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Existence of solutions for perturbed fourth order elliptic equations with variable exponents

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Abstract. Using variational methods, we study the existence and multiplicity of solutions for a class of fourth order elliptic equations of the form

$$\begin{cases} \Delta^2_{p(x)} u - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \Delta_{p(x)} u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain, $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2} \Delta u)$ is the operator of fourth order called the $p(x)$-biharmonic operator, $\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$ is the $p(x)$-Laplacian, $p : \overline{\Omega} \to \mathbb{R}$ is a log-Hölder continuous function, $M : [0, +\infty) \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are two continuous functions satisfying some certain conditions.

Keywords: fourth order elliptic equations, Kirchhoff type problems, variable exponents, variational methods.

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1 Introduction

In this paper, we are interested in the existence of weak solutions for the following fourth order elliptic equations

$$\begin{cases} \Delta^2_{p(x)} u - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \Delta_{p(x)} u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain, $\Delta^2_{p(x)} u = \Delta\left(|\Delta u|^{p(x)-2} \Delta u\right)$ is the operator of fourth order called the $p(x)$-biharmonic operator, $\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$ is the $p(x)$-Laplacian, the exponent $p : \overline{\Omega} \to \mathbb{R}$ is log-Hölder continuous, that is, there exists $\overline{c} > 0$ such that $|p(x) - p(y)| \leq -\frac{\overline{c}}{\log|x-y|}$ for all $x, y \in \overline{\Omega}$ with $0 < |x - y| \leq \frac{1}{2}$ and $1 < p^- :=$

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$\inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < \frac{N}{2}$, the function $M \in C([0,+\infty),\mathbb{R})$ may be degenerate at zero and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following subcritical growth condition:

(F_0) There exists $C > 0$ such that

$$|f(x,t)| \leq C(1 + |t|^{q(x)-1}), \quad \forall x \in \overline{\Omega}, \quad t \in \mathbb{R},$$

where $q \in C_+(\overline{\Omega})$ and $q(x) < p_2^*(x) = \frac{Np(x)}{N-2p(x)}$ for all $x \in \overline{\Omega}$.

We point out that if $p(.)$ is a constant then problem (1.1) has been studied by many authors in recent years, we refer to some interesting papers [4, 11, 21, 26, 27, 31, 32, 36–38, 40]. In [38], Wang and An considered the following fourth-order elliptic equation

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u) & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain, $M : [0,+\infty) \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are two continuous functions. This problem is related to the stationary analog of the evolution equation of Kirchhoff type

$$u_{tt} + \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,t), \tag{1.3}$$

where Δ^2 is the biharmonic operator, ∇u denotes the spatial gradient of u, see [8] for the meaning of the problem from the point of view of physics and engineering. By assuming that M is bounded on $[0,+\infty)$ and the nonlinear term f satisfies the Ambrosetti–Rabinowitz type condition, Wang et al. obtained in [38] at least one nontrivial solution for problem (1.2) using the mountain pass theorem. Moreover, the authors also showed the existence at least two solutions in the case when f is asymptotically linear at infinity. After that, Wang et al. [37] studied problem (1.2) in the case when M is unbounded function, i.e. $M(t) = a + bt$, where $a > 0$, $b \geq 0$ by using the mountain pass techniques and the truncation method. Some extensions regarding these results can be found in [4, 11, 21, 31, 36, 40] in which the authors considered problem (1.2) in $\mathbb{R}^N$ or the nonlinearities involved critical exponents. In [26,27,32], problem (1.1) was studied in the general case when $p(.) = p \in (1,+\infty)$ is a constant.

In recent years, the study of differential equations and variational problems with nonstandard $p(x)$-growth conditions has received more and more interest. The reason of such interest starts from the study of the role played by their applications in mathematical modelling of non-Newtonian fluids, in particular, the electrorheological fluids and of other phenomena related to image processing, elasticity and the flow in porous media, we refer the readers to [5,35,43] for more details. Some results on problems involving $p(x)$-Laplace operator or $p(x)$-biharmonic operator can be found in [6, 7, 9, 10, 12, 16, 17, 28, 30, 33]. These types of operators where $p(.)$ is a continuous function possess more complicated properties than the constant cases, mainly due to the fact that they are not homogeneous. We also find that Kirchhoff type problems with variable exponents has received a lot of attention in recent years, see for example [1, 2, 13–15, 18–20, 41].

Motivated by the contributions cited above, in this paper we study the existence and multiplicity of solutions for perturbed fourth order elliptic equations with variable exponents of the form (1.1). More precisely, we consider problem (1.1) in two case when f is sublinear or superlinear at infinity. In the sublinear case, we obtain an existence result using the minimum principle while in the superlinear case we prove some existence and multiplicity results with the help of the Mountain Pass Theorem, Fountain Theorem and Dual Fountain Theorem. To the best of our knowlegde, the present paper is the first contribution to the study of this type of problems in Sobolev spaces with variable exponents.

2 Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books of Diening et al. [22] and Musielak [34], the papers of Fan et al. [24,25], Zang et al. [42], Ayoujil et al. [6,7] and Boureanu et al. [10]. Set

$$C_+(\overline{\Omega}) := \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \quad \text{and} \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \text{a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf\left\{ \mu > 0 :\ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{2.1}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{2.2}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow \ \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

As in the constant case, for any positive integer k, the Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \le k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)},$$

also becomes a separable and reflexive Banach space. Due to the log-Hölder continuity of the exponent p, the space $C^\infty(\Omega)$ is dense in $W^{k,p(x)}(\Omega)$. Moreover, we have the following embedding results.

Proposition 2.1 (see [24, 25]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

where $p_k^*(x) = \frac{Np(x)}{N - kp(x)}$ if $kp(x) < N$ and $p_k^*(x) = +\infty$ if $kp(x) > N$. If we replace $\le$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space

$$X = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$$

equipped with the norm $\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$ or $\|u\|_X = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{\alpha=2} |D^\alpha u|_{p(x)}$.

