Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **50**, 1–15; https://doi.org/10.14232/ejqtde.2019.1.50 www.math.u-szeged.hu/ejqtde/

The bifurcation of limit cycles of two classes of cubic isochronous systems

Yi Shao✉[1], **Yongzeng Lai**[2] and **Chunxiang A**[1]

[1]School of Mathematics and Statistics, Zhaoqing University, Guangdong, 526061, P.R. China
[2]Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada, N2L 3C5

Received 29 January 2019, appeared 3 August 2019
Communicated by Gabriele Villari

Abstract. In this paper, we study the bifurcation of limit cycles of the periodic annulus of two classes of cubic isochronous systems. By using complete elliptic integrals of the first, second kinds and the Chebyshev criterion, we show that the upper bound for the number of limit cycles which appear from the periodic annuli of the two systems are at least three under cubic perturbations. Moreover, there exists a perturbation that give rise to exactly i limit cycles bifurcating from the period annulus for each $i = 0, 1, 2, 3$.

Keywords: limit cycle, period annulus, cubic perturbations.

2010 Mathematics Subject Classification: 34C05, 34A34, 34C14.

1 Introduction

This paper is concerned with the bifurcation of limit cycles of the centers of cubic isochronous systems, more concretely, differential systems of form

$$\begin{aligned} \dot{x} &= -y + P_n(x, y), \\ \dot{y} &= x + Q_n(x, y), \end{aligned} \tag{1.1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are real polynomials of degree n. In this paper we restrict ourselves to the case $n = 3$ and nonlinear isochrones of the above system, that are degrees for which the centers and the isochrones have been classified (see [3]).

The above problem belongs to the context of the second part of the Hilbert's 16th Problem. Until now the problem still remains to be unsolved even though a lot of work to be done in recent decades. Arnold [1] proposed a weaker version of this problem, the so-called infinitesimal Hilbert's 16th problem, that is to study the number of isolated zeros of the Abelian integrals.

We consider a polynomial system of the form

$$\begin{aligned} \dot{x} &= \frac{H_y(x, y)}{R(x, y)} + \varepsilon f(x, y), \\ \dot{y} &= -\frac{H_x(x, y)}{R(x, y)} + \varepsilon g(x, y), \end{aligned} \tag{1.2}$$

✉Corresponding author. Email: mathsyishao@126.com

where $H(x,y)$ is the first integral of System (1.2) for $\varepsilon = 0$ with integrating factor $R(x,y)$, $f(x,y)$ and $g(x,y)$ are polynomials of degree n in x, y and ε is a small parameter. The Abelian integral is defined as

$$I(h) = \oint_{\Gamma_h} R(x,y)(f(x,y)dy - g(x,y)dx), \tag{1.3}$$

where $\{\Gamma_h : h \in (a,b)\}$ is the family of ovals contained in the level curves $H(x,y) = h$ for $h \in (a,b)$.

Suppose that System (1.2) for $\varepsilon = 0$ has at least one center surrounded by the compact connected component of real curve $H(x,y) = h$. Let $d(h,\varepsilon)$ be defined on a section to the flow, which is parametrized by the Hamiltonian value h, then the Abelian integral $I(h)$ in (1.3) gives the first order approximation of the displacement function of the perturbed system, that is

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + O(\varepsilon^3). \tag{1.4}$$

Hence, if $I(h) = M_1(h)$ is not identically zero, then the number of isolated zeros of $M_1(h)$ gives an upper bound of the number of limit cycles of System (1.2). However if $I(h) \equiv 0$, then we need to compute the second order Melnikov function $M_2(h)$. We call $M_k(h), k = 1, 2, \dots$ the Melnikov functions and the first non-vanish Melnikov function is called a generating function.

In the past decades, many scholars studied limit cycles that bifurcate from periodic orbits of a center for a quadratic system, readers are referred to papers [8, 13, 15, 18, 21]. In the meanwhile, there are more studies on the bifurcation of limit cycles for other systems, see recently published papers [12], [20] and references therein. Besides, many researchers study the number of limit cycles produced from periodic orbits of the unperturbed cubic system. Dumortier and Li have made a complete investigate for Liénard system of degree 3 in a series of papers (see [4–7]). Gasull et al. [9] estimated an upper bound for the number of limit cycles from cubic isochronous System S_1^* (see [3]) under a small polynomial perturbation of degree $n \geq 9$. Wu and Zhao [19] investigated the bifurcation of limit cycles of a cubic isochronous center under cubic perturbations. In [16], the authors estimate the maxmum number of limit cycles which is bifurcated from the periodic annulus of cubic isochronous centers, and the orbits of these centers are formed by conics inside the class of all polynomial systems of degree n.

In the papers [3], the authors gave the following four classes of cubic systems with homogeneous nonlinearities

$$S_1^*: \quad \begin{aligned} \dot{x} &= -y - 3xy^2 + x^3, \\ \dot{y} &= x + 3x^2y - y^3, \end{aligned} \tag{1.5}$$

$$S_2^*: \quad \begin{aligned} \dot{x} &= -y + x^2y, \\ \dot{y} &= x + xy^2. \end{aligned} \tag{1.6}$$

$$S_3^*: \quad \begin{aligned} \dot{x} &= -y + 3x^2y, \\ \dot{y} &= x - 2x^3 + 9xy^2, \end{aligned} \tag{1.7}$$

and

$$\bar{S}_3^*: \quad \begin{aligned} \dot{x} &= -y - 3x^2y, \\ \dot{y} &= x + 2x^3 - 9xy^2. \end{aligned} \tag{1.8}$$

The origins of these four systems are all isochronous center. Shao and Wu [17] investigated the bifurcation of limit cycles from Systems S_3^* and $\bar{S}_3^*$. In this paper, we mainly intend to study the number of limit cycles produced from periodic annulus of cube isochronous Systems S_1^* and S_2^*. It is easy to know that System S_1^* has a first integral

$$H_1(x,y) = \frac{(x^2+y^2)^2}{1+4xy} = h, \qquad h \in (0,+\infty) \tag{1.9}$$

with integrating factor $R_1(x,y) = -\frac{4(x^2+y^2)}{(1+4xy)^2}$, and System S_2^* has a first integral

$$H_2(x,y) = \frac{x^2+y^2}{1-x^2} = h, \qquad h \in (0,+\infty) \tag{1.10}$$

with integrating factor $R_2(x,y) = -\frac{2}{(1-x^2)^2}$.

