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A Morse inequality for a fourth order elliptic equation on a bounded domain

Khadijah Sharaf[✉1] and **Hichem Hajaiej**[1,2]

[1]Department of mathematics, King Abdulaziz University,
P.O. 80230, Jeddah, Kingdom of Saudi Arabia
[2]California State University Los Angeles, 5151 University Drive, Los Angeles, USA

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Abstract. This article deals with the existence of multiple solutions of a fourth order elliptic equation with a critical nonlinearity on a bounded domain of $\mathbb{R}^n, n \geq 5$. We develop an approach to overcome the lack of compactness of the problem and we establish under a generic hypothesis a Morse inequality providing a lower bound of the number of solutions.

Keywords: fourth order PDE, critical nonlinearity, variational problem, critical points at infinity.

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1 Introduction and main results

Let Ω be a smooth bounded domain of $\mathbb{R}^n, n \geq 5$ and let $K : \Omega \to \mathbb{R}$ be a given function. We are interested in constructing a smooth positive function u on Ω satisfying

$$\begin{cases} \Delta^2 u = K(x)\, u^{\frac{n+4}{n-4}}, \\ u > 0 \ \text{ in } \Omega, \\ \Delta u = u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Equation (1.1) is heavily connected to the celebrated problem of prescribing Q-curvature on closed Riemannian manifolds. See [3, 9–11, 14–17] and the references therein for details.

Problem (1.1) has a variational structure. The solutions correspond to positive critical points of the functional:

$$J(u) = \frac{\int_\Omega (\Delta u)^2}{\left(\int_\Omega K(x)\, u^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}}}$$

defined on the function space:

$$\Sigma = \left\{ u \in H_2^2(\Omega) \cap H_1^0(\Omega)\,,\ \text{s.t. } \|u\| = 1 \right\},$$

✉Corresponding author. Email: kh-sharaf@yahoo.com

where

$$\|u\| = \Big(\int_\Omega (\Delta u(x))^2 dx\Big)^{\frac{1}{2}}.$$

One can see that, u is a critical point of J in $\Sigma^+ = \big\{ u \in \sum,\ u > 0 \big\}$, if and only if $J(u)^{\frac{n-4}{8}}.u$ is a solution of (1.1). Problem (1.1) is delicate from the variational viewpoint since the functional J does not satisfy the Palais–Smale condition on Σ^+ (P.S. in short): There exist sequences along which J is bounded, its gradient goes to zero and the sequences do not converge. This is a consequence of the lack of compactness of the embedding $H_2^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-4}}(\Omega)$. Consequently, challenging situations where critical points at infinity are limits of non-compact flow-lines of the gradient vector field $(-\partial J)$, occur.

In [18] and [25], the authors showed the existence of solutions of (1.1), provided $K \equiv 1$. Their results hinge on the shape of Ω. When $K \neq 1$, some existence results can be found for example in [1], [9], and [13].

Recently in [1] Abdelhedi, Chtioui and Hajaiej established compactness and existence results for (1.1) under the following three conditions:

(A) $\frac{\partial K}{\partial \nu}(x) \neq 0,\ \forall x \in \partial\Omega.$

Here ν is the unit outward normal vector on $\partial\Omega$.

$(f)_\beta$ K is a C^1-positive function on $\overline{\Omega}$ such that at any critical point y of K, there exists a real number $\beta = \beta(y)$ satisfying

$$K(x) = K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta + o(|x-y|^\beta), \qquad \forall x \in B(y, \rho_0),$$

where ρ_0 is a positive fixed constant, $b_k = b_k(y) \in \mathbb{R} \setminus \{0\}$, $\forall k = 1 \ldots, n$, and

$$-\frac{n-4}{n}\frac{c_1}{K(y)}\sum_{k=1}^{n} b_k(y) + c_2 \frac{n-4}{2} H(y,y) \neq 0, \qquad \forall y \in \mathcal{K}_{n-4},$$

where $\mathcal{K}_{n-4} := \{y \in \Omega, \nabla K(y) = 0 \text{ and } \beta(y) = n-4\}$. Here $c_1 = \int_{\mathbb{R}^n} \frac{|z_1|^{n-4}}{(1+|z|^2)^n} dz$, $c_2 = \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^{\frac{n+4}{2}}}$ and $H(\cdot,\cdot)$ is the regular part of the Green function $G(\cdot,\cdot)$ of the bilaplacian under the Navier boundary condition and

(A') $\beta(y) = \beta \in (1, n-4]$ at any y such that $\nabla K(y) = 0$.

Many interesting studies were dedicated to the problem (1.1) and its related Q-curvature problem on closed manifolds under the above $(f)_\beta$-condition. See for example [19], [14] and [12] on the standard n-dimensional sphere $n \geq 5$, treating respectively the case of $\beta \in]n-4, n[$, $\beta \in]1, n-4]$ and $\beta = n$. Concerning the problem on bounded domains case, we refer to [1]. We point out that $(f)_\beta$-condition covers the famous non degeneracy condition corresponding to the case of $\beta = 2$ and used in several works on (1.1) and its related curvature, see for example [2], [8], [13], [17] and [16].

According to the above results, we observe that the flatness order β does not exceed the value of n; the dimension of the associated domain. In this paper, we provide new existence results to the problem and we establish a lower bound of the number of solutions thanks to a Morse inequality. Our results are new and important as it address the case of β-flatness condition for any $\beta \geq n-4$. To state our existence results, we need to introduce some notations and assumptions: Let

$$\mathcal{K}^+_{n-4} = \left\{ y \in \mathcal{K}_{n-4}, -\frac{n-4}{n}\frac{c_1}{K(y)}\sum_{k=1}^{n} b_k(y) + c_2\frac{n-4}{2}H(y,y) > 0 \right\},$$

and

$$\mathcal{K}_{>n-4} = \{y \in \Omega, \nabla K(y) = 0,\ \beta(y) > n-4\}.$$

For any p-tuple of distinct points $\tau_p = (y_{\ell_1}, \dots, y_{\ell_p}) \in (\mathcal{K}^+_{n-4} \cup \mathcal{K}_{>n-4})^p, 1 \leq p$, we define a symmetric matrix $M(\tau_p) = (m_{ij})_{1\leq i,j\leq p}$ defined by:

$$m_{ii} = m(y_{\ell_i}, y_{\ell_i}) = \begin{cases} -\dfrac{1}{K(y_{\ell_i})^{\frac{n-4}{4}}}\left(\dfrac{n-4}{n}\dfrac{c_1}{K(y_{\ell_i})}\displaystyle\sum_{k=1}^{n} b_k(y_{\ell_i}) - c_2\dfrac{n-4}{2}H(y_{\ell_i}, y_{\ell_i})\right) & \text{if } \beta(y_{\ell_i}) = n-4, \\ \dfrac{n-4}{2}\dfrac{c_2}{K(y_{\ell_i})^{\frac{n-4}{4}}}H(y_{\ell_i}, y_{\ell_i}) & \text{if } \beta(y_{\ell_i}) > n-4, \end{cases}$$

$\forall i = 1, \dots, p$ and

$$m_{ij} = m(y_{\ell_i}, y_{\ell_j}) = -\frac{n-4}{2}c_2\frac{G(y_{\ell_i}, y_{\ell_j})}{\left(K(y_{\ell_i})K(y_{\ell_j})\right)^{\frac{n-4}{8}}}, \quad \text{for } 1 \leq i \neq j \leq p.$$

(B) Assume that the least eigenvalue $\rho(\tau_p)$ of $M(\tau_p)$ is non zero for any $\tau_p \in (\mathcal{K}^+_{n-4} \cup \mathcal{K}_{>n-4})^p, p \geq 1$.