According to [42], the norm $\|.\|_X$ is equivalent to the norm $|\Delta.|_{p(x)}$ in the space X. Consequently, the norms $\|.\|_{2,p(x)}$, $\|.\|_X$ and $|\Delta.|_{p(x)}$ are equivalent. For this reason, we can consider in the space X the following equivalent norms

$$\|u\| = |\Delta u|_{p(x)} + |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf\left\{\mu > 0 : \ \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\mu}\right|^{p(x)}\right) dx \le 1\right\}.$$

Let us define the functional $\Lambda : X \to \mathbb{R}$ by

$$\Lambda(u) = \int_\Omega \left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)}\right) dx, \qquad u \in X, \tag{2.4}$$

then using similar arguments as in [10, Proposition 1] we obtain the following modular-type inequalities.

Proposition 2.2. *For $u, u_n \in X$ and the functional $\Lambda : X \to \mathbb{R}$ define as in (2.4), we have the following assertions:*

(1) $\|u\| < 1$ (*respectively* $= 1; > 1$) $\Longleftrightarrow \Lambda(u) < 1$ (*respectively* $= 1; > 1$);

(2) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \Lambda(u) \le \|u\|^{p^-}$;

(3) $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le \Lambda(u) \le \|u\|^{p^+}$;

(4) $\|u_n\| \to 0$ (*respectively* $\to \infty$) $\Longleftrightarrow \Lambda(u_n) \to 0$ (*respectively* $\to \infty$) *as* $n \to \infty$.

3 Main results

In this section, we will discuss the existence and multiplicity of weak solutions of problem (1.1). Let us denote by c_i, $i = 1, 2, \dots$ general positive constants whose value may change from line to line. We will look for weak solutions of problem (1.1) in the space $X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$ with the norm mentioned as in Section 2. First, let us make the definition of a weak solution of problem (1.1) as follows.

Definition 3.1. We say that $u \in X$ is a weak solution of problem (1.1) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\,dx + M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx - \int_\Omega f(x,u)v\,dx = 0$$

for all $v \in X$.

Let us define the functional $J : X \to \mathbb{R}$ by

$$\begin{aligned} J(u) &= \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right) - \int_\Omega F(x,u)\,dx \\ &= \Phi(u) - \Psi(u), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \Phi(u) &= \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right), \\ \Psi(u) &= \int_\Omega F(x,u)\,dx, \quad u \in X \end{aligned} \tag{3.2}$$

and $\widehat{M}(t) = \int_0^t M(s)\,ds$.

Using some simple computations, we can show that $J \in C^1(X, \mathbb{R})$ and its derivative is given by the formula

$$\begin{aligned} J'(u)(v) = &\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\,dx + M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx \\ &- \int_\Omega f(x,u)v\,dx \end{aligned}$$

for all $u, v \in X$. Thus, we will seek weak solutions of problem (1.1) as the critical points of the functional J. We first obtain an existence result for problem (1.1) in the case when f is sublinear at infinity. We also consider case when the Kirchhoff function M are allowed to be degenerate at zero.

Theorem 3.2. *Assume that the condition* (F_0) *hold with* $1 < q^- \leq q^+ < p^-$. *Moreover, we assume that the following conditions hold:*

(M_1') *There exist* $m_0', t_0 > 0$ *such that*

$$M(t) \geq m_0', \qquad \forall t \geq t_0;$$

(M_2') *There exists* $\alpha > 1$ *such that*

$$\lim_{t \to 0} \frac{M(t)}{t^{\alpha - 1}} = 0;$$

(F_0') *There exist positive constants* $C_0, \delta > 0$ *and a subset* $\Omega_0 \subset \Omega$, *a function* $r \in C_+(\overline{\Omega})$, $r(x) < p(x)$ *for all* $x \in \overline{\Omega}$, *such that*

$$|F(x,t)| \geq C_0 |t|^{r(x)}$$

for all $x \in \Omega_0$ *and* $|t| < \delta$.

Then problem (1.1) *has a nontrivial weak solution.*

Proof. From (F_0), there exists $c_1 > 0$ such that

$$|F(x,t)| \leq c_1 \left(|t| + |t|^{q(x)} \right), \qquad \forall x \in \Omega,\ t \in \mathbb{R}. \tag{3.3}$$

We also obtain from (M_1') and (M_2') that

$$\widehat{M}(t) \geq m_0' t,\ \forall t \geq t_0, \qquad \widehat{M}(t) \leq \epsilon t^{\alpha}, \quad \forall t \in (0, t_\epsilon), \tag{3.4}$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$ and t_ϵ is a positive constant depending on $\epsilon > 0$.

For t_0 given as above, let us define the set

$$\widehat{X} := \left\{ u \in X : \min\{ |\nabla u|_{p(x)}^{p^+}, |\nabla u|_{p(x)}^{p^-} \} \geq p^+ t_0 \right\}.$$

Then it follows that $\widehat{X}$ is a closed subspace of the reflexive Banach space X, so $\widehat{X}$ is a reflexive Banach space too. Moreover, for any $u \in \widehat{X}$, we have

$$\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx \geq \frac{1}{p^+} \min \left\{ |\nabla u|_{p(x)}^{p^+}, |\nabla u|_{p(x)}^{p^-} \right\} \geq t_0.$$

By relations (3.3) and (3.4), by the Sobolev embedding, we deduce that for any $u \in \widehat{X}$ with $\|u\| > 1$ large enough,

$$\begin{aligned} J(u) &= \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx + \widehat{M} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx \right) - \int_\Omega F(x,u)\, dx \\ &\geq \frac{1}{p^+} \int_\Omega |\Delta u|^{p(x)}\, dx + \frac{m_0'}{p^+} \int_\Omega |\nabla u|^{p(x)}\, dx - c_1 \int_\Omega |u|^{q(x)}\, dx - c_1 \int_\Omega |u|\, dx \\ &\geq \frac{\min\{1, m_0'\}}{p^+} \|u\|^{p^-} - c_2 \|u\|^{q^+} - c_2 \|u\|. \end{aligned}$$