We consider the following perturbations of Systems S_1^* and S_2^*

$$\begin{aligned} \dot{x} &= -y - 3xy^2 + x^3 + \varepsilon f(x,y), \\ \dot{y} &= x + 3x^2y - y^3 + \varepsilon g(x,y), \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} \dot{x} &= -y + x^2y + \varepsilon f(x,y), \\ \dot{y} &= x + xy^2 + \varepsilon g(x,y), \end{aligned} \tag{1.12}$$

where $f(x,y)$, $g(x,y)$ are cubic polynomials in x, y and ε is enough small. It follows from (1.3) that the Abelian integrals of Systems (1.11) and (1.12) are

$$I_m(h) = \oint_{\Gamma_h} R_m(x,y)f(x,y)dy - R_m(x,y)g(x,y)dx, \qquad m = 1,2, \tag{1.13}$$

where $\Gamma_h = \{H_m(x,y) = h : h \in (0,+\infty)\}$, $m = 1,2$ are families of periodic orbits surrounding the center $(0,0)$.

The next theorem is the main result of this paper.

Theorem 1.1. *For the cubic perturbed Systems* (1.11) *and* (1.12)*, if each Abelian integral of $I_1(h)$ and $I_2(h)$ is not identically zero, then the maximum number of zeros (taking into account of the multiplicity) of $I_1(h)$ and $I_2(h)$ in* (1.13) *are both equal to three on $h \in (0,+\infty)$. Moreover, for each $i = 0,1,2,3$, there exist perturbations such that $I_1(h)$ and $I_2(h)$ have exactly i zeros.*

Since the Abelian integrals $I_1(h)$ and $I_2(h)$ are not identically zero, and they are the first order Melnikov functions, we have the following theorem.

Theorem 1.2. *The upper bound for the number of limit cycles of Systems* (1.11) *and* (1.12) *bifurcating from the periodic orbits of Systems S_1^* and S_2^* are at least three if each Abelian integrals of $I_1(h)$ and $I_2(h)$ in* (1.13) *is not identically zero. Moreover, for each $i = 0,1,2,3$, there exists a perturbation such that there are exactly i limit cycles produced by the periodic annulus of each System S_1^* and S_2^*.*

The rest of this paper is organized as follows. In Section 2, we will introduce definition of Chebyshev system and some properties of complete elliptic integrals of the first and the second kinds. In Section 3, first, we will transform the Abelian integrals $I_1(h_1)$ and $I_2(h_2)$ in (1.13) into a linear combination of four terms and prove that the four terms form an extended complete Chebychev system. Then, by using complete elliptic integrals of the first and the second kinds, the Chebyshev criterion and some purely algebraic computations, we will complete the proof of Theorem 1.1.

2 Preliminary results and properties

In order to prove Theorem 1.1, we need some preliminary results on the elliptic integrals of the first, second kinds $K(k)$, $E(k)$ and extended complete Chebychev system.

Definition 2.1. The complete normal elliptic integrals of the first and the second kinds are defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{and} \quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}d\theta, \tag{2.1}$$

respectively, which are analytic functions for $k \in (-1,1)$.

Lemma 2.2. *The complete elliptic integrals of the first kind $K(k)$ and the second kind $E(k)$ have the following properties.*

(a) ([2]) The elliptic integrals $K(k)$ and $E(k)$ satisfy the following Picard–Fuchs equations

$$\frac{dK}{dk} = \frac{E-(1-k^2)K}{k(1-k^2)}, \qquad \frac{dE}{dk} = \frac{E-K}{k}. \tag{2.2}$$

(b) ([11]) The power series of the elliptic integrals $K(k)$ and $E(k)$ at $k=0$ are

$$K(k) = \frac{\pi}{2}\sum_{i=0}^{\infty}\left(\frac{(2i-1)!!}{(2i)!!}\right)^2 k^{2i} \quad and \quad E(k) = -\frac{\pi}{2}\sum_{i=0}^{\infty}\left(\frac{(2i-1)!!}{(2i)!!}\right)^2 \frac{k^{2i}}{2i-1}, \tag{2.3}$$

respectively, where $k \in (-1,1)$ and the double factorial of integer $n(n \geq -1)$ is defined as

$$n!! = \begin{cases} n(n-2)\cdots 5\cdot 3\cdot 1, & \text{if } n>0 \text{ and } n \text{ is odd}, \\ n(n-2)\cdots 6\cdot 4\cdot 2, & \text{if } n>0 \text{ and } n \text{ is even}, \\ 1, & \text{if } n=-1,0. \end{cases}$$

(c) ([9]) The asymptotic expansions of $K(k)$ and $E(k)$ near $k=1$ are

$$\begin{aligned} K(k) &= \log 4 - \frac{1}{2}\log(1-k^2) + O(|(\log(1-k^2))(1-k^2)|), \\ E(k) &= 1 + \frac{1}{2}\left[\log 4 - \frac{1}{2}\log(1-k^2) - \frac{1}{2}\right](1-k^2) + O(|(\log(1-k^2))(1-k^2)^2|), \end{aligned} \tag{2.4}$$

respectively.

Next, we will introduce definition of an extended complete Chebyshev system (ECT-system) and its properties.

Definition 2.3 ([10])**.** Let $g_0(x), g_1(x), \ldots, g_{n-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$.

(1) $(g_0(x), g_1(x), \ldots, g_{n-1}(x))$ is called an extend complete Chebyshev system (an ECT-system) on L if for all $k=1,2,\ldots,n$, any nontrivial linear combination

$$c_0g_0(x) + c_1g_1(x) + \cdots + c_{k-1}g_{k-1}(x)$$

has at most $k-1$ isolated zeros on L counted with multiplicities.