For any $\tau_p = (y_{\ell_1}, \dots, y_{\ell_p}) \in (\mathcal{K}_{n-4} \cup \mathcal{K}_{>n-4})^p, p \geq 1$, we define

$$i(\tau_p) = p - 1 + \sum_{i=1}^{p} n - \widetilde{i}(y_{\ell_i}),$$

where $\widetilde{i}(y) = \sharp\{b_k(y),\ 1 \leq k \leq n,\ s.t.\ b_k(y) < 0\}$.

We now state our multiplicity result.

Theorem 1.1. *Let $K : \Omega \to \mathbb{R}$ be a function satisfying $(A), (B)$ and $(f)_\beta, \beta \in [n-4, \infty)$. If there exists an integer $k_0 \in \mathbb{N}$ such that*

(i) $i(\tau_p) \neq k_0 + 1,\ \forall \tau_p \in \mathcal{K}^\infty$, *where*

$$\mathcal{K}^\infty := \left\{ (y_{\ell_1}, \dots, y_{\ell_p}) \in (\mathcal{K}^+_{n-4} \cup \mathcal{K}_{>n-4})^p, p \geq 1,\ y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j \text{ and } \rho(y_{\ell_1}, \dots, y_{\ell_p}) > 0 \right\}.$$

(ii) All the critical points of J of indices $\leq k_0 + 1$ are non degenerate. Then

$$N_{k_0+1} \geq \left| 1 - \sum_{\tau_p \in \mathcal{K}^\infty, i(\tau_p) \leq k_0} (-1)^{i(\tau_p)} \right|,$$

where N_{k_0+1} is the number of solutions of (1.1) having their Morse indices $\leq k_0 + 1$.

We point out that Morse inequalities for Morse functions provide a lower bound for the number of the associated critical points. Therefore, Theorem 1.1 can be considered as a sort of Morse type inequality, since it provides a lower bound of the number of solutions and consequently a lower bound of the number of critical points of J. Notice also by the Sard–Smale theorem, see [23], the critical points of J are non degenerate for generic K. In the sense that for any C^1-function K_0, there exits a C^1-function K close to K_0 (in the C^1 sense) such that J has only non degenerate critical points.

An immediate corollary of Theorem 1.1 is the following result which prove the existence of at least one solution without assuming that (1.1) has only non degenerate solutions.

Theorem 1.2. *Assume that K satisfies $(A), (B)$, $(f)_\beta$, $\beta \in [n-4, \infty)$ and the condition (i) of the above theorem. If*

$$\sum_{\tau_p \in \mathcal{K}^\infty, i(\tau_p) \le k_0} (-1)^{i(\tau_p)} \ne 1,$$

then (1.1) has a solution of index $\le k_0 + 1$.

Observe that the integer $k_0 = \max\{i(\tau_p), \tau_p \in \mathcal{K}^\infty\}$ satisfies the condition (i) of the above Theorems. Therefore, the following two results are consequences of Theorem 1.1 and Theorem 1.2.

Theorem 1.3. *Assume $(A), (B)$ and $(f)_\beta, \beta \in [n-4, \infty)$. For generic K it holds*

$$N \ge \Big|1 - \sum_{\tau_p \in \mathcal{K}^\infty} (-1)^{i(\tau_p)}\Big|,$$

where N is the number of solutions of (1.1).

Theorem 1.4. *Under the assumptions $(A), (B)$ and $(f)_\beta$, $\beta \in [n-4, \infty)$. If*

$$\sum_{\tau_p \in \mathcal{K}^\infty} (-1)^{i(\tau_p)} \ne 1,$$

then (1.1) has at least one solution.

Our method is inspired by Bahri's principle of critical points theory at infinity [4]. The most important novelty of the present work is the extension of existence and multiplicity results of [1, 14] and [19], to any order of flatness larger than $n-4$. The main analysis difficulty in our statement comes from the divergence of integrals for β large. This leads to get new estimates for the the associated Euler–Lagrange functional and its derivatives. Using these estimates, we construct a suitable pseudo-gradient, completely different from the one of [1] allowing us to describe the lack of compactness of our problem and identify the critical points at infinity of the associated variational structure. We then use topological arguments to prove our results. In the next section, we will state some preliminaries related to the variational structure associated to problem (1.1). In Section 3, we will study the concentration phenomenon of the problem and identify the critical points at infinity of J and in Section 4, we will prove our existence results.

2 Variational structure

In this section, we state some preliminary tools of the variational structure associated to (1.1). For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = c_n \left(\frac{\lambda}{1+\lambda^2|x-a|^2} \right)^{\frac{n-4}{2}}, \tag{2.1}$$

where c_n is a positive constant chosen such that $\delta_{a,\lambda}$ is the family of solutions of the following problem (see [22]):

$$\Delta^2 u = |u|^{\frac{8}{n-4}} u\,, \quad u > 0 \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

Let $P\delta_{a,\lambda}$ the unique solution of

$$\begin{cases} \Delta^2 P\delta_{a,\lambda} = \delta_{a,\lambda}{}^{\frac{n+4}{n-4}} & \text{in } \Omega \\ \quad P\delta_{a,\lambda} = \Delta P\delta_{a,\lambda} = 0 & \text{on } \partial\Omega. \end{cases}$$

For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, we define the following set of potential critical points at infinity associated to J:

$$V(p,\varepsilon) = \begin{cases} u \in \Sigma^+, s.t.,\ \exists\ a_1,\dots,a_p \in \Omega, \exists\ \lambda_1,\dots,\lambda_p > \varepsilon^{-1} \text{ and} \\ \alpha_1,\dots,\alpha_p > 0 \text{ with } \left\| u - \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} \right\| < \varepsilon,\ \varepsilon_{ij} < \varepsilon\ \ \forall i \neq j, \\ \lambda_i d_i > \varepsilon^{-1} \text{ and } \left| J^{\frac{n}{n-4}}(u) \alpha_i^{\frac{8}{n-4}} K(a_i) - 1 \right| < \varepsilon\ \ \forall i = 1,\dots,p. \end{cases}$$

Here, $d_i = d(a_i, \partial\Omega)$ and $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2 \right)^{\frac{4-n}{2}}$.

Let w be a critical point of J in Σ^+. Define

$$V(p,\varepsilon,w) = \Big\{ u \in \Sigma^+,\ s.t. \text{ there exists } \alpha_0 > 0 \text{ satisfying } u - \alpha_0 w \in V(p,\varepsilon) \\ \text{and } \left| \alpha_0^{\frac{8}{n-4}} J(u)^{\frac{n}{n-4}} - 1 \right| < \epsilon \Big\}.$$

The following proposition describes the failure of the $(P.S.)$-condition of J.

Proposition 2.1 ([5,24]). *Let $(u_k)_k$ be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exists a positive integer p, a sequence (ε_k) with $\varepsilon_k \to 0$ as $k \to +\infty$ and an extracted subsequence of $(u_k)_k$'s, again denoted $(u_k)_k$, such that $u_k \in V(p,\varepsilon_k,w), \forall k$, where w is a solution of* (1.1) *or zero.*

The following proposition gives a parametrization of $V(p,\varepsilon,w)$.

Proposition 2.2 ([5]). *For all $p \in \mathbb{N}^*$, there exists $\varepsilon_p > 0$ such that for any $\varepsilon \leq \varepsilon_p$ and any u in $V(p,\varepsilon,w)$, the problem*

$$\min \left\{ \left\| u - \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} - \alpha_0(w+h) \right\|,\ \alpha_i > 0, \lambda_i > 0, a_i \in \Omega, h \in T_w(W_u(w)) \right\}.$$

admits a unique solution (α,λ,a,h). Thus, we can uniquely write u as follows

$$u = \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + \alpha_0(w+h) + v,$$

where $v \in H_2^2(\Omega) \cap H_0^1(\Omega) \cap T_w(W_s(w))$ and satisfies

$$(V_0) \qquad \langle v,\psi \rangle = 0 \quad \text{for } \psi \in \left\{ w, h, P\delta_i, \frac{\partial P\delta_i}{\partial\lambda_i}, \frac{\partial P\delta_i}{\partial a_i}, i = 1,\dots,p \right\}.$$

Here, $P\delta_i = P\delta_{a_i,\lambda_i}$ *and* $\langle \cdot, \cdot \rangle$ *denotes the inner product on* $H_2^2(\Omega)$ *defined by*

$$\langle u, v \rangle = \int_\Omega \Delta u \Delta v.$$

The following proposition deals with the v-part of u and shows that is negligible with respect to the concentration phenomenon.