Since $1 < q^+ < p^-$ it follows that the functional J is coercive in $\widehat{X}$. Moreover, we find that J is weakly lower semicontinuous in $\widehat{X}$ and thus, J attains its infimum in $\widehat{X}$ and there exists $u_0 \in \widehat{X}$ such that

$$J(u_0) = \inf_{u \in \widehat{X}} J(u).$$

Next, we show that $u_0 \neq 0$ i.e. u_0 is a nontrivial weak solution of problem (1.1). Let $x_0 \in \Omega_0$. Since $p, r \in C_+(\overline{\Omega})$, we can choose $\rho > 0$ small enough such that $B_\rho(x_0) \subset \Omega_0$ and $p_0^- := \min_{x \in B_\rho(x_0)} p(x) > r_0^+ := \max_{x \in B_\rho(x_0)} r(x)$. Now, let us choose $\phi \in C_0^\infty(\Omega)$ with $|\phi| \leq 1$, $\|\phi\|_{W^{2,p(x)}(B_\rho(x_0)) \cap W_0^{1,p(x)}(B_\rho(x_0))} \leq c(\rho)$ and $|\phi|_{L^{r(x)}(B_\rho(x_0))} > 0$. Then, for any $0 < t < \delta$ we deduce from (F_0') and (3.4) that

$$\begin{aligned}
J(t\phi) &= \int_\Omega \frac{1}{p(x)} |\Delta(t\phi)|^{p(x)}\, dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla(t\phi)|^{p(x)}\, dx\right) - \int_\Omega F(x, t\phi)\, dx \\
&\leq \frac{t^{p_0^-}}{p^-} \int_\Omega |\Delta\phi|^{p(x)}\, dx + \frac{\epsilon t^{\alpha p_0^-}}{(p^-)^\alpha} \left(\int_\Omega |\nabla\phi|^{p(x)}\, dx\right)^\alpha - C_0 t^{r_0^+} \int_{B_\rho(x_0)} |\phi|^{r(x)}\, dx \\
&\leq \frac{t^{p_0^-}}{p^-} \max\{c(\rho)^{p_0^-}, c(\rho)^{p_0^+}\} + \frac{\epsilon t^{\alpha p_0^-}}{(p^-)^\alpha} \max\{c(\rho)^{\alpha p_0^-}, c(\rho)^{\alpha p_0^+}\} - C_0 t^{r_0^+} \int_{B_\rho(x_0)} |\phi|^{r(x)}\, dx.
\end{aligned}$$

Since $r_0^+ < p_0^- < \alpha p_0^-$, we get $J(t_1\phi) < 0$ by taking $0 < t_1 < \delta$ small enough. Hence, $J(u_0) \leq J(t_1\phi) < 0$. Therefore, $u_0 \in \widehat{X} \subset X$ is a nontrivial critical point of J and problem (1.1) has a nontrivial weak solution. □

In the next part of this paper, we will study the existence and multiplicity of weak solutions for problem (1.1) in the case when f is superlinear at infinity. In the sequel, we always assume that the following conditions hold:

(M_1) There exists $m_0 > 0$ such that

$$M(t) \geq m_0, \quad \forall t \geq 0;$$

(M_2) There exists $\mu \in (0, 1)$ such that

$$\widehat{M}(t) \geq (1 - \mu) M(t) t, \quad \forall t \geq 0,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$.

Definition 3.3. A functional J is said to satisfy the Palais–Smale condition (or (PS) condition) in a space X, if any sequence $\{u_n\} \subset X$ such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Lemma 3.4. *If M satisfies (M_1)–(M_2), f satisfies (F_0) and the Ambrosetti–Rabinowitz type condition, namely,*

(F_1) *there exist $T_0 > 0$ and $\theta > \frac{p^+}{1-\mu}$ such that*

$$0 < \theta F(x, t) \leq f(x, t)t, \quad \forall x \in \Omega, \quad |t| \geq T_0,$$

then the functional J satisfies the (PS) condition.

Proof. Suppose that $\{u_n\} \subset X$, $|J(u_n)| \leq c$ and $J'(u_n) \to 0$ in X^* as $n \to \infty$. We will show that $\{u_n\}$ is bounded in X. By contradiction, we assume that $\|u_n\| \to +\infty$. For n large enough, by the conditions (F_1), (M_1), (M_2) and Proposition 2.2 we have

$$\begin{aligned}
c+\|u_n\| &\geq J(u_n)-\frac{1}{\theta}J'(u_n)(u_n)\\
&=\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx+\widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)-\int_\Omega F(x,u_n)\,dx\\
&\quad-\frac{1}{\theta}\int_\Omega |\Delta u_n|^{p(x)}\,dx-\frac{1}{\theta}M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx\\
&\quad+\frac{1}{\theta}\int_\Omega f(x,u_n)u_n\,dx\\
&\geq \left(\frac{1}{p^+}-\frac{1}{\theta}\right)\int_\Omega |\Delta u_n|^{p(x)}\,dx+(1-\mu)M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\\
&\quad-\frac{1}{\theta}M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx\\
&\quad+\int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n-F(x,u_n)\right)\,dx\\
&\geq \left(\frac{1}{p^+}-\frac{1}{\theta}\right)\int_\Omega |\Delta u_n|^{p(x)}\,dx+m_0\left(\frac{1-\mu}{p^+}-\frac{1}{\theta}\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx\\
&\quad+\int_{\{x\in\Omega:\ |u_n|\geq T_0\}}\left(\frac{1}{\theta}f(x,u_n)u_n-F(x,u_n)\right)\,dx-c_3\\
&\geq c_4\|u_n\|^{p^-}-c_3,
\end{aligned}$$

where $c_4=\min\left\{\frac{1}{p^+}-\frac{1}{\theta},m_0\left(\frac{1-\mu}{p^+}-\frac{1}{\theta}\right)\right\}>0$ since $\theta>\frac{p^+}{1-\mu}>p^+>1$.