(2) The continuous Wronskian of $(g_0(x), g_1(x), \ldots, g_{k-1}(x))$ at $x \in L$ is defined as

$$W[g_0, g_1, \ldots, g_{k-1}](x) = \mathrm{Det}(g_j^{(i)}(x))_{0\le i,j\le k-1} = \begin{vmatrix} g_0(x) & \cdots & g_{k-1}(x) \\ g_0'(x) & \cdots & g_{k-1}'(x) \\ \vdots & \ddots & \vdots \\ g_0^{(k-1)}(x) & \cdots & g_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

Lemma 2.4 ([10])**.** *$(g_0(x), g_1(x), \ldots, g_{n-1}(x))$ is an ECT-system on L if and only if, for each $1 \le k \le n$, $k \in \mathbb{N}$,*

$$W[g_0, g_1, \ldots, g_{k-1}](x) \neq 0 \quad \text{for all } x \in L.$$

Lemma 2.5. *If $(g_0(x), g_1(x), \ldots, g_{n-1}(x))$ is an ECT-system on L, then, for each $1 \le k \le n$, $k \in \mathbb{N}$, there exists a linear combination*

$$c_0 g_0(x) + c_1 g_1(x) + \cdots + c_{k-1} g_{k-1}(x)$$

with exactly k simple zeros on L (see [14] for instance).

We still need the following lemmas in the process of proving the case S_1^* in Theorem 1.1. For the completeness and the convenience of reading, in Lemma 2.6 we part use for reference the proof of Lemma 6 in [9].

Lemma 2.6. *Let $\varphi(x)$ be a continuous function and i, j be integers. Then*

(1) If $i + j$ is odd, then

$$\int_0^{2\pi} \varphi(\sin 2\theta) \cos^i \theta \sin^j \theta d\theta = 0.$$

(2) If $i + j = 2N$ is even, then there exist real constants $c_0, c_1, \ldots, c_N$ and $\tilde{c}_0, \tilde{c}_1, \ldots, \tilde{c}_N$ such that

$$\begin{aligned}\int_0^{2\pi} \varphi(\sin 2\theta) \cos^i \theta \sin^j \theta d\theta &= \sum_{l=0}^{N} c_l \int_{-\pi}^{\pi} \varphi(\cos\theta) \cos^l \theta d\theta \\ &= \sum_{l=0}^{N} \tilde{c}_l \int_{-\pi}^{\pi} \varphi(\sin\theta) \sin^l \theta d\theta.\end{aligned}$$

Proof. (1) Since $i + j$ is odd, let $\theta = \pi + \alpha$, then

$$I = \int_0^{2\pi} \varphi(\sin 2\theta) \cos^i \theta \sin^j \theta d\theta = (-1)^{i+j} \int_0^{2\pi} \varphi(\sin 2\alpha) \cos^i \alpha \sin^j \alpha d\alpha = -I,$$

it shows that $I = 0$.

(2) If $i + j = 2N$ is even, let $\theta = \frac{\pi}{4} - \alpha$, then

$$\begin{aligned}&\int_0^{2\pi} \varphi(\sin 2\theta) \cos^i \theta \sin^j \theta d\theta \\ &\quad = \int_0^{2\pi} \varphi(\cos 2\alpha) \left(\frac{1}{\sqrt{2}} \cos\alpha + \frac{1}{\sqrt{2}} \sin\alpha\right)^i \left(\frac{1}{\sqrt{2}} \cos\alpha - \frac{1}{\sqrt{2}} \sin\alpha\right)^j d\alpha \\ &\quad = \sum_{l=0}^{2N} \bar{c}_l \int_{-\pi}^{\pi} \varphi(\cos 2\alpha)(\sin\alpha)^{2N-l} \cos^l \alpha d\alpha\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{N} \bar{c}_{2l} \int_{-\pi}^{\pi} \varphi(\cos 2\alpha)(\sin\alpha)^{2N-2l}(\cos\alpha)^{2l} d\alpha \\
&= \sum_{l=0}^{N} \bar{c}_{2l} \int_{-\pi}^{\pi} \varphi(\cos 2\alpha)\left(\frac{1-\cos 2\alpha}{2}\right)^{N-l}\left(\frac{1+\cos 2\alpha}{2}\right)^{l} d\alpha \\
&= \sum_{l=0}^{N} c_l \int_{-\pi}^{\pi} \varphi(\cos 2\alpha)(\cos 2\alpha)^{l} d\alpha = \sum_{l=0}^{N} c_l \int_{-\pi}^{\pi} \varphi(\cos\theta)(\cos\theta)^{l} d\theta \quad (2\alpha = \theta) \\
&= \sum_{l=0}^{N} \tilde{c}_l \int_{-\pi}^{\pi} \varphi(\sin\theta)(\sin\theta)^{l} d\theta,
\end{aligned}
$$

where the value of the constants might not be the same from one expression to the other. The proof of the lemma is finished. □

Lemma 2.7. *Define*

$$\Phi_m = \Phi_m(h) = \int_{-\pi}^{\pi} (\sin\theta)^{2m} \sqrt{h^2 \sin^2\theta + h} d\theta, \qquad m = 0, 1.$$

Then

$$\Phi_0 = \frac{4k}{1-k^2} E, \qquad \Phi_1 = \frac{4}{3k(1-k^2)}((k^2-1)K + (k^2+1)E),$$

where $k^2 = h/(1+h)$.

Proof. By Lemma 2.6, we have that

$$
\begin{aligned}
\Phi_0 &= \int_{-\pi}^{\pi} \sqrt{h^2 \sin^2\theta + h} d\theta = \int_{-\pi}^{\pi} \sqrt{h^2 \cos^2\theta + h} d\theta \\
&= \sqrt{h^2 + h} \int_{-\pi}^{\pi} \sqrt{1 - \frac{h}{1+h} \sin^2\theta} d\theta \\
&= \frac{k}{1-k^2} \int_{-\pi}^{\pi} \sqrt{1 - k^2 \sin^2\theta} d\theta = \frac{4k}{1-k^2} E.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\Phi_1 &= \int_{-\pi}^{\pi} \sin^2\theta \sqrt{h^2 \sin^2\theta + h} d\theta = \int_{-\pi}^{\pi} \cos^2\theta \sqrt{h^2 \cos^2\theta + h} d\theta \\
&= \frac{k}{1-k^2} \int_{-\pi}^{\pi} (1 - \sin^2\theta) \sqrt{1 - k^2 \sin^2\theta} d\theta \\
&= \frac{k}{1-k^2} \left(\int_{-\pi}^{\pi} \sqrt{1 - k^2 \sin^2\theta} d\theta - \int_{-\pi}^{\pi} \sin^2\theta \sqrt{1 - k^2 \sin^2\theta} d\theta \right).
\end{aligned}
$$