Proposition 2.3 ([4, 5])**.** *There is a* $\mathcal{C}^1$*-map which to each* $(\alpha_i, a_i, \lambda_i, h)$ *such that* $\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + \alpha_0(w+h)$ *belongs to* $V(p, \varepsilon, w)$ *associates* $\overline{v} = \overline{v}(\alpha_i, a_i, \lambda_i, h)$ *such that* $\overline{v}$ *is the unique solution of the following minimization problem*

$$\min \left\{ J\left(\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + \alpha_0(w+h) + v \right),\ v \text{ satisfies } (V_0) \right\}.$$

In addition, there exists a change of variables $v - \overline{v} \to V$ *such that*

$$J\left(\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + \alpha_0(w+h) + v \right) = J\left(\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + \alpha_0(w+h) + \overline{v} \right) + \|V\|^2.$$

The estimate of $\|\overline{v}\|$ is given in the following lemma.

Lemma 2.4 ([14, p. 3020])**.** *There exists* $c > 0$ *independent of* u *such that the following holds*

$$\|\overline{v}\| \leq c \sum_{i=1}^p \left[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n+4}{2n}}}{\lambda_i^{\frac{n+4}{2}}} \right] + c \begin{cases} \sum_{k \neq r} \varepsilon_{k\,r}^{\frac{n+4}{2(n-4)}} \left(\log \varepsilon_{kr}^{-1} \right)^{\frac{n+4}{2n}}, & \text{if } n \geq 12 \\ \sum_{k \neq r} \varepsilon_{k\,r} \left(\log \varepsilon_{kr}^{-1} \right)^{\frac{n-4}{n}}, & \text{if } n < 12. \end{cases}$$

We now state the definition of critical point at infinity.

Definition 2.5 ([4])**.** A critical point at infinity of J is a limit of a non-compact flow line $u(s)$ of the gradient vector field $(-\partial J)$. By Propositions 2.1 and 2.2, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^p \alpha_i(s) P\delta_{a_i(s),\lambda_i(s)} + v(s).$$

Denoting by $y_i = \lim_{s \to +\infty} a_i(s)$ and $\alpha_i = \lim_{s \to +\infty} \alpha_i(s)$, we then denote by

$$\sum_{i=1}^p \alpha_i P\delta_{y_i,\infty} \quad \text{or} \quad (y_1, \ldots, y_p)_\infty$$

such a critical point at infinity.

3 Concentration phenomenon and critical points at infinity

In this section, we study the concentration phenomenon of the problem and we provide the description of the critical points at infinity under $(f)_\beta$-condition, $\beta \in [n-4, \infty)$.

Theorem 3.1. *Assume* $(A), (B)$ *and* $(f)_\beta$*,* $\beta \in [n-4, \infty)$*. There exists a decreasing pseudo-gradient* W *in* $V(p, \varepsilon)$ *satisfying the following*

(i)
$$\langle \partial J(u), W(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{\min(n,\beta)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big),$$

(ii)
$$\Big\langle \partial J(u+\overline{v}), W(u)+\frac{\partial \overline{v}}{\partial(\alpha_i,a_i,\lambda_i)}(W(u))\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{\min(n,\beta)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big).$$

In addition, W is bounded and the only case where $\lambda_i(s), i=1,\dots,p$, tend to ∞ is when $a_i(s)$ goes to $y_{\ell_i}, \forall i=1,\dots,p$ such that $(y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{K}^\infty$.

The proof of Theorem 3.1 is based on the following sequence of lemmas which describe the concentration phenomenon in particular regions of $V(p,\varepsilon)$ and hint the concentration of the required pseudo-gradient W. Let $\delta>0$ small enough, setting:

$$V_1(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}+v\in V(p,\varepsilon),\ a_i\in B(y_{\ell_i},\rho_0), \lambda_i^{n-4}|a_i-y_{\ell_i}|^\beta<\delta, \forall i=1,\dots,p, \text{ with } (y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{K}^\infty\Big\},$$

$$V_2(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}+v\in V(p,\varepsilon),\ a_i\in B(y_{\ell_i},\rho_0), \nabla K(y_{\ell_i})=0, \lambda_i^{n-4}|a_i-y_{\ell_i}|^\beta<\delta, m(y_{\ell_i},y_{\ell_i})>0, \forall i=1,\dots,p,\ y_{\ell_i}\neq y_{\ell_j}\ \forall j\neq i, \text{ and } \rho(y_{\ell_1},\dots,y_{\ell_p})<0\Big\},$$

$$V_3(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}+v\in V(p,\varepsilon),\ a_i\in B(y_{\ell_i},\rho_0), \nabla K(y_{\ell_i})=0, \lambda_i^{n-4}|a_i-y_{\ell_i}|^\beta<\delta, \forall i=1,\dots,p,\ y_{\ell_i}\neq y_{\ell_j}\ \forall j\neq i, \text{ and there exists } i_1\in\{1,\dots,p\}, \text{ s.t. } m(y_{\ell_{i_1}},y_{\ell_{i_1}})<0\Big\},$$

$$V_4(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}+v\in V(p,\varepsilon),\ a_i\in B(y_{\ell_i},\rho_0), \nabla K(y_{\ell_i})=0, \forall i=1,\dots,p, y_{\ell_i}\neq y_{\ell_j}\ \forall j\neq i, \text{ and there exists } i_1\in\{1,\dots,p\}, \textit{ s.t. } \lambda_{i_1}^{n-4}|a_{i_1}-y_{\ell_{i_1}}|^\beta\geq\delta, \Big\},$$

$$V_5(p,\varepsilon)=V(p,\varepsilon)\setminus\cup_{i=1}^{4}V_i(p,\varepsilon).$$

Lemma 3.2. *There exists a pseudo-gradient W_1 in $V_1(p,\varepsilon)$ such that for any $u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}\in V_1(p,\varepsilon)$, we have*

$$\langle \partial J(u), W_1(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{n-4}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i}\varepsilon_{ij}\right).$$

W_1 is bounded and the concentration components $\lambda_i(s)$ of the associated flow lines increase and go to $+\infty, i=1,\dots,p$.

Proof. Let $u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}\in V(p,\varepsilon)$. We increase all the $\lambda_i, i=1,\dots,p$ with respect to the differential equation
$$\dot{\lambda}_i=\lambda_i,\quad \forall i=1,\dots,p.$$
The corresponding vector field is
$$W_1(u)=\sum_{i=1}^{p}\alpha_i\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}\dot{\lambda}_i.$$

Recall that the variation of J with respect to $\lambda_i, i = 1, \dots, p$ was given in ([9, Proposition 3.3]) under the so-called non-degeneracy condition. In the same way, we state here this variation under $(f)_\beta$-condition, $\beta \in [n-4, \infty)$. We have the following two estimates.