Dividing by $\|u_n\|^{p^-}$ in the last inequality and letting $n\to\infty$ we obtain a contradiction. It follows that the sequence $\{u_n\}$ is bounded in X. Without loss of generality, we assume that $\{u_n\}$ converges weakly to u in X. Then $\{u_n\}$ converges strongly to u in $L^{r(x)}(\Omega)$ for all $r(x)<p_2^*(x)$. Since $J'(u_n)\to 0$ in X^* we deduce that $J'(u_n)(u_n-u)\to 0$ as $n\to\infty$. We also have $J'(u)(u_n-u)\to 0$ as $n\to\infty$ because $\{u_n\}$ converges weakly to u in X. Thus,

$$\lim_{n\to\infty}(J'(u_n)-J'(u))(u_n-u)=0. \tag{3.5}$$

Using (F_0) and the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_\Omega (f(x,u_n)-f(x,u))(u_n-u)\,dx\right|\\
&\quad\leq \int_\Omega |f(x,u_n)-f(x,u)||u_n-u|\,dx\\
&\quad\leq C\int_\Omega\left(2+|u_n|^{q(x)-1}+|u|^{q(x)-1}\right)|u_n-u|\,dx\\
&\quad\leq 2C\left(2+\left||u_n|^{q(x)-1}\right|_{q'(x)}+\left||u|^{q(x)-1}\right|_{q'(x)}\right)|u_n-u|_{q(x)}\\
&\quad\to 0,\quad q'(x)=\frac{q(x)}{q(x)-1}
\end{aligned}$$

when $n\to+\infty$. This implies that

$$\lim_{n\to\infty}\int_\Omega (f(x,u_n)-f(x,u))(u_n-u)\,dx=0. \tag{3.6}$$

Since the sequence $\{u_n\}$ converges weakly to $u \in X = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$, it is bounded in X and converges weakly to u in $W_0^{1,p(x)}(\Omega)$, so we deduce that

$$\lim_{n\to\infty} \left[M\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right) - M\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)\right] \int_\Omega |\nabla u|^{p(x)-2}\nabla u(\nabla u_n - \nabla u)\,dx = 0. \tag{3.7}$$

Let us recall the following elementary inequalities (see [6])

$$\left(|\xi|^{s-2}\xi - |\zeta|^{s-2}\zeta\right)(\xi-\zeta) \ge \frac{1}{2^s}|\xi-\zeta|^s, \quad s \ge 2, \tag{3.8}$$

$$\left(|\xi|^{s-2}\xi - |\zeta|^{s-2}\zeta\right)(\xi-\zeta)\left(|\xi|+|\zeta|\right)^{2-s} \ge (s-1)|\xi-\zeta|^2, \quad 1 < s < 2 \tag{3.9}$$

for all $\xi, \zeta \in \mathbb{R}^N$. Put

$$U_{p(x)} := \{x \in \Omega : \; p(x) \ge 2\}, \qquad V_{p(x)} := \{x \in \Omega : \; 1 < p(x) < 2\}, \tag{3.10}$$

then, it follows from (3.8) and (3.9) that

$$\int_{U_{p(x)}} |\Delta u_n - \Delta u|^{p(x)}\,dx \le c_5 \int_\Omega A^{(1)}(\Delta u_n, \Delta u)\,dx, \tag{3.11}$$

$$\int_{U_{p(x)}} |\nabla u_n - \nabla u|^{p(x)}\,dx \le c_5 \int_\Omega A^{(N)}(\nabla u_n, \nabla u)\,dx, \tag{3.12}$$

$$\int_{V_{p(x)}} |\Delta u_n - \Delta u|^{p(x)}\,dx \le c_6 \int_\Omega \left(A^{(1)}(\Delta u_n, \Delta u)\right)^{\frac{p(x)}{2}} \left(C^{(1)}(\Delta u_n, \Delta u)\right)^{(2-p(x))\frac{p(x)}{2}}\,dx, \tag{3.13}$$

$$\int_{V_{p(x)}} |\nabla u_n - \nabla u|^{p(x)}\,dx \le c_6 \int_\Omega \left(A^{(N)}(\nabla u_n, \nabla u)\right)^{\frac{p(x)}{2}} \left(C^{(N)}(\nabla u_n, \nabla u)\right)^{(2-p(x))\frac{p(x)}{2}}\,dx, \tag{3.14}$$

where $A^{(i)}, C^{(i)} : \mathbb{R}^i \times \mathbb{R}^i \to \mathbb{R}$, $i = 1, N$ are defined by the following formulas

$$A^{(i)}(\xi,\zeta) := \left(|\xi|^{p(x)-2}\xi - |\zeta|^{p(x)-2}\zeta\right)(\xi-\zeta), \qquad C^{(i)}(\xi,\zeta) := |\xi| + |\zeta|$$

for all $\xi, \zeta \in \mathbb{R}^i$, $i = 1, N$. Now, from the definition of the functional J and relations (3.5)–(3.7), we have

$$\begin{aligned}
0 &\le \int_\Omega (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u)(\Delta u_n - \Delta u)\,dx \\
&\quad + M\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)\int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)\,dx \\
&= (J'(u_n) - J'(u))(u_n - u) + \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)\,dx \\
&\quad + \left[M\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right) - M\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)\right]\int_\Omega |\nabla u|^{p(x)-2}\nabla u(\nabla u_n - \nabla u)\,dx \\
&\to 0
\end{aligned}$$

when $n \to \infty$. By (M_1), it follows that

$$\lim_{n\to\infty} \int_\Omega A^{(1)}(\Delta u_n, \Delta u)\,dx = \lim_{n\to\infty} \int_\Omega A^{(N)}(\nabla u_n, \nabla u)\,dx = 0. \tag{3.15}$$