Denote

$$V_1 = \int_{-\pi}^{\pi} \sin^2\theta \sqrt{1 - k^2 \sin^2\theta} d\theta,$$

and $\tau = \sin\theta$, then

$$V_1 = 4 \int_0^1 \tau^2 \sqrt{\frac{1 - k^2\tau^2}{1 - \tau^2}} d\tau = 4(F_2 - k^2 F_4),$$

where

$$F_2 = \int_0^1 \frac{\tau^2}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} d\tau = \frac{1}{k^2}(K - E).$$

It follows from formula (320.05) of [2] that

$$F_4 = \frac{2(1+k^2)F_2 - F_0}{3k^2}, \qquad F_0 = K.$$

Therefore, we have that

$$\Phi_1 = \frac{k}{1-k^2}(4E - V_1) = \frac{4}{3k(1-k^2)}((k^2-1)K + (k^2+1)E).$$

This finishes the proof of the lemma. □

3 Proof of Theorem 1.1

In this Section, by using lemmas given in Section 2 and computing the maximum number of zeros of corresponding Abelian integrals $I_1(h)$ and $I_2(h)$ in (1.13), we will prove Theorem 1.1 for Systems S_1^* and S_2^* separately.

3.1 Proof of the case S_1^*

To simplify calculation and apply Lemma 2.7, we change the first integral $H_1(x,y) = h, h \in (0,+\infty)$ and integrating factor $R_1(x,y)$ in (1.9) into

$$H(x,y) = \frac{1}{H_1(x,y)} = \frac{1+4xy}{(x^2+y^2)^2} = \frac{1}{h}, \qquad R(x,y) = \frac{4}{(x^2+y^2)^3}, \tag{3.1}$$

respectively. Moreover, we rewrite Abelian integrals $I_1(h)$ in (1.13) as

$$I_1(h) = \oint_{\Gamma_h} R(x,y)f(x,y)dy - R(x,y)g(x,y)dx. \tag{3.2}$$

Since the origin is an elementary center, we can see that $(x,y) \neq (0,0)$ in the first integral $H(x,y) = \frac{1}{h}, h \in (0,+\infty)$ and $R_1(x,y)$, which has not effect on the number of zeros of Abelian integrals $I_1(h)$ in (3.2).

We change the Abelian integral $I_1(h)$ in (3.2) into a linear combination of four terms, and have the following proposition.

Proposition 3.1. *The generating function $I_1(h)$ defined by (3.2) can be expressed as*

$$I_1(h)) = k^{-3}(\alpha_0 J_0(k) + \alpha_1 J_1(k) + \alpha_2 J_2(k) + \alpha_3 J_3(k)), \qquad k \in (-1,1), \tag{3.3}$$

where

$$J_0(k) = k, \qquad J_1(k) = k^3, \qquad J_2(k) = k^2 E, \qquad J_3(k) = (k^2-1)K + (k^2+1)E,$$

$h = k^2/(1-k^2)$ and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are any constants.

Proof. Denote $\Gamma_h = \{H(x,y) = 1/h : h \in (0,+\infty)\}$ all periodic annuli surrounding the origin of System S_1^*. In polar coordinates $x = r\cos\theta, y = r\sin\theta$, it follows from (1.10) that the periodic orbits Γ_h can be written as

$$r = r(h,\theta) = \sqrt{h\sin 2\theta + \sqrt{h^2\sin^2 2\theta + h}}.$$

By using (3.2) and Green's formula, we can rewrite the Abelian integral of System S_1^* as

$$\begin{aligned} I_1(h) &= \int_{\Gamma_h} \frac{4f(x,y)}{(x^2+y^2)^3}dy - \frac{4g(x,y)}{(x^2+y^2)^3}dx \\ &= \iint_{\Omega_h,\sigma} \left[\frac{4(f_x(x,y)+g_y(x,y))}{(x^2+y^2)^3} - \frac{24(xf(x,y)+yg(x,y))}{(x^2+y^2)^4}\right]dxdy - Q_\sigma, \end{aligned}$$

where $\Omega_h, \sigma = \Omega_h \setminus D_\sigma$, D_σ is a disk small enough with $(0,0)$ as center point and σ as radius, and Ω_h is the simple connected region enclosed by Γ_h,

$$Q_\sigma = \int_{r=\sigma} \frac{4f(x,y)}{(x^2+y^2)^3}dy - \frac{4g(x,y)}{(x^2+y^2)^3}dx,$$

where r is polar radius and $0 < \sigma \ll h$. Let

$$4[(x^2+y^2)(f_x(x,y)+g_y(x,y)) - 6(xf(x,y)+yg(x,y))] = \sum_{i+j=1}^{4} d_{i,j}x^iy^j,$$

then

$$\begin{aligned} I_1(h) &= \sum_{i+j=1}^{4} d_{i,j} \iint_{\Omega_h,\sigma} \frac{x^iy^j}{(x^2+y^2)^4}dxdy - Q_\sigma \\ &= \sum_{i+j=1}^{4} d_{i,j} \int_0^{2\pi} \int_\sigma^{r(h,\theta)} r^{i+j-7}\cos^i\theta\sin^j\theta drd\theta - Q_\sigma \\ &= \sum_{i+j=1}^{4} \bar{d}_{i,j} \int_0^{2\pi} (r(h,\theta))^{i+j-6}\cos^i\theta\sin^j\theta d\theta - C_\sigma \\ &= \sum_{i+j=1}^{4} \bar{d}_{i,j} I_{i,j} - C_\sigma, \end{aligned}$$

where $\bar{d}_{i,j} = \frac{1}{i+j-6}d_{i,j}$,

$$I_{i,j} = \int_0^{2\pi} (r(h,\theta))^{i+j-6}\cos^i\theta\sin^j\theta d\theta, \qquad C_\sigma = \sum_{i+j=1}^{4} (\bar{d}_{i,j}\sigma^{i+j-6}\int_0^{2\pi}\cos^i\theta\sin^j\theta d\theta) + Q_\sigma$$

and C_σ is a constant which does not depend on h.