$$\begin{aligned}
&\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \right\rangle \\
&\quad = 2c_2 J(u) \sum_{j\neq i} \alpha_i \alpha_j \left(-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \frac{n-4}{2} \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right) \\
&\qquad + 2\alpha_i^2 J(u) \begin{cases} \dfrac{n-4}{n} c_1 \dfrac{\sum_{k=1}^n b_k(y_{\ell_i})}{K(a_i)\lambda_i^{\beta(y_{\ell_i})}} - c_2 \dfrac{n-4}{2} \dfrac{H(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-4}}, & \text{if } \beta(y_{\ell_i}) = n-4 \\ -c_2 \dfrac{n-4}{2} \dfrac{H(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-4}}, & \text{if } \beta(y_{\ell_i}) > n-4 \end{cases} \\
&\qquad + O\left(|a_i - y_{\ell_i}|^\beta\right) + o\left(\sum_{j\neq i} \left(\varepsilon_{ij} + \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right) \right) + o\left(\sum_{j=1}^p \frac{1}{(\lambda_j d(a_j, \partial\Omega))^{n-4}} \right)
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \right\rangle \\
&\quad = 2c_2 J(u) \sum_{j\neq i} \alpha_i \alpha_j \left(-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \frac{n-4}{2} \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right) \\
&\qquad + O\left(\sum_{j=2}^{[\min(n,\beta)]} \frac{|a_i - y_{\ell_i}|^{\beta - j}}{\lambda_i^j} \right) + O\left(\frac{1}{\lambda_i^\beta} \right) + o\left(\sum_{j\neq i} \left(\varepsilon_{ij} + \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right) \right).
\end{aligned} \tag{3.2}$$

Here c_1 and c_2 are defined in the first section. The complete proof of (3.1) and (3.2) was given in [1]. Observe that for any $u \in V_1(p, \varepsilon)$ we have

$$|a_i - y_{\ell_i}|^\beta = o\left(\frac{1}{\lambda_i^{n-4}} \right), \quad \text{as } \delta \text{ small.}$$

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \frac{n-4}{2} \frac{1}{(|a_i - a_j|^2 \lambda_i \lambda_j)^{\frac{n-4}{2}}} + o\left(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right), \quad \text{since } |a_i - a_j| \geq \rho_0.$$

Therefore,

$$\begin{aligned}
-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \frac{n-4}{2} \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} &= \frac{n-4}{2} \left[\frac{1}{|a_i - a_j|^{n-4}} - H(a_i,a_j) \right] \frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} + o\left(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right) \\
&= \frac{n-4}{2} \left[\frac{1}{|y_{\ell_i} - y_{\ell_j}|^{n-4}} - H(y_{\ell_i}, y_{\ell_j}) \right] \frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} + o\left(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right) \\
&= \frac{n-4}{2} \frac{G(y_{\ell_i}, y_{\ell_j})}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} + o\left(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}} \right).
\end{aligned}$$

Therefore,

$$\langle \partial J(u), W_1(u)\rangle = 2J(u)\sum_{i=1}^{p}\sum_{j\neq i}\alpha_i\alpha_j\frac{n-4}{2}c_2\frac{G(y_{\ell_i},y_{\ell_j})}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}}$$
$$+2J(u)\sum_{i=1}^{p}\alpha_i^2\begin{cases}\frac{n-4}{n}c_1\frac{\sum_{k=1}^n b_k(y_{\ell_i})}{K(a_i)\lambda_i^{\beta(y_{\ell_i})}}-c_2\frac{n-4}{2}\frac{H(y_{\ell_i},y_{\ell_i})}{\lambda_i^{n-4}}, & \text{if } \beta(y_{\ell_i})=n-4\\ -c_2\frac{n-4}{2}\frac{H(y_{\ell_i},y_{\ell_i})}{\lambda_i^{n-4}}, & \text{if } \beta(y_{\ell_i})>n-4\end{cases}$$
$$+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-4}}\right).$$

Since $J(u)^{\frac{n}{n-4}}\alpha_i^{\frac{8}{n-4}}K(a_i)=1+o(1), \forall i=1,\dots,p$, we get

$$\begin{aligned}\langle \partial J(u), W_1(u)\rangle =& -2J(u)^{\frac{4-n}{4}}\left[\sum_{i=1}^{p}\sum_{j\neq i}\frac{m(y_{\ell_i},y_{\ell_j})}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}}+\sum_{i=1}^{p}\frac{m(y_{\ell_i},y_{\ell_i})}{\lambda_i^{n-4}}\right]+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-4}}\right)\\ =& -2J(u)^{\frac{4-n}{4}}\left(\frac{1}{\lambda_1^{\frac{n-4}{2}}},\dots,\frac{1}{\lambda_p^{\frac{n-4}{2}}}\right)M(y_{\ell_1},\dots,y_{\ell_p})\left(\frac{1}{\lambda_1^{\frac{n-4}{2}}},\dots,\frac{1}{\lambda_p^{\frac{n-4}{2}}}\right)^t\\ &+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-4}}\right).\end{aligned}$$

Here $M(y_{\ell_1},\dots,y_{\ell_p})$ is defined in the first section. Using now the fact that $\rho(y_{\ell_1},\dots,y_{\ell_p})$ is the least eigenvalue of $M(y_{\ell_1},\dots,y_{\ell_p})$, we derive that

$$\begin{aligned}\langle \partial J(u), W_1(u)\rangle &\leq -\rho(y_{\ell_1},\dots,y_{\ell_p})\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-4}}\\ &\leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{n-4}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i}\varepsilon_{ij}\right),\end{aligned}$$

since $\rho(y_{\ell_1},\dots,y_{\ell_p})>0$, $\frac{|\nabla K(a_i)|}{\lambda_i}=o(\frac{1}{\lambda_i^{n-4}})$ and $\varepsilon_{ij}\sim\frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}}$. This concludes the proof of Lemma 3.2. □

Lemma 3.3. *There exists a pseudo-gradient W_2 in $V_2(p,\varepsilon)$ such that for any $u=\sum_{i=1}^p\alpha_i P\delta_{(a_i,\lambda_i)}\in V_2(p,\varepsilon)$, we have*

$$\langle \partial J(u), W_2(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{n-4}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i}\varepsilon_{ij}\right).$$

W_2 is bounded and $\max_{1\leq i\leq p}\lambda_i(s)$ remains bounded along the associated flow lines.

Proof. Let $u=\sum_{i=1}^p\alpha_i P\delta_{(a_i,\lambda_i)}\in V_2(p,\varepsilon)$. We set in this region $W_2^1=-\sum_{i=1}^p\alpha_i\lambda_i\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}$. Using the same techniques of Lemma 3.2, we have:

$$\langle \partial J(u), W_2^1(u)\rangle = 2J(u)^{\frac{4-n}{4}}\left(\frac{1}{\lambda_1^{\frac{n-4}{2}}},\dots,\frac{1}{\lambda_p^{\frac{n-4}{2}}}\right)M(y_{\ell_1},\dots,y_{\ell_p})\left(\frac{1}{\lambda_1^{\frac{n-4}{2}}},\dots,\frac{1}{\lambda_p^{\frac{n-4}{2}}}\right)^t$$
$$+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-4}}\right).$$

Let $e = (e_1, \dots, e_p) \in \mathbb{R}^p$ be a unit eigenvector associated to $\rho(y_{\ell_1}, \dots, y_{\ell_p})$. For $\gamma > 0$ small enough, we denote by $B(e, \gamma)$ the ball in S^{p-1} of center e and radius γ satisfying $\forall X \in B(e, \gamma)$:

$$XM(y_{\ell_1}, \dots, y_{\ell_p})X^t < \frac{1}{2}\rho(y_{\ell_1}, \dots, y_{\ell_p}).$$

In the next, we denote by

$$\Gamma = \left(\frac{1}{\lambda_1^{\frac{n-4}{2}}}, \dots, \frac{1}{\lambda_p^{\frac{n-4}{2}}} \right).$$

Thus, if $\frac{\Gamma}{\|\Gamma\|} \in B(e, \gamma)$, we get

$$\begin{aligned}
\langle \partial J(u), W_2^1(u) \rangle &\le \frac{1}{2}\rho(y_{\ell_1}, \dots, y_{\ell_p}) \left\| \left(\frac{1}{\lambda_1^{\frac{n-4}{2}}}, \dots, \frac{1}{\lambda_p^{\frac{n-4}{2}}} \right) \right\|^2 \\
&\le -c \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-4}} \\
&\le -c \left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).
\end{aligned}$$

Therefore, we take $W_2 = W_2^1$ in this region as the required vector field.