For this reason, we can assume that $0 \le \int_\Omega A^{(1)}(\Delta u_n, \Delta u)\,dx < 1$. If $\int_\Omega A^{(1)}(\Delta u_n, \Delta u)\,dx = 0$ then $A^{(1)}(\Delta u_n, \Delta u) = 0$ since $A^{(1)}(\Delta u_n, \Delta u) \ge 0$ in Ω. If $0 < \int_\Omega A^{(1)}(\Delta u_n, \Delta u)\,dx < 1$, then thanks to the Young inequality

$$ab \le \frac{a^r}{r} + \frac{b^{r'}}{r'}, \quad \forall a, b > 0, \qquad \frac{1}{r} + \frac{1}{r'} = 1, \quad r, r' \in (1, +\infty),$$

with

$$a = \left(A^{(1)}(\Delta u_n, \Delta u)\right)^{\frac{p(x)}{2}} \left(\int_{V_{p(x)}} A^{(1)}(\Delta u_n, \Delta u)\,dx\right)^{\frac{-p(x)}{2}}, \qquad b = \left(C^{(1)}(\Delta u_n, \Delta u)\right)^{(2-p(x))\frac{p(x)}{2}},$$

$$r = \frac{2}{p(x)}, \qquad r' = \frac{2}{2-p(x)},$$

we conclude that

$$\begin{aligned}
&\left(\int_{V_{p(x)}} A^{(1)}(\Delta u_n, \Delta u)\,dx\right)^{-\frac{1}{2}} \int_{V_{p(x)}} \left(A^{(1)}(\Delta u_n, \Delta u)\right)^{\frac{p(x)}{2}} \left(C^{(1)}(\Delta u_n, \Delta u)\right)^{(2-p(x))\frac{p(x)}{2}} dx \\
&\le \int_{V_{p(x)}} \left(A^{(1)}(\Delta u_n, \Delta u)\right)^{\frac{p(x)}{2}} \left(\int_{V_{p(x)}} A^{(1)}(\Delta u_n, \Delta u)\,dx\right)^{-\frac{p(x)}{2}} \left(C^{(1)}(\Delta u_n, \Delta u)\right)^{(2-p(x))\frac{p(x)}{2}} dx \\
&\le \int_{V_{p(x)}} \left(A^{(1)}(\Delta u_n, \Delta u) \left(\int_{V_{p(x)}} A^{(1)}(\Delta u_n, \Delta u)\,dx\right)^{-\frac{1}{2}} + \left(C^{(1)}(\Delta u_n, \Delta u)\right)^{p(x)}\right) dx \\
&\le 1 + \int_\Omega \left(C^{(1)}(\Delta u_n, \Delta u)\right)^{p(x)} dx.
\end{aligned}$$

Hence, by relation (3.13),

$$\frac{1}{c_6} \int_{V_{p(x)}} |\Delta u_n - \Delta u|^{p(x)}\,dx \le \left(\int_{V_{p(x)}} A^{(1)}(\Delta u_n, \Delta u)\,dx\right)^{\frac{1}{2}} \left(1 + \int_\Omega \left(C^{(1)}(\Delta u_n, \Delta u)\right)^{p(x)} dx\right). \tag{3.16}$$

We also have

$$\frac{1}{c_6} \int_{V_{p(x)}} |\nabla u_n - \nabla u|^{p(x)}\,dx \le \left(\int_{V_{p(x)}} A^{(N)}(\nabla u_n, \nabla u)\,dx\right)^{\frac{1}{2}} \left(1 + \int_\Omega \left(C^{(N)}(\nabla u_n, \nabla u)\right)^{p(x)} dx\right). \tag{3.17}$$

By (3.11), (3.13), (3.15) and (3.16), we have

$$\int_\Omega |\Delta u_n - \Delta u|^{p(x)}\,dx = \int_{U_{p(x)}} |\Delta u_n - \Delta u|^{p(x)}\,dx + \int_{V_{p(x)}} |\Delta u_n - \Delta u|^{p(x)}\,dx \to 0 \tag{3.18}$$

when $n \to \infty$. Similarly, from (3.12), (3.14), (3.15) and (3.17) we have

$$\int_\Omega |\nabla u_n - \nabla u|^{p(x)}\,dx = \int_{U_{p(x)}} |\nabla u_n - \nabla u|^{p(x)}\,dx + \int_{V_{p(x)}} |\nabla u_n - \nabla u|^{p(x)}\,dx \to 0. \tag{3.19}$$

Therefore,

$$\|u_n - u\|^{p^+} \le \int_\Omega \left(|\Delta u_n - \Delta u|^{p(x)} + |\nabla u_n - \nabla u|^{p(x)}\right) dx \to 0$$

when $n \to \infty$. So, the sequence $\{u_n\}$ converges strongly to $u \in X$ and the functional J satisfies the (PS) condition in X. □

Lemma 3.5. *If M satisfies (M_1), (M_2) and f satisfies (F_0), (F_1) and the following condition*

(F_2) $f(x,t) = o(|t|^{p^+-1})$ *for $x \in \Omega$ uniformly,*

where $q^- > p^+$, then problem (1.1) has a nontrivial weak solution.

Proof. Our idea is to apply the mountain pass theorem [3]. By Lemma 3.4, J satisfies the Palais–Smale condition in X. Since $p^+ < q^- \le q(x) < p_2^*(x)$, the embedding $X \hookrightarrow L^{p^+}(\Omega)$ is continuous and compact and then there exists $c_7 > 0$ such that

$$|u|_{p^+} \le c_7\|u\|, \qquad \forall u \in X. \tag{3.20}$$

Let $\epsilon > 0$ be small enough such that $\epsilon c_7^{p^+} < \frac{1}{2p^+}\min\{1, m_0\}$. By the assumptions (F_0) and (F_2), there exists $c_\epsilon > 0$ depending on ϵ such that

$$|F(x,t)| \le \epsilon|t|^{p^+} + c_\epsilon|t|^{q(x)}, \qquad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.21}$$

Hence, for all $u \in X$ with $\|u\| < 1$, we have

$$\begin{aligned} J(u) &\ge \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right) - \int_\Omega F(x,u)\,dx \\ &\ge \frac{1}{p^+}\int_\Omega |\Delta u|^{p(x)}\,dx + \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \epsilon\int_\Omega |u|^{p^+}\,dx - c_\epsilon\int_\Omega |u|^{q(x)}\,dx \\ &\ge \frac{\min\{1, m_0\}}{p^+}\|u\|^{p^+} - \epsilon c_7^{p^+}\|u\|^{p^+} - \overline{c}_\epsilon\|u\|^{q^-} \\ &\ge \frac{\min\{1, m_0\}}{2p^+}\|u\|^{p^+} - \overline{c}_\epsilon\|u\|^{q^-}, \end{aligned}$$

where $\overline{c}_\epsilon$ is a positive constant. Since $q^- > p^+$, we conclude that there exist $\alpha > 0$ and $\rho > 0$ such that $J(u) \ge \alpha > 0$ for all $u \in X$ with $\|u\| = \rho$.

