & 2(n-k+a) \cdot g_2(a). \end{aligned}$$

Since the sum of two convex sequences is also convex, function $g_1(a) + (n-k+a) \cdot g_2(a) = f(n,k,r,a)$ is also convex for integers $\max\{0, k-n\} \leq a \leq k$. This proves the claim. □