If $\frac{\Gamma}{\|\Gamma\|} \notin B(e, \gamma)$, in this case we move $\frac{\Gamma}{\|\Gamma\|}$ along the path $c(t) = \frac{(1-t)\Gamma + te}{\|(1-t)\Gamma + te\|}$. Observe that all $\lambda_i, i = 1, \dots, p$ remain bounded along this path. Therefore, the Palais–Smale condition is satisfied along this piece of flow line. Let in this case

$$W_2(u) = \sum_{i=1}^{p} \alpha_i \dot{\lambda}_i \frac{\partial P\delta_{(a_i, \lambda_i)}}{\partial \lambda_i}.$$

where

$$\dot{\lambda}_i = -\frac{4}{n-4} \|\Gamma\| \lambda_i^{\frac{n}{2}} \left(\frac{\|\Gamma\| e_i - \Gamma_i}{\|c(0)\|} - \frac{c_i(0) < \|\Gamma\| e - \Gamma, c(0) >}{\|c(0)\|^3} \right).$$

Here c_i and e_i are the i^{th} component of c and e respectively. Notice that we can choose e such that $e_i > 0, \forall i = 1, \dots, p$. This is due to the fact that $m(y_{\ell_i}, y_{\ell_i}) > 0, \forall i = 1, \dots, p$. Using the estimates (3.1), we have

$$\begin{aligned}
\langle \partial J(u), W_2(u) \rangle &\le \frac{1}{2} \| \Gamma \|^2 \frac{\partial}{\partial t} \left(\frac{c(t) M(y_{\ell_1}, \dots, y_{\ell_p}) c(t)^t}{\|c(t)\|^2} \right)_{/t=0} \\
&\le -c \| \Gamma \|^2,
\end{aligned}$$

since $\frac{\partial}{\partial t} \left(\frac{c(t) M(y_{\ell_1}, \dots, y_{\ell_p}) c(t)^t}{\|c(t)\|^2} \right)_{/t=0} \le -c$, see [7, p. 650]. Therefore,

$$\langle \partial J(u), W_2(u) \rangle \le -c \Big(\sum_{i=1}^{p} \Big(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i} \Big) + \sum_{j \neq i} \varepsilon_{ij} \Big). \qquad \square$$

Lemma 3.4. *There exists a pseudo-gradient W_3 in $V_3(p,\varepsilon)$ such that for any $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V_3(p,\varepsilon)$, we have*

$$\langle \partial J(u), W_3(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{j\neq i}\varepsilon_{ij}\Big).$$

W_3 is bounded and $\max_{1\leq i\leq p}\lambda_i(s)$ remains bounded along the associated flow lines.

Proof. Let $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V_3(p,\varepsilon)$ and let $i_1,\dots,i_\ell$ be the indices such that $m(y_{i_j}, y_{i_j}) < 0$. We point out that the only cases where $m(y,y)$ is negative is when $\beta(y) = n-4$. Otherwise $m(y,y) \sim H(y,y)$ is therefore positive. Define

$$I = \Big\{ i, 1\leq i\leq p,\ \lambda_i \geq \frac{1}{2}\min_{1\leq j\leq \ell} \lambda i_j \Big\} \quad \text{and} \quad J = \{1,\dots,p\}\setminus I.$$

Let $M_J = (m_{ij})_{1\leq i,j\leq \sharp J}$ be the matrix defined by:

$$m_{ii} = mm(y_{\ell_i}, y_{\ell_i}), \quad \forall i\in J \quad \text{and} \quad m_{ij} = mm(y_{\ell_i}, y_{\ell_j}), \quad \forall 1\leq i\neq j\leq \sharp J.$$

Observe that m_{ii} is positive $\forall i\in J$. Thus, we can apply the arguments of Lemmas 3.2 and 3.3. Let $\rho(M_J)$ be the least eigenvalue of M_J. Define for $m>0$ and small

$$W_3^1 = m\Big((1+\operatorname{sign}\rho(M_J))W_1\Big(\sum_{i\in J}\alpha_i P\delta_{(a_i,\lambda_i)}\Big) + (1-\operatorname{sign}\rho(M_J))W_2\Big(\sum_{i\in J}\alpha_i P\delta_{(a_i,\lambda_i)}\Big)\Big),$$

where $\operatorname{sign}\rho(M_J) = 1$ if $\rho(M_J)>0$ and $\operatorname{sign}\rho(M_J) = -1$ if $\rho(M_J)<0$. Using Lemmas 3.2 and 3.3 we have

$$\langle \partial J(u), W_3^1(u)\rangle \leq -c\Big(\sum_{i\in J}\Big(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{j\neq i, i,j\in J}\varepsilon_{ij}\Big) + O\Big(\sum_{i\in J, j\in I}\varepsilon_{ij}\Big).$$

Observe that our upper bound is limited to those indices $i\in J$. We must add the indices $i\in I$. For this let

$$W_3^2(u) = -\sum_{j=1}^{\ell}\alpha_{i_j}\lambda_{i_j}\frac{\partial P\delta_{(a_{i_j},\lambda_{i_j})}}{\partial\lambda_{i_j}}.$$

Using (3.1) and the fact that $m(y_{i_j}, y_{i_j}) < 0, \forall 1\leq j\leq \ell$, we get

$$\begin{aligned}\langle \partial J(u), W_3^2(u)\rangle &\leq -c\Big(\sum_{j=1}^{\ell}\Big(\frac{1}{\lambda_{i_j}^{n-4}} + \frac{|\nabla K(a_{i_j})|}{\lambda_{i_j}}\Big) + \sum_{j=1}^{\ell}\sum_{k\neq i_j}\varepsilon_{i_j k}\Big)\\ &\leq -c\Big(\sum_{i\in I}\Big(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{i\in I, j\in J}\varepsilon_{ij}\Big).\end{aligned}$$

Therefore, for m small, we derive that

$$\langle \partial J(u), W_3(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{j\neq i}\varepsilon_{ij}\Big),$$

where $W_3(u) = W_3^1(u) + W_3^2(u)$. □

Lemma 3.5. *There exists a pseudo-gradient W_4 in $V_4(p,\varepsilon)$ such that for any $u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)} \in V_4(p,\varepsilon)$, we have*

$$\langle \partial J(u), W_4(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{\min(\beta,n)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big).$$

W_4 is bounded and $\max_{1\leq i\leq p}\lambda_i(s)$ remains bounded along the associated flow lines.

Proof. Let $u=\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)} \in V_4(p,\varepsilon)$. Let $L=\big\{j,\ 1\leq j\leq p,\ \lambda_j^{n-4}|a_j-y_{\ell_j}|^{\beta}\geq\delta\big\}$. We claim the following:

(C) $\forall j_1 \in L, \exists$ a pseudo-gradient Y_{j_1} such that

$$\langle \partial J(u), Y_{j_1}(u)\rangle \leq -c\left(\frac{1}{\lambda_{j_1}^{\min(\beta,n)}}+\frac{|\nabla K(a_{j_1})|}{\lambda_{j_1}}+\sum_{j\neq j_1}\varepsilon_{jj_1}\right)+o\left(\sum_{k\neq r}\varepsilon_{kr}\right).$$

The proof of (C) depends to the fourth following cases. Let $i\in L$.