On the other hand, from (F_1) it follows that

$$F(x,t) \ge c_8|t|^\theta - c_9, \qquad \forall x \in \Omega,\ t \in \mathbb{R}. \tag{3.22}$$

From (M_2) we can easily obtain that

$$\widehat{M}(t) \le \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\mu}}}t^{\frac{1}{1-\mu}} = c_{10}t^{\frac{1}{1-\mu}}, \qquad \forall t > t_0, \tag{3.23}$$

where t_0 is an arbitrary positive constant. Hence, for $w \in X\backslash\{0\}$ and $t > 1$, we have

$$\begin{aligned} J(tw) &= \int_\Omega \frac{1}{p(x)}|t\Delta w|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|t\nabla w|^{p(x)}\,dx\right) - \int_\Omega F(x,tw)\,dx \\ &\le \frac{t^{p^+}}{p^-}\int_\Omega |\Delta w|^{p(x)}\,dx + c_{10}t^{\frac{p^+}{1-\mu}}\left(\int_\Omega |\nabla w|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - c_8t^\theta\int_\Omega |w|^\theta\,dx - c_9 \\ &\to -\infty \quad \text{as } t \to +\infty, \end{aligned}$$

due to $\theta > \frac{p^+}{1-\mu} > p^+$. Since $J(0) = 0$, we conclude that J satisfies all assumptions of the mountain pass theorem [3]. So, J admits at least one nontrivial critical point and problem (1.1) has a nontrivial weak solution. □

In what follows, we will study the multiplicity of weak solutions for problem (1.1) by using the Fountain Theorem and the Dual Fountain Theorem. For the reader's convenience, we recall these results as follows.

As we stated in Section 2, $X = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$ is a reflexive and separable Banach space, so there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\mathrm{span}\{\mathrm{e_j} :\ \mathrm{j} = 1, 2, \ldots, \}}, \qquad X^* = \overline{\mathrm{span}\{\mathrm{e_j^*} :\ \mathrm{j} = 1, 2, \ldots, \}},$$

and

$$\left\langle e_i, e_j^* \right\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

For each $j, k = 1, 2, \ldots,$ let us define $X_j = \mathrm{span}\{\mathrm{e_j}\}$, $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \oplus_{j=k}^\infty X_j$. We first have the following lemma which will be used in the proof of our main results.

Lemma 3.6. *If $s \in C_+(\overline{\Omega})$, $s(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$ denote*

$$\beta_k = \sup\left\{ |u|_{s(x)} :\ \|u\| = 1,\ u \in Z_k \right\},$$

then $\lim_{k\to\infty} \beta_k = 0$.

Proof. It is clear that $0 < \beta_k \le \beta_{k+1}$, so $\beta_k \to \beta \ge 0$. Let $u_k \in Z_k$ be such that $\|u_k\| = 1$ and $0 \le \beta_k - |u_k|_{s(x)} < \frac{1}{k}$. Then, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$ such that $\{u_k\}$ converges weakly to u in X and

$$\lim_{k\to\infty} \left\langle e_j^*, u_k \right\rangle = \left\langle e_j^*, u \right\rangle = 0, \qquad j = 1, 2, \ldots,$$

which implies that $u = 0$ and thus, $\{u_k\}$ converges weakly to 0 in X. Since the embedding $X \hookrightarrow L^{s(x)}(\Omega)$ is compact, $\{u_k\}$ converges strongly to 0 in $L^{s(x)}(\Omega)$. Therefore, we have $\beta_k \to 0$ as $k \to \infty$. $\square$

Proposition 3.7 (see [39, Fountain Theorem]). *Assume that $(X, \|\cdot\|)$ is a separable Banach space, $J \in C^1(X, \mathbb{R})$ is an even functional satisfying the (PS) condition. Moreover, for each $k = 1, 2, \ldots,$ there exist $\rho_k > r_k > 0$ such that*

(A_1) $\inf_{\{u\in Z_k:\ \|u\|=r_k\}} J(u) \to +\infty$ *as* $k \to \infty$;

(A_2) $\max_{\{u\in Y_k:\ \|u\|=\rho_k\}} J(u) \le 0$.

Then J has a sequence of critical values tending to $+\infty$.

Definition 3.8. We say that J satisfies the $(\mathrm{PS})_c^*$ condition (with respect to (Y_n)) if any sequence $\{u_{n_j}\} \subset X$ such that $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$ as $n_j \to +\infty$, contains a subsequence converging to a critical point of J.

Proposition 3.9 (see [39, Dual Fountain Theorem]). *Assume that $(X, \|\cdot\|)$ is a separable Banach space, $J \in C^1(X, \mathbb{R})$ is an even functional satisfying the $(\mathrm{PS})_c^*$ condition. Moreover, for each $k = 1, 2, \ldots,$ there exist $\rho_k > r_k > 0$ such that*

(B_1) $\inf_{\{u\in Z_k:\ \|u\|=\rho_k\}} J(u) \ge 0$;

(B_2) $b_k := \max_{\{u\in Y_k:\ \|u\|=r_k\}} J(u) < 0$;

(B_3) $d_k := \inf_{\{u\in Z_k:\ \|u\|\le\rho_k\}} J(u) \to 0$ *as* $k\to\infty$.

Then J has a sequence of negative critical values tending to 0.

Theorem 3.10. *Assume that the conditions (M_1), (M_2), (F_0), (F_1) hold, and f satisfies*

(F_3) $f(x,-t) = -f(x,t)$ *for all* $x\in\Omega$ *and* $t\in\mathbb{R}$.

Then problem (1.1) *has a sequence of weak solutions $\{\pm u_k\}_{k=1}^{\infty}$ such that $J(\pm u_k)\to+\infty$ as $k\to+\infty$.*

Proof of Theorem 3.10. According to (F_3) and Lemma 3.4, J is an even functional and satisfies the (PS) condition. We will prove Theorem 3.10 by using the Fountain Theorem, see Proposition 3.7. Indeed, we will show that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_1) and (A_2) hold. Thus, the assertion of conclusion can be obtained.