By Lemma 2.6, we know that $I_{i,j} = 0$ if $i+j$ is odd. For $0 < i+j = 2N \leq 4$, we have that

$$\begin{aligned} I_{i,j} &= \int_{-\pi}^{\pi} \left(\sqrt{h\sin 2\theta + \sqrt{h^2\sin^2 2\theta + h}}\right)^{2N-6} \cos^i\theta\sin^j\theta d\theta \\ &= \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{h^2\sin^2 2\theta + h} + h\sin 2\theta}\right)^{3-N} \cos^i\theta\sin^j\theta d\theta \\ &= \sum_{l=0}^{N} c_l \int_{-\pi}^{\pi} \sin^l\theta \left(\frac{\sqrt{h^2\sin^2\theta + h} - h\sin\theta}{h}\right)^{3-N} d\theta \\ &= h^{N-3}\sum_{l=0}^{N} c_l \int_{-\pi}^{\pi} \sin^l\theta \left[\sum_{s=0}^{3-N} (h^2\sin^2\theta + h)^{\frac{3-N-s}{2}} C_{3-N}^s (-h)^s \sin^s\theta\right] d\theta. \end{aligned}$$

Denote

$$\bar{I}_1 = \sum_{i+j=2} c_i I_{i,j}, \qquad \bar{I}_2 = \sum_{i+j=4} \bar{c}_i I_{i,j},$$

where $c_i, \bar{c}_i$ are constants. By direct computation, we obtain that

$$\begin{aligned}\bar{I}_1 &= 2\pi c_0(1+h^{-1}) - 2c_1 h^{-1}\Phi_1,\\ \bar{I}_2 &= -\bar{c}_1\pi + \bar{c}_0 h^{-1}\Phi_0 + \bar{c}_2 h^{-1}\Phi_1.\end{aligned}$$

Hence the Abelian integral $I_1(h)$ of System S_1^* can be expressed as

$$\begin{aligned}I_1(h) &= \bar{d}_1\bar{I}_1 + \bar{d}_2\bar{I}_2 + C_\sigma\\ &= \pi(2\bar{d}_1c_0 - \bar{d}_2\bar{c}_1 + C_\sigma) + 2\pi\bar{d}_1c_0h^{-1} + \bar{d}_2\bar{c}_0h^{-1}\Phi_0 + (\bar{d}_2\bar{c}_2 - 2\bar{d}_1c_0)h^{-1}\Phi_1,\end{aligned}$$

where $\bar{d}_i, i = 1,2$ are constants. Substituting Φ_0, Φ_1 in Lemma 2.7 and $h = k^2/(1-k^2)$ into the above formula, we obtain (3.3). Thus, the proof of Proposition 3.1 is finished. □

By applying Definition 2.3 and Lemma 2.4, we need to check that (J_0, J_1, J_2, J_3) is an ECT-system for $k \in (0,1)$. So we shall verify that there are no zeros for the Wronskian $W[J_i](k)$ $(i = 0,1,2,3)$ in the interval $(0,1)$. By direct calculation, we have the following lemma.

Lemma 3.2. *(J_0, J_1, J_2, J_3) in (3.3) is an ECT-system for $k \in (0,1)$.*

Proof. From Proposition 3.1, it is easy to know that

$$W[J_0](k) = k \neq 0 \quad \text{and} \quad W[J_0, J_1](k) = 2k^3 \neq 0$$

for any $k \in (0,1)$. By using (2.2) and taking the derivatives of $K(k)$ and $E(k)$, we have that

$$\frac{d^2K}{dk^2} = \frac{(1-3k^2+2k^4)K + (3k^2-1)E}{k^2(k^2-1)^2}, \tag{3.4}$$

and

$$\frac{d^2E}{dk^2} = \frac{(k^2-1)K + E}{k^2(k^2-1)}. \tag{3.5}$$

Substituting (2.2), (3.3) and (3.5) into the Wronskian of (J_0, J_1, J_2), one obtains

$$W[J_0, J_1, J_2](k) = \frac{2k^3}{k^2-1}E(k).$$

Noticing that $0 < k^2 < 1$ and $E(k) > 0$, we have $W[J_0, J_1, J_2](k) \neq 0$ for all $k \in (0,1)$.

Next, we will compute the four-dimensional Wronskian $W[J_0, J_1, J_2, J_3](k)$. First of all, by taking the derivatives of $K''(k)$ and $E''(k)$, we have

$$\frac{d^3K}{dk^3} = \frac{(2-6k^2+10k^4-6k^6)K + (-2+5k^2-11k^4)E}{k^3(-1+k^2)^3}, \tag{3.6}$$

and

$$\frac{d^3E}{dk^3} = \frac{(-2+5k^2-3k^4)K + 2(1-2k^2)E}{k^3(k^2-1)^2}. \tag{3.7}$$

Then, applying (2.2), (3.4)–(3.7), we can factorize the Wronskian of (J_0, J_1, J_2, J_3) in the following form

$$\begin{aligned} W[J_0, J_1, J_2, J_3](k) &= -\frac{6[(k^2-1)(K(k))^2 + (4-2k^2)K(k)E(k) - 3(E(k))^2]}{(k^2-1)^2} \\ &= -\frac{6[(k^2-1)K(k) + \varphi_-(k)E(k)][(k^2-1)K(k) + \varphi_+(k)E(k)]}{(k^2-1)^3}, \end{aligned} \tag{3.8}$$

where

$$\varphi_\pm(k) = (2-k^2) \pm \sqrt{1-k^2+k^4}.$$

Since $K(k)$ and $E(k)$ are both even functions in k, Hence, by (3.8), we need only to show that $W[J_0, J_1, J_2, J_3](k)$ does not vanish for any $k \in (0,1)$.

Denote

$$\phi_\pm(k) = [(k^2-1)K(k) + \varphi_\pm(k)E(k)]. \tag{3.9}$$

Using Lemma 2.2 and by computation, we have that

$$\phi_+'(k) = -\frac{\mu(k)[\sqrt{1-k^2+k^4}(K(k)-E(k)) + k^2E(k)]}{k\sqrt{1-k^2+k^4}},$$

where $\mu(k) = 1 - 2k^2 + \sqrt{1-k^2+k^4}$. It is easy to verify that function $\mu(k)$ is monotonically decreasing and $\mu(k) > 0$ for $k \in (0,1)$. It follows from Definition 2.1 that $K(k) > E(k) > 0$ for all $k \in (0,1)$, that is $\phi_+'(k) < 0$. Moreover $\lim_{k\to 1^-} \phi_+(k) = 2\lim_{k\to 1^-} E(k) > 0$, which shows that $\phi_+(k) \neq 0$ for any $k \in (0,1)$.