Case 1: If $\beta(y_{\ell_i})=n-4$ and $\lambda_i|a_i-y_{\ell_i}|\leq\frac{1}{\delta}$. Define in this case

$$X_i(u)=\alpha_i\sum_{k=1}^{n}b_k\int_{\mathbb{R}^n}\frac{|x_k+\lambda_i(a_i-y_{\ell_i})_k|^{\beta}x_k}{(1+|x|^2)^{n+1}}dx\frac{1}{\lambda_i}\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial(a_i)_k}.$$

The variation of J with respect to $(a_i)_k, 1\leq k\leq n$; the k^{th} coordinate of a_i is given by the following two estimates:

$$\begin{aligned}\left\langle \partial J(u),\alpha_i\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial(a_i)_k}\right\rangle = &-c\alpha_i^2 J(u)\frac{b_k}{\lambda_i K(a_i)}\beta\,\mathrm{sign}(a_i-y_{\ell_i})_k|(a_i-y_{\ell_i})_k|^{\beta-1}\\ &+O\left(\sum_{j=2}^{[\min(n,\beta)]}\frac{|a_i-y_{\ell_i}|^{\beta-j}}{\lambda_i^j}\right)+O\left(\frac{1}{\lambda_i^{\min(n,\beta)}}\right)\\ &+O\left(\frac{1}{\lambda_i^{n-1}}\right)+O\left(\sum_{j\neq i}|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}|\right).\end{aligned}\tag{3.3}$$

$$\begin{aligned}\left\langle \partial J(u),\alpha_i\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial(a_i)_k}\right\rangle = &-(n-4)\alpha_i^2 J(u)\frac{b_k}{K(a_i)\lambda_i^{\beta}}\int_{\mathbb{R}^n}|x_k+\lambda_i(a_i-y_{\ell_i})_k|^{\beta}\\ &\times\frac{x_k}{(1+|x|^2)^{n+1}}dx+o\left(\frac{1}{\lambda_i^{\beta}}\right)+O\left(\sum_{j\neq i}|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}|\right).\end{aligned}\tag{3.4}$$

See [1,9]. The last estimate yields

$$\Big\langle \partial J(u), X_i(u)\Big\rangle \leq -\frac{c}{\lambda_i^{n-4}}\Big(\int_{\mathbb{R}^n}\frac{|x_{k_a}+\lambda_i(a_i-y_{\ell_i})_{k_a}|^{\beta}x_{k_a}}{(1+|x|^2)^{n+1}}dx\Big)^2+o\left(\frac{1}{\lambda_i^{n-4}}\right)+o\left(\sum_{j\neq i}\varepsilon_{ij}\right),$$

since $|\frac{1}{\lambda_k}\frac{\partial\varepsilon_{kr}}{\partial a_k}|=o(\varepsilon_{kr}), \forall k\neq r$ such that $|a_k-a_r|\geq\rho_0$. Here k_a satisfies $|(a_i-y_{\ell_i})_{k_a}|=\max_{1\leq k\leq n}|(a_i-y_{\ell_i})_k|$. Using the fact that $\lambda_i|a_i-y_{\ell_i}|\geq\delta$, therefore $\lambda_i|a_{i_{k_a}}-y_{\ell_{i_{k_a}}}|\geq c(\delta)>0$. We obtain

$$\Big(\int_{\mathbb{R}^n}\frac{|x_{k_a}+\lambda_i(a_i-y_{\ell_i})_{k_a}|^{\beta}x_{k_a}}{(1+|x|^2)^{n+1}}dx\Big)^2\geq c,$$

and thus

$$\begin{aligned}\left\langle \partial J(u), X_i(u)\right\rangle &\leq -\frac{c}{\lambda_i^{n-4}} + o\left(\sum_{j\neq i}\varepsilon_{ij}\right)\\ &\leq -c\left(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + o\left(\sum_{j\neq i}\varepsilon_{ij}\right).\end{aligned}$$

This is comes from the fact that $|\nabla K(a_i)| = O(|a_i - y_{\ell_i}|^{\beta-1})$.

To add $-\sum_{j\neq i}\varepsilon_{ij}$ to the upper bound of the last estimates, we define $Z_i(u) = -\alpha_i\lambda_i\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}$. Using (3.1) and

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \sim -c\varepsilon_{ij}, \quad \forall i\neq j, \text{ such that } |a_i - a_j| \geq \rho_0,$$

we get

$$\left\langle \partial J(u), Z_i(u)\right\rangle \leq -c\sum_{j\neq i}\varepsilon_{ij} + O\left(\frac{1}{\lambda_i^{n-4}}\right).$$

Therefore,

$$\left\langle \partial J(u), X_i(u) + mZ_i(u)\right\rangle \leq -c\left(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i}\varepsilon_{ij}\right) + o\left(\sum_{k\neq r}\varepsilon_{kr}\right),$$

for m small enough.

Case 2: If $\beta(y_{\ell_i}) = n-4$ and $\lambda_i|a_i - y_{\ell_i}| \geq \frac{1}{\delta}$. In this case we define:

$$\hat{X}_i(u) = \alpha_i\sum_{k=1}^{n} b_k \text{ sign } (a_i - y_{\ell_i})_k \frac{1}{\lambda_i}\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial(a_i)_k}.$$

Using (3.3), we have

$$\left\langle \partial J(u), \hat{X}_i(u)\right\rangle = -c\sum_{k=1}^{n} b_k^2\frac{|(a_i - y_{\ell_i})_k|^{\beta-1}}{\lambda_i} + O\left(\sum_{s=2}^{n-4}\frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s}\right) + O\left(\frac{1}{\lambda_i^{\beta}}\right) + o\left(\sum_{k\neq r}\varepsilon_{kr}\right).$$

Using the fact that

$$\frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s} = o\left(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\right), \qquad \forall s\geq 2, \text{ as } \delta \text{ small}, \tag{3.5}$$

$$\frac{1}{\lambda_i^{\beta}} = o\left(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\right), \quad \text{as } \delta \text{ small}, \tag{3.6}$$

we get

$$\left\langle \partial J(u), \hat{X}_i(u)\right\rangle \leq -c\frac{|(a_i - y_{\ell_i})_k|^{\beta-1}}{\lambda_i} + o\left(\sum_{k\neq r}\varepsilon_{kr}\right).$$

Now using $Z_i(u)$ be the vector field defined in the first case and (3.2), we have:

$$\left\langle \partial J(u), Z_i(u)\right\rangle \leq -c\sum_{j\neq i}\varepsilon_{ij} + O\left(\sum_{s=2}^{n-4}\frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s}\right) + O\left(\frac{1}{\lambda_i^{\beta}}\right) + o\left(\sum_{k\neq r}\varepsilon_{kr}\right).$$

By (3.5) and (3.6), we have:

$$\Big\langle \partial J(u), Z_i(u) \Big\rangle \le -c\sum_{j\neq i}\varepsilon_{ij} + o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big) + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big).$$

Therefore,

$$\Big\langle \partial J(u), \hat{X}_i(u) + Z_i(u) \Big\rangle \le -c\Big(\sum_{j\neq i}\varepsilon_{ij} + \frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big) + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big).$$

Using again (3.5) and (3.6) and the fact that $|\nabla K(a_i)| \sim |a_i - y_{\ell_i}|^{\beta-1}$, we get

$$\Big\langle \partial J(u), \hat{X}_i(u) + Z_i(u) \Big\rangle \le -c\Big(\sum_{j\neq i}\varepsilon_{ij} + \frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big).$$

Case 3: If $\beta(y_{\ell_i}) > n-4$. We use $\hat{X}_i$; the vector field defined in the second case. We have:

$$\Big\langle \partial J(u), \hat{X}_i(u) \Big\rangle \le -c\sum_{k=1}^{n} b_k^2 \frac{|(a_i - y_{\ell_i})_k|^2}{\lambda_i} + O\Big(\sum_{s=2}^{[\min(n,\beta)]} \frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s}\Big) + O\Big(\frac{1}{\lambda_i^{\beta}}\Big) + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big).$$

Observe that

$$\frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big), \qquad \forall s \ge 2, \text{ as } \lambda \to +\infty. \tag{3.7}$$

Indeed,

$$\frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s}\frac{\lambda_i}{|a_i - y_{\ell_i}|^{\beta-1}} = \frac{1}{|a_i - y_{\ell_i}|^{s-1}} \le \Big(\frac{1}{\delta}\Big)^{\beta}\frac{1}{\lambda^{(s-1)(1-\frac{n-4}{\beta})}}.$$

In the same way, we have:

$$\frac{1}{\lambda_i^{\beta}} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big), \quad \text{as } \lambda \to +\infty, \tag{3.8}$$

$$\frac{1}{\lambda_i^{n}} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big), \quad \text{as } \lambda \to +\infty. \tag{3.9}$$

Therefore,

$$\Big\langle \partial J(u), \hat{X}_i(u) \Big\rangle \le -c\frac{|(a_i - y_{\ell_i})_k|^{\beta-1}}{\lambda_i} + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big).$$

Now let Z_i be the vector field defined in the above cases. By using (3.2), (3.7) and (3.8), we have:

$$\Big\langle \partial J(u), Z_i(u) \Big\rangle \le -c\sum_{j\neq i}\varepsilon_{ij} + o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big) + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big).$$

Therefore, we get

$$\Big\langle \partial J(u), Y_i(u) \Big\rangle \le -c\Big(\sum_{j\neq i}\varepsilon_{ij} + \frac{1}{\lambda_i^{\min(n,\beta)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big),$$

where $Y_i(u) = \hat{X}_i(u) + Z_i(u)$. Hence claim (C) follows.