(A_1): Using (M_1) and (3.3), for any $u\in Z_k$,

$$
\begin{aligned}
J(u) &= \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right) - \int_\Omega F(x,u)\,dx\\
&\ge \frac{\min\{1,m_0\}}{p^+}\|u\|^{p^-} - c_1\int_\Omega (|t| + |t|^{q(x)})\,dx\\
&\ge \frac{\min\{1,m_0\}}{p^+}\|u\|^{p^-} - c_{11}|u|_{q(x)}^{q(\xi)} - c_{11}\|u\|, \qquad \text{where } \xi\in\Omega\\
&\ge \begin{cases} \frac{\min\{1,m_0\}}{p^+}\|u\|^{p^-} - c_{11} - c_{11}\|u\| & \text{if } |u|_{q(x)}\le 1,\\ \frac{\min\{1,m_0\}}{p^+}\|u\|^{p^-} - c_{12}\beta_k^{q^+}\|u\|^{q^+} - c_{11}\|u\| & \text{if } |u|_{q(x)} > 1\end{cases}\\
&\ge \frac{\min\{1,m_0\}}{p^+}\|u\|^{p^-} - c_{12}\beta_k^{q^+}\|u\|^{q^+} - c_{11}\|u\| - c_{13},
\end{aligned}
$$

where

$$
\beta_k = \sup\left\{|u|_{q(x)} :\ \|u\| = 1,\ u\in Z_k\right\} \to 0 \quad \text{as } k\to\infty. \tag{3.24}
$$

Now, we deduce from (3.24) that for any $u\in Z_k$, $\|u\| = r_k = \left(\frac{c_{12}q^+\beta_k^{q^+}}{\min\{1,m_0\}}\right)^{\frac{1}{p^--q^+}}$,

$$
\begin{aligned}
J(u) &\ge \frac{\min\{1,m_0\}}{p^+}\|u\|^{p^-} - c_{12}\beta_k^{\alpha^+}\|u\|^{q^+} - c_{11}\|u\| - c_{11}\\
&= \frac{\min\{1,m_0\}}{p^+}\left(\frac{c_{12}q^+\beta_k^{q^+}}{\min\{1,m_0\}}\right)^{\frac{p^-}{p^--q^+}} - c_{12}\beta_k^{q^+}\left(\frac{c_{12}q^+\beta_k^{q^+}}{\min\{1,m_0\}}\right)^{\frac{q^+}{p^--q^+}}\\
&\quad - c_{11}\left(\frac{c_{12}q^+\beta_k^{q^+}}{\min\{1,m_0\}}\right)^{\frac{1}{p^--q^+}} - c_{13}\\
&= \min\{1,m_0\}\left(\frac{1}{p^+} - \frac{1}{q^+}\right)\left(\frac{c_{12}q^+\beta_k^{q^+}}{\min\{1,m_0\}}\right)^{\frac{p^-}{p^--q^+}} - c_{11}\left(\frac{c_{12}q^+\beta_k^{q^+}}{\min\{1,m_0\}}\right)^{\frac{1}{p^--q^+}} - c_{13},
\end{aligned}
$$

converging to $+\infty$ as $k\to+\infty$, because $p^+ < q^- \le q(x) < p_*(x)$ and $\beta_k\to 0$ as $k\to\infty$, see Lemma 3.6.

(A_2): Using (3.22), (3.23), (M_2), for any $w \in Y_k$ with $\|w\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned} J(tw) &= \int_\Omega \frac{1}{p(x)} |t\Delta w|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |t\nabla w|^{p(x)}\,dx\right) - \int_\Omega F(x, tw)\,dx \\ &\le \frac{t^{p^+}}{p^-} \int_\Omega |\Delta w|^{p(x)}\,dx + c_{10} t^{\frac{p^+}{1-\mu}} \left(\int_\Omega |\nabla w|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - c_8 t^\theta \int_\Omega |w|^\theta\,dx - c_9 \\ &\le \frac{\rho_k^{p^+}}{p^-} \int_\Omega |\Delta w|^{p(x)}\,dx + c_{10} \rho_k^{\frac{p^+}{1-\mu}} \left(\int_\Omega |\nabla w|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - c_8 \rho_k^\theta \int_\Omega |w|^\theta\,dx - c_9. \end{aligned}$$

Since $\theta > \frac{p^+}{1-\mu} > p^+$ and $\dim(Y_k) = k$, it is easy to see that $J(u) \to -\infty$ as $\|u\| \to +\infty$ for $u \in Y_k$. Conclusion of Theorem 3.10 is reached by the Fountain Theorem. □

Theorem 3.11. *Assume that the conditions (M_1), (M_2), (F_0) and (F_1), (F_2), (F_3) are satisfied. Moreover, we assume that*

(F_4) *$f(x,t) \ge C|t|^{\gamma(x)-1}$, $t \to 0$, where $p^+ < \gamma^- \le \gamma^+ < \frac{p^-}{1-\mu}$ for all $x \in \Omega$ and $t \in \mathbb{R}$.*

Then problem (1.1) has a sequence of weak solutions $\{\pm v_k\}_{k=1}^\infty$ such that $J(\pm v_k) < 0$ and $J(\pm v_k) \to 0$ as $k \to +\infty$.

In order to prove Theorem 3.11, we need to verify the following lemma.