We next show that $\phi_-(k) \neq 0$ for any $k \in (0,1)$. In fact, by Lemma 2.2 (c), it is easy to know that $\lim_{k\to 1^-} \phi_-(k) = 0$. From Lemma 2.2 (b), we can find that $\lim_{k\to 0^+} \phi_-(k) = 0$ and

$$\phi_-(k) = -\frac{201326592\pi}{2147483648}k^4 + o(k^4) < 0 \quad \text{for } k \approx 0.$$

Taking the derivative of $\phi_-(k)$, we get that

$$\phi_-'(k) = \frac{\lambda(k)[\sqrt{1-k^2+k^4}(K(k)-E(k)) - k^2E(k)]}{k\sqrt{1-k^2+k^4}}, \tag{3.10}$$

where $\lambda(k) = -1 + 2k^2 + \sqrt{1-k^2+k^4}$. It is easy to verify that function $\lambda(k)$ is monotonically increasing and $\lambda(k) > 0$ for all $k \in (0,1)$. Denote

$$v(k) = \sqrt{1-k^2+k^4}(K(k)-E(k)) - k^2E(k).$$

By use Lemma 2.2 (b), we have that

$$v(k) = -\frac{268435456\pi}{1073741824}k^2 + o(k^2) < 0 \quad \text{for } k \approx 0.$$

Since $v(k) < 0$ for $k \approx 0$, this implies that $\phi_-'(k) < 0$ for $k \approx 0$. By contradiction, suppose that there exists some $k_0 \in (0,1)$ such that $\phi_-(k_0) = 0$. Then, it follows from (3.9) that

$$K(k_0) = \frac{\left(\sqrt{1-k_0^2+k_0^4} + k_0^2 - 2\right)E(k_0)}{k_0^2 - 1}.$$

Substituting $K(k_0)$ into $v(k)$, it is easy to get that

$$v(k_0) = \frac{\left(1 - \sqrt{1 - k_0^2 + k_0^4}\right) E(k_0)}{k_0^2 - 1}.$$

We can see that $v(k_0) < 0$ for $k_0 \in (0,1)$. It follows from (3.10) that $\phi'_-(k_0) < 0$ for $k_0 \in (0,1)$, that is to say that $\phi_-(k)$ is monotonically decreasing in small neighborhood of k_0, which contradicts the fact that $\phi_-(k) < 0$ for any $k \in (0, k_0)$ and $\phi_-(k_0) = 0$. Hence, $\phi_-(k) \neq 0$ for any $k \in (0,1)$.

Summarizing above analyses, we know that $W[J_0, J_1, J_2, J_3](k)$ is non-zero for all $k \in (0,1)$. In short, (J_0, J_1, J_2, J_3) is an ECT-system on $(0,1)$ and the proof of Lemma 3.2 is completed. □

Proof of the case S_1^* in Theorem 1.1. It follows from Definition 2.3, Lemmas 2.4 and 3.2 that the Abelien integral $I_1(h)$ of System S_1^* has at most three zeros for any $h \in (0, +\infty)$. Moreover, by Lemma 2.5, for each $i = 0,1,2,3$, there exists a perturbation such that $I_1(h)$ has exactly i zeros.

3.2 Proof of the case S_2^*

Due to the differences between the two first integrals and the two integrating factors, the complete elliptic integrals do not emerge from the Abelien integral $I_2(h)$ in (1.13). Hence the proof of System S_2^* is easier than that of System S_1^*. But the basic method is similar. We still need to compute the number of zeros of the Abelien integral $I_2(h)$ by using the Chebyshev criterion.

In order to simplify the expression of the Abelien integral $I_2(h)$, we need the following Lemma (cf. Lemma 4.1 in [10]).

Lemma 3.3. *Let Γ_h be the periodic orbits inside the level curve $a(x) + b(x)y^2 = h$. If there exists a function $G(x)$ such that $\frac{G(x)}{(a(x))'}$ is analytic at $x = 0$, then, for any $i \in \mathbb{N}$,*

$$\int_{\Gamma_h} G(x) y^{i-2} dx = \int_{\Gamma_h} P(x) y^i dx,$$

where $P(x) = \frac{2}{i}\left(\frac{b(x)G(x)}{a'(x)}\right)' - \left(\frac{b'(x)G(x)}{a'(x)}\right)$.

Now we rewrite the first integral $H_2(x,y)$ of System S_2^* as

$$H_2(x,y) = \frac{x^2}{1-x^2} + \frac{1}{1-x^2} y^2 = a(x) + b(x)y^2 = h, \qquad h \in (0, +\infty). \tag{3.11}$$

It is easy to verify that $H(0,0) = 0$. By Lemma 3.2, we will change the Abelian integral $I_2(h)$ in (1.13) to a linear combination of four basic integrals.

Proposition 3.4. *The generating function $I_2(h)$, defined by (1.13), of S_2^* can be changed to*

$$I_2(h) = \beta_0 \bar{J}_0(h) + \beta_1 \bar{J}_1(h) + \beta_2 \bar{J}_2(h) + \beta_3 \bar{J}_3(h), \tag{3.12}$$

where

$$\bar{J}_0(h) = \int_{\Gamma_h} \frac{y^3}{(1-x^2)^2} dx, \qquad \bar{J}_1(h) = \int_{\Gamma_h} \frac{y^3}{(1-x^2)^3} dx,$$

$$\bar{J}_2(h) = \int_{\Gamma_h} \frac{x^2 y}{(1-x^2)^2} dx, \qquad \bar{J}_3(h) = \int_{\Gamma_h} \frac{y}{(1-x^2)^2} dx,$$

and $\beta_i, i = 0,1,2,3$ are any real constants.