Let us denote by $\lambda_{i_0}^{\beta} = \min_{i\in L} \lambda_i^{\beta}$ and define

$$\widetilde{L} = \left\{ j,\ 1 \le j \le p,\ \lambda_j^{\beta} \ge \frac{1}{2}\lambda_{i_0}^{\beta} \right\}.$$

We have:

$$\left\langle \partial J(u), \sum_{i\in L} Y_i(u) \right\rangle \le -c\left(\sum_{i\in\widetilde{L}} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i\in L, j\ne i} \varepsilon_{ij} \right) + o\left(\sum_{k\ne r} \varepsilon_{kr} \right).$$

Using the preceding computation, we have for $m > 0$ and small:

$$\left\langle \partial J(u), \sum_{i\in L} Y_i(u) + m \sum_{i\in\widetilde{L}\setminus L} Z_i(u) \right\rangle \le -c\left(\sum_{i\in\widetilde{L}} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i\in\widetilde{L}, j\ne i} \varepsilon_{ij} \right) + o\left(\sum_{k\ne r} \varepsilon_{kr} \right).$$

Observe now that $\bar{u} = \sum_{i\notin\widetilde{L}} \alpha_i P\delta_{(a_i,\lambda_i)} \in V_i(\sharp\widetilde{L}^c, \varepsilon), i = 1,2,3,$ (defined in the above lemmas). Let $\widetilde{W}(u) = W_i(\bar{u})$ where $W_i(\bar{u})$ is the corresponding vector field in $V_i(\sharp\widetilde{L}^c, \varepsilon)$. It satisfy:

$$\left\langle \partial J(u), \widetilde{W}(u) \right\rangle \le -c\left(\sum_{i\notin\widetilde{L}} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i,j\notin\widetilde{L}, j\ne i} \varepsilon_{ij} \right) + O\left(\sum_{i\notin\widetilde{L}, j\in\widetilde{L}} \varepsilon_{ij} \right).$$

For $\widetilde{m} > 0$ and small, setting

$$W_u(u) = \widetilde{m}\widetilde{W}(u) + \sum_{i\in L} Y_i(u) + m \sum_{i\in\widetilde{L}\setminus L} Z_i(u),$$

we have

$$\langle \partial J(u), W_u(u) \rangle \le -c\left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j\ne i} \varepsilon_{ij} \right).$$ □

Lemma 3.6. *There exists a pseudo-gradient W_5 in $V_5(p,\varepsilon)$ such that for any $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V_5(p,\varepsilon)$, we have*

$$\langle \partial J(u), W_5(u) \rangle \le -c\left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j\ne i} \varepsilon_{ij} \right).$$

W_5 is bounded and $\max_{1\le i\le p} \lambda_i(s)$ remains bounded along the associated flow lines.

Proof. We divide $V_5(p,\varepsilon)$ into two regions:

$$R_1 = \Big\{ u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon), a_i \in B(y_{\ell_i}, \rho_0), \nabla K(y_{\ell_i}) = 0, \forall i = 1,\dots,p$$
$$\text{and there exists } j \ne i \text{ such that } y_{\ell_i} = y_{\ell_j} \Big\},$$

$$R_2 = \Big\{ u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon),\ \exists i,\ 1 \le i \le p,\ a_i \notin \cup_{y,\nabla K(y)=0} B(y,\rho_0) \Big\}$$

We will give the construction of W_5 in R_1. The construction in R_2 proceeds under the assumption (A) as in [9]. Let $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in R_1$. To any index $i, i = 1,\dots,p$, we define

$$B_i = \{ j, 1 \le j \le p,\ a_j \in B(y_{\ell_i}, \rho_0) \}.$$

We suppose that $B_{i_1}, \dots, B_{i_\ell}$ are the sets such that $\sharp B_{i_k} > 1, \forall k = 1, \dots, \ell$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth positive function such that

$$\begin{cases} \phi(t) = 0 & \text{if } |t| < \eta, \\ \phi(t) = 1 & \text{if } |t| \geq \eta. \end{cases}$$

Here η is a small positive function. For $j \in B_{i_k}$ define:

$$\bar{\phi}(\lambda_j) = \sum_{i \neq j \in B_{i_k}} \phi\left(\frac{\lambda_j}{\lambda_i}\right).$$

Setting

$$W_5^1(u) = -\sum_{k=1}^{\ell} \sum_{j \in B_{i_k}} \alpha_j \bar{\phi}(\lambda_j) \lambda_j \frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

Using (3.2) we obtain

$$\left\langle \partial J(u), W_5^1(u) \right\rangle \leq c \sum_{k=1}^{\ell} \left(\sum_{j \in B_{i_k}} \bar{\phi}(\lambda_j) \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \sum_{j \in B_{i_k}} O\left(\sum_{s=1}^{[\min(\beta,n)]} \frac{|a_j - y_{\ell_j}|^{\beta - s}}{\lambda_j^s} \right) \right).$$

For $j \in B_{i_k}$ such that $\bar{\phi}(\lambda_j) \neq 0$, there exists $i_0 \neq j \in B_{i_k}$ such that

$$\frac{1}{\lambda_j^\beta} = o(\varepsilon_{ji_0}) \quad \text{and} \quad \frac{1}{\lambda_j^n} = o(\varepsilon_{ji_0}).$$

Observe that if $i \in B_{i_k}^c$, then $|a_i - a_j| \geq \rho_0$. Therefore,

$$\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -c\varepsilon_{ij} \quad \text{and} \quad \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\varepsilon_{ij}.$$

Thus,

$$\left\langle \partial J(u), W_5^1(u) \right\rangle \leq -c \sum_{k=1}^{\ell} \left(\sum_{j \in B_{i_k}} \bar{\phi}(\lambda_j) \left(\sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_j^{\min(\beta,n)}} \right) + \sum_{j \in B_{i_k}} O\left(\sum_{s=1}^{[\min(\beta,n)]} \frac{|a_j - y_{\ell_j}|^{\beta - s}}{\lambda_j^s} \right) \right).$$

Let j_0 denote the index such that

$$\lambda_{j_0}^{\min(n,\beta)} = \min\left\{ \lambda_i^{\min(n,\beta)},\ 1 \leq i \leq p \right\}.$$

We have two cases:

Case 1: There exists $j \in B_{i_k}, k = 1, \dots, \ell$, with $\bar{\phi}(\lambda_j) \neq 0$ such that $\frac{\lambda_{j_0}^{\min(n,\beta)}}{\lambda_j^{\min(n,\beta)}} \geq m$, where m is a fixed positive constant small enough. In this case we get:

$$\left\langle \partial J(u), W_5^1(u) \right\rangle \leq -c \sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \sum_{i \neq j} \varepsilon_{ij} \right) + \sum_{k=1}^{\ell} \sum_{j \in B_{i_k}} O\left(\sum_{s=1}^{[\min(\beta,n)]} \frac{|a_j - y_{\ell_j}|^{\beta - s}}{\lambda_j^s} \right).$$

Therefore,

$$\left\langle \partial J(u), W_5^1(u) + m_1 \sum_{i=1}^{p} \hat{X}_i(u) \right\rangle \leq -c \left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j} \varepsilon_{ij} \right),$$

for $m_1 > 0$ small enough.