Lemma 3.12. *Assume that the conditions (M_1), (M_2), (F_0) and (F_1) are satisfied. Then the functional J satisfies the $(\mathrm{PS})_c^*$ condition.*

Proof. Let $\{u_{n_j}\} \subset X$ be such that $u_{n_j} \in Y_{n_j}$ and $J(u_{n_j}) \to 0$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$ as $n_j \to \infty$. Similar to the process of verifying the (PS) condition in the proof of Lemma 3.4, we can get the boundedness of $\{\|u_{n_j}\|\}$. Going, if necessary, to a subsequence, we can assume that $\{u_{n_j}\}$ converges weakly to u in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence,

$$\begin{aligned} \lim_{n_j \to \infty} J'(u_{n_j})(u_{n_j} - u) &= \lim_{n_j \to \infty} J'(u_{n_j})(u_{n_j} - v_{n_j}) + \lim_{n_j \to \infty} J'(u_{n_j})(v_{n_j} - u) \\ &= \lim_{n_j \to \infty} (J|_{Y_{n_j}})'(u_{n_j})(u_{n_j} - v_{n_j}) = 0. \end{aligned}$$

From the proof of Lemma 3.4, J' is of (S_+) type, so we can conclude that $u_{n_j} \to u$ as $n_j \to \infty$, furthermore we have $J'(u_{n_j}) \to J'(u)$.

Let us prove $J'(u) = 0$, i.e., u is a critical point of J. Indeed, taking arbitrarily $w_k \in Y_k$, notice that when $n_j \ge k$ we have

$$\begin{aligned} J'(u)(w_k) &= (J'(u) - J'(u_{n_j}))(w_k) + J'(u_{n_j})(w_k) \\ &= (J'(u) - J'(u_{n_j}))(w_k) + (J|_{Y_{n_j}})'(u_{n_j})(w_k). \end{aligned}$$

Going to limit in the right hand-side of above equation reaches $J'(u)(w_k) = 0$ for all $w_k \in Y_k$. Thus, $J'(u) = 0$ and the functional J satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in \mathbb{R}$. □

Proof of Theorem 3.11. From (F_0), (F_1), (F_3) and Lemma 3.12, we know that J is an even functional and satisfies the $(\mathrm{PS})_c^*$ condition, the assertion of conclusion can be obtained from Dual Fountain Theorem, see Proposition 3.9.

(B_1): For any $v \in Z_k$, $\|v\| = 1$ and $0 < t < 1$, using (M_1) and (3.21), we have

$$\begin{aligned} J(tv) &= \int_\Omega \frac{1}{p(x)} |\Delta tv|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla tv|^{p(x)}\,dx\right) - \int_\Omega F(x,tv)\,dx \\ &\geq \frac{\min\{1,m_0\}}{p^+} t^{p^+} \|v\|^{p^+} - \epsilon t^{p^+} \int_\Omega |v|^{p^+}\,dx - c_\epsilon t^{q^-} \int_\Omega |v|^{q(x)}\,dx \\ &\geq \left(\frac{\min\{1,m_0\}}{p^+} - \epsilon c_{12}\right) \|v\|^{p^+} t^{p^+} - \begin{cases} c_{13}\beta_k^{q^-} t^{q^-} \|v\|^{q^-} & \text{if } |v|_{q(x)} \leq 1, \\ c_{13}\beta_k^{q^+} t^{q^-} \|v\|^{q^+} & \text{if } |v|_{q(x)} > 1. \end{cases} \end{aligned}$$

Let $0 < \epsilon < \frac{\min\{1,m_0\}}{2c_{12}p^+}$ we have

$$J(tv) \geq \frac{\min\{1,m_0\}}{2p^+} t^{p^+} - \begin{cases} c_{13}\beta_k^{q^-} t^{q^-} & \text{if } |v|_{q(x)} \leq 1, \\ c_{13}\beta_k^{q^+} t^{q^-} & \text{if } |v|_{q(x)} > 1. \end{cases}$$

Since $q^- > p^+$, taking $\rho_k = t$ small enough and sufficiently large k, for $v \in Z_k$ with $\|v\| = 1$, we have $J(tv) \geq 0$. So for sufficiently large k,

$$\inf_{\{u \in Z_k:\ \|u\| = \rho_k\}} J(u) \geq 0,$$

i.e., (B_1) is satisifed.

(B_2): For $v \in Y_k$, $\|v\| = 1$ and $0 < t < \rho_k < 1$, we have

$$\begin{aligned} J(tv) &= \int_\Omega \frac{1}{p(x)} |\Delta tv|^{p(x)}\,dx + \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla tv|^{p(x)}\,dx\right) - \int_\Omega F(x,tv)\,dx \\ &\leq \frac{t^{p^-}}{p^-} \int_\Omega |\Delta v|^{p(x)}\,dx + c_{10} t^{\frac{p^-}{1-\mu}} \left(\int_\Omega |\nabla v|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - Ct^{\gamma^+} \int_\Omega |v|^{\gamma(x)}\,dx. \end{aligned}$$

Condition $\gamma^+ < \frac{p^-}{1-\mu}$ implies that there exists a constant $r_k \in (0, \rho_k)$ such that $J(tv) < 0$ when $t = r_k$. Hence, we get

$$b_k := \max_{\{u \in Y_k:\ \|u\| = r_k\}} J(u) < 0,$$

so (B_2) is satisfied.

(B_3): Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$ we have

$$d_k := \inf_{\{u \in Z_k:\ \|u\| \leq \rho_k\}} J(u) \leq b_k := \max_{\{u \in Y_k:\ \|u\| = r_k\}} J(u) < 0. \tag{3.25}$$

From (3.25), for $v \in Z_k$, $\|v\| = 1$, $0 \leq t \leq \rho_k$ and $u = tv$ we have

$$\begin{aligned} J(u) &= J(tv) \\ &\geq \frac{\min\{1,m_0\}}{2p^+} t^{p^+} - \begin{cases} c_{14}\beta_k^{q^-} t^{q^-} & \text{if } |v|_{q(x)} \leq 1, \\ c_{14}\beta_k^{q^+} t^{q^-} & \text{if } |v|_{q(x)} > 1 \end{cases} \\ &\geq - \begin{cases} c_{14}\beta_k^{q^-} t^{q^-} & \text{if } |v|_{q(x)} \leq 1, \\ c_{14}\beta_k^{q^+} t^{q^-} & \text{if } |v|_{q(x)} > 1. \end{cases} \end{aligned}$$

Hence, $d_k \to 0$ as $k \to \infty$, i.e., (B_3) is satisfied. Conclusion of Theorem 3.11 is reached by the Dual Fountain Theorem. □

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