Proof. From (1.11), we notice that the level curve $H_2(x,y)=h$ is symmetrical with respect to the x axis and the y axis, that is, $H_2(x,-y)=H_2(-x,y)=H_2(x,y)$. Hence

$$\int_{\Gamma_h}\frac{x^iy^j}{(1-x^2)^2}dy=0,\quad \text{if } i \text{ is even},$$
$$\int_{\Gamma_h}\frac{x^iy^j}{(1-x^2)^2}dx=0,\quad \text{if } j \text{ is even}.$$

On the other hand, using integration by parts we can verify that

$$\int_{\Gamma_h}\frac{xy}{(1-x^2)^2}dy=\int_{\Gamma_h}\frac{(1+3x^2)y^2}{2(1-x^2)^3}dx=0,$$
$$\int_{\Gamma_h}\frac{xy}{(1-x^2)^2}dx=\int_{\Gamma_h}\frac{1}{2(x^2-1)}dy=0.$$

By applying Green's formula and direct computation we obtain that

$$I_2(h)=\int_{\Gamma_h}\frac{(c_0+c_1x^2)y+c_2y^3}{(1-x^2)^2}dx+\int_{\Gamma_h}\frac{(c_3x^2+c_4x^4)y+c_5x^2y^3}{(1-x^2)^3}dx,$$

where $c_i(i=0,1,\ldots,5)$ are any constants. It follows from Lemma 3.2 that

$$\int_{\Gamma_h}\frac{3x^2y}{(1-x^2)^3}dx=\int_{\Gamma_h}\frac{y^3}{(1-x^2)^3}dx.$$

Moreover, it can be verified that $\int_{\Gamma_h}x^4ydx/(1-x^2)^3$ and $\int_{\Gamma_h}x^2y^3dx/(1-x^2)^3$ can be expressed as linear combination of $\int_{\Gamma_h}y^3dx/(1-x^2)^2$, $\int_{\Gamma_h}y^3dx/(1-x^2)^3$ and $\int_{\Gamma_h}x^2ydx/(1-x^2)^2$. These facts imply that (3.12) hold. The proof of the proposition is complete. □

Next, we will prove that $(\bar{J}_0,\bar{J}_1,\bar{J}_2,\bar{J}_3)$ is an ECT-system for $h\in(0,+\infty)$. There exists an analytic involution $\sigma(x)=-x$ such that $a(x)=a(\sigma(x))$ for all $x\in(-1,1)$, since System S_2^* is symmetrical with respect to the x-axis and the y-axis. Hence we can apply the following Lemma (cf. Theorem B in [10]).

Lemma 3.5. *Denote*

$$\bar{I}_i(h)=\int_{\Gamma_h}\psi_i(x)y^5dx,\qquad i=0,1,2,3,$$

where Γ_h is the set of periodic orbit surrounding the orign inside the level curve $\{a(x)+b(x)y^2=h\}$ for each $h\in(0,+\infty)$. If $\sigma(x)$ is the involution($\sigma(x)=-x$) and

$$\mu_i(x)=\left(\frac{\psi_i}{a'b^{\frac{5}{2}}}\right)(x)-\left(\frac{\psi_i}{a'b^{\frac{5}{2}}}\right)(\sigma(x)),$$

then $(\bar{I}_0,\bar{I}_1,\bar{I}_2,\bar{I}_3)$ is an ECT-system on $h\in(0,+\infty)$ if $(\mu_0,\mu_1,\mu_2,\mu_3)$ is an ECT-system on $x\in(0,1)$.

To apply Lemma 3.5, we need to transform $(\bar{J}_0,\bar{J}_1,\bar{J}_2,\bar{J}_3)$ in (3.12) into $(\bar{I}_0,\bar{I}_1,\bar{I}_2,\bar{I}_3)$. Firstly, it follows from (3.11) and Lemma 3.3 that

$$\begin{aligned}\bar{J}_0(h)&=\frac{1}{h}\int_{\Gamma_h}\frac{(a(x)+b(x)y^2)y^3}{(1-x^2)^2}dx=\frac{1}{h}\int_{\Gamma_h}\frac{a(x)y^3+b(x)y^5}{(1-x^2)^2}dx\\&=\frac{1}{h}\bar{I}_0(h)=\frac{1}{h}\int_{\Gamma_h}\psi_0(x)y^5dx,\end{aligned}$$

where

$$\psi_0(x) = \frac{2(x^2-3)}{5(x^2-1)^3}. \tag{3.13}$$

In the same way, we obtain that

$$\bar{J}_i(h) = \frac{1}{h}\bar{I}_i(h) = \frac{1}{h}\int_{\Gamma_h} \psi_i(x) y^5 dx, \qquad i = 1,2,3,$$

where

$$\psi_1(x) = \frac{6}{5(x^2-1)^4}, \qquad \psi_2(x) = -\frac{2(4x^4-9x^2+3)}{15(x^2-1)^3}, \qquad \psi_3(x) = \frac{4}{15x^2(x^2-1)^3}. \tag{3.14}$$

Thus, we can see that $(\bar{J}_0, \bar{J}_1, \bar{J}_2, \bar{J}_3)$ is an ECT-system on $(0,+\infty)$ if and only if $(\bar{I}_0, \bar{I}_1, \bar{I}_2, \bar{I}_3)$ is an ECT-system on $(0,+\infty)$. By Lemma 3.5, we need only to prove that $(\mu_0, \mu_1, \mu_2, \mu_3)$ is an ECT-system $x \in (0,1)$. This is done in Lemma 3.6 below.

Lemma 3.6. *$(\mu_0, \mu_1, \mu_2, \mu_3)$ is an ECT-system on $(0,1)$.*

Proof. It follows from Lemma 3.5 that

$$\mu_i(x) = \frac{(1-x^2)^4\sqrt{1-x^2}\,\psi_i(x)}{x}, \qquad i = 0,1,2,3. \tag{3.15}$$

Substituting (3.13) and (3.14) into (3.15), we can get that

$$\begin{aligned}
\mu_0(x) &= \frac{2\sqrt{1-x^2}(x^4-4x^2+3)}{5x},\\
\mu_1(x) &= \frac{6\sqrt{1-x^2}}{5x},\\
\mu_2(x) &= \frac{2\sqrt{1-x^2}(x^2-1)(4x^4-9x^2+3)}{15x},\\
\mu_3(x) &= \frac{4\sqrt{1-x^2}(x^2-1)}{15x^3}.
\end{aligned}$$

Applying Lemma 2.4 again, for any $x \in (0,1)$, it is clear that

$$W[\mu_0](x) = \mu_0(x) \neq 0.$$

Similarly, by direct computation, we obtain that

$$\begin{aligned}
W[\mu_0,\mu_1](x) &= \frac{48(x^2-1)(x^2-2)}{25x},\\
W[\mu_0,\mu_1,\mu_2](x) &= -\frac{512}{125}(x^2-1)\sqrt{1-x^2}(3x^4-12x^2+10),
\end{aligned}$$

and

$$W[\mu_0,\mu_1,\mu_2,\mu_3](x) = \frac{65536(x^2-1)^3(3x^2-5)}{625x^6}.$$

It is easy to see that $W[\mu_0,\mu_1](x), W[\mu_0,\mu_1,\mu_2](x)$ and $W[\mu_0,\mu_1,\mu_2,\mu_3](x)$ do not vanish for any $x \in (0,1)$. Thus, the proof of Lemma 3.6 is finished. □

Proof of the case S_2^* in Theorem 1.1. By using Proposition 3.4, Lemmas 3.5 and 3.6, it is easy to know that the Abelien integral $I_2(h)$ of System S_2^* has at most three zeros. Moreover, by Lemma 2.5, for each $i = 0,1,2,3$, there exists a perturbation such that $I_2(h)$ has exactly i zeros.

Acknowledgements

Acknowledgments are due to the referees for their useful suggestions and valuable comments. This research is partially supported by NSF of China (Nos. 11571379, 11661017, 71801186) and NSF of Guangdong Province of China (No. 2017A030310660).

References

[1] V. I. Arnold et al., Some unsolved problems in the theory of differential equations and mathematical physics, *Russian Math. Surveys* **44**(1989), No. 4, 157–171. https://doi.org/10.1070/RM1989v044n04ABEH002139; MR1023106; Zbl 0703.35002

[2] P. F. Byid, M. D. Friedman, *Handbook of elliptic integrals for engineer and physicists*, Springer-Verlag, Berlin, 1954. https://doi.org/10.1007/978-3-642-52803-3; MR0060642

[3] J. Chavarriga, M. Sabatini, A survey of isochronous centers, *Qual. Theory Dyn. Syst.* **1**(1999), No. 1, 1–70. https://doi.org/10.1007/BF02969404; MR1747197

[4] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: (I) Saddle loop and two saddle cycle, *J. Differential Equations* **176**(2001), No. 1, 114–157. https://doi.org/10.1006/jdeq.2000.3977; MR1861185; Zbl 1004.34018

[5] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: (II) Cuspidal loop, *J. Differential Equations* **175**(2001), No. 2, 209–243. https://doi.org/10.1006/jdeq.2000.3978; MR1855970; Zbl 1034.34036

[6] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: (III) Global center, *J. Differential Equations* **188**(2003), No. 2, 473–511. https://doi.org/10.1016/S0022-0396(02)00110-9; MR1954291; Zbl 1056.34044

[7] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: (IV) Figure eight-loop, *J. Differential Equations* **188**(2003), No. 2, 512–554. https://doi.org/10.1016/S0022-0396(02)00110-9; MR1954292; Zbl 1057.34015

[8] S. Gautier, L. Gavrilov, I. D. Iliev, Perturbations of quadratic centers of genus one, *Discret Contin. Dyn. Syst.* **25**(2009), No. 2, 511–535. https://doi.org/10.3934/dcds.2009.25.511; MR2525189; Zbl 1178.34037

[9] A. Gasull, W. G. Li, J. Llibre, Z. F. Zhang, Chebyshev property of complete elliptic integrals and its application to Abelian integrals, *Pac. J. Math.* **202**(2002), No. 2, 341–361. https://doi.org/10.2140/pjm.2002.202.341; MR1887769; Zbl 1086.34523

[10] M. Grau, F. Mañosas, J. Villadelprat, A Chebyshev criterion for Abelian integrals, *Trans. Amer. Math. Soc.* **363**(2011), 109–129. https://doi.org/10.1090/S0002-9947-2010-05007-X; MR2719674; Zbl 1217.34052

[11] M. Grau, J. Villadelprat, Bifurcation of critical periods from Pleshkan's isochrones, *J. Lond. Math. Soc.* **81**(2010), No. 1, 142–160. https://doi.org/10.1112/jlms/jdp062; MR2580458; Zbl 1196.34048

[12] M. Han, L. Sheng, X. Zhang, Bifurcation theory for finitely smooth planar autonomous differential systems, *J. Differential Equations* **264**(2018), No. 5, 3596–3618. https://doi.org/10.1016/j.jde.2017.11.025; MR3741399; Zbl 0682.4723

[13] I. D. Iliev, Perturbations of quadratic centers, *Bull. Sci. Math.* **122**(1998), No. 2, 107–161. https://doi.org/10.1016/S0007-4497(98)80080-8; MR1612784; Zbl 0920.34037

[14] S. Karlin, W. Studden, *Tchebycheff systems: with applications in analysis and statistics*, Pure and Applied Mathematics, Vol. XV, Interscience Publishers, New York, 1966. https://doi.org/10.1137/1009050; MR0204922; Zbl 0153.38902

[15] C. Li, J. Llibre, The cyclicity of period annulus of a quadratic reversible Lotka–Volterra system, *Nonlinearity* **22**(2009), No. 12, 2971–2979. https://doi.org/10.1088/0951-7715/22/12/009; MR2557456; Zbl 1193.34062

[16] C. Li, W. Li, J. Llibre, Z. Zhang, Linear estimation of the number of zeros of Abelian integrals for some cubic isochronous centers, *J. Differential Equations* **180**(2002), No. 2, 307–333. https://doi.org/10.1006/jdeq.2001.4064; MR1894015; Zbl 1010.34026

[17] Y. Shao, K. Wu, The cyclicity of the period annulus of two classes of cubic isochronous systems, *J. Appl. Anal. Comput.* **3**(2013), No. 3, 279–290. https://doi.org/10.11948/2013020; MR3109979; Zbl 1305.34051

[18] Y. Shao, Y. Zhao, The cyclicity and period function of a class of quadratic reversible Lotka–Volterra system of genus one, *J. Math. Anal. Appl.* **377**(2011), No. 2, 817–827. https://doi.org/10.1016/j.jmaa.2010.11.048; MR2769177; Zbl 1219.34044

[19] K. Wu, Y. Zhao, The cyclicity of the period annulus of the cubic isochronous center, *Ann. Mat. Pura Appl.* **191**(2012), No. 3, 459–467. https://doi.org/10.1007/s10231-011-0190-5; MR2958343; Zbl 1264.34062

[20] J. Yang, P. Yu, M. Han, Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle of order m, *J. Differential Equations* **266**(2019), No. 1, 455–492. https://doi.org/10.1016/j.jde.2018.07.042; MR3870568; Zbl 1406.34068

[21] H. Żoladek, Quadratic systems with centers and their perturbations, *J. Differential Equations* **109**(1994), No. 2, 223–273. https://doi.org/10.1006/jdeq.1994.1049; MR1273302; Zbl 0797.34044