Case 2: $\forall j \in B_{i_k}, k = 1,\dots,\ell$, we have $\frac{\lambda_{j_0}^{\min(n,\beta)}}{\lambda_j^{\min(n,\beta)}} < m$ or if there exists $j \in B_{i_k}, k = 1,\dots,\ell$, with $\frac{\lambda_{j_0}^{\min(n,\beta)}}{\lambda_j^{\min(n,\beta)}} \geq m$ we have $\bar{\phi}(\lambda_j) = 0$. Define

$$E = \left\{ k,\ \frac{\lambda_k^{\min(n,\beta)}}{\lambda_{j_0}^{\min(n,\beta)}} < \frac{1}{m} \right\} \cup \{k,\ \bar{\phi}(\lambda_k) = 0\} \cup \left(\cup_{k=1}^{\ell} B_{i_k} \right)^c.$$

For all $k \neq j \in E$, we have $a_k \in B(y_{\ell_k}, \rho_0)$ and $a_j \in B(y_{\ell_j}, \rho_0)$ with $y_{\ell_j} \neq y_{\ell_k}$. Let $\bar{u} = \sum_{i\in E} \alpha_i P\delta_{a_i,\lambda_i}$. $\bar{u}$ lies in $V_i(\sharp E, \varepsilon), i = 1,2,3,4$, (defined in the above lemmas). We denote W_i the related vector field in $V_i(\sharp E, \varepsilon)$. We have:

$$\left\langle \partial J(u), W_i(u) \right\rangle \leq -c \left(\sum_{i\in E} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i\neq j, i,j\in E} \varepsilon_{ij} \right) + O\left(\sum_{i\in E, j\notin E} \varepsilon_{ij} \right).$$

Therefore

$$\left\langle \partial J(u), W_5^1(u) + m_1 (W_i(u) + \sum_{i=1}^{p} \hat{X}_i(u) \right\rangle \leq -c \left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i\neq j} \varepsilon_{ij} \right). \quad \square$$

Proof of Theorem 3.1. It follows from Proposition 2.3 and Lemmas 3.2–3.6. $\square$

Corollary 3.7. *Under the assumptions $(A), (B)$ and $(f)_\beta, \beta \in [n-4, \infty)$, the critical points at infinity in $V(p,\varepsilon), p \geq 1$ are*

$$(y_{\ell_1}, \dots, y_{\ell_p})_\infty := \sum_{i=1}^{p} \frac{1}{K(y_{\ell_i})^{\frac{n-4}{2}}} P\delta_{(y_{\ell_i}, \infty)},$$

where $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{K}^\infty$. The index of $(y_{\ell_1}, \dots, y_{\ell_p})_\infty$ is $i(y_{\ell_1}, \dots, y_{\ell_p})_\infty = p - 1 + \sum_{i=1}^{p} n - \widetilde{i}(y_{\ell_i})$.

The following result exclude the possibility of existence of critical points at infinity in $V(p,\varepsilon,w)$ when $w \neq 0$.

Theorem 3.8. *Let w be a critical point of J in Σ^+. Assume $(A), (B)$ and $(f)_\beta, \beta \in (\frac{n-4}{2}, \infty)$. There exists a decreasing pseudo-gradient W on $V(p,\varepsilon,w)$ satisfying the following*

(i)
$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\frac{n-4}{2}}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j\neq i} \varepsilon_{ij} + \|h\|^2 \right),$$

(ii)
$$\left\langle \partial J(u+\bar{v}), W(u) + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W(u)) \right\rangle \leq -c \left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\frac{n-4}{2}}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j\neq i} \varepsilon_{ij} + \|h\|^2 \right).$$

(iii) W is bounded and all λ_i'(s), $i = 1,\dots,p$, decrease along the W flow lines.

Proof. The proof of Theorem 3.8 proceeds exactly as the one of Theorem 3.1 of [14]. $\square$

As a consequence of Theorem 3.8, we have the following result.

Corollary 3.9. *Under the assumptions of Theorem 3.8, there is no critical points at infinity of J in $V(p,\varepsilon,w)$.*

4 Proof of results

Assume that K satisfies $(A),(B)$ and $(f)_\beta, \beta \geq n-4$. For any $\tau_p = (y_{\ell_1}, \ldots, y_{\ell_p}) \in \mathcal{K}^\infty$, we denote $\tau_{p_\infty} = (y_{\ell_1}, \ldots, y_{\ell_p})_\infty$ the corresponding critical point at infinity. The dimension of the unstable manifold at infinity $W_u(\tau_p)_\infty$ of τ_{p_∞} is equal to $i(\tau_p)_\infty = p - 1 + \sum_{j=1}^{p} n - \widetilde{i}(y_{\ell_i})$. Define

$$Y_{k_0}^\infty := \bigcup_{\tau_p \in \mathcal{K}^\infty,\, i(\tau_p)_\infty \leq k_0} W_u(\tau_p)_\infty.$$

$Y_{k_0}^\infty$ is a manifold in dimension less or equal to k_0 in Σ^+. For simplicity, we assume that $Y_{k_0}^\infty$ is in dimension k_0. For $\lambda >> 1$ and $v_0 \in Y_{k_0}^\infty$ we set

$$\begin{aligned} \Theta_{\lambda,v_0} : [0,1] \times Y_{k_0}^\infty &\longrightarrow \Sigma^+ \\ (s,v) &\longmapsto \Theta_{\lambda,v_0}(s,v) = \frac{sv + (1-s)v_0}{\|sv + (1-s)v_0\|}. \end{aligned}$$

Therefore, $\Theta_{\lambda,v_0}([0,1] \times Y_{k_0}^\infty)$ is a contraction of $Y_{k_0}^\infty$ in Σ^+ of dimension $k_0 + 1$. $\Theta_{\lambda,v_0}([0,1] \times Y_{k_0}^\infty)$ can be deformed by using the gradient flow-lines of J. Once the possibility of existence of mixed critical points at infinity of J is excluded, see Corollary 3.9, the only critical points at infinity of J are τ_{p_∞}, where $\tau_p \in \mathcal{K}^\infty$. Observe that by a dimension argument, the stable manifold at infinity of each τ_{p_∞} of index larger or equal to $k_0 + 2$ can be avoided along such a deformation. Therefore, by the deformation lemma of Bahri and Rabinowitz [6] we have

$$\Theta_{\lambda,v_0}([0,1] \times Y_{k_0}^\infty) \quad \text{retracts on} \quad \bigcup_{\tau_p \in \mathcal{K}^\infty,\, i(\tau_p)_\infty \leq k_0+1} W_u(\tau_p)_\infty \cup \bigcup_{\partial J(w)=0,\, i(w) \leq k_0+1} W_u(w). \tag{4.1}$$

Observe that from the deformation retract (4.1) and from the condition (i) of Theorems 1.1 and 1.2, the functional J admits at least a critical point w in Σ^+. Otherwise, it follows from (4.1) that:

$$\Theta_{\lambda,v_0}([0,1] \times Y_{k_0}^\infty) \quad \text{retracts on } Y_{k_0}^\infty.$$

By applying the Euler–Poincaré characteristic, we get

$$1 = \sum_{\tau_p \in \mathcal{K}^\infty, i(\tau_p)_\infty \leq k_0} (-1)^{i(\tau_p)_\infty}$$

which is a contradiction. This completes the proof of Theorem 1.2 and therefore the proof of Theorem 1.4.

Now for generic K, we can assume that all the critical points of J are non degenerate. This is a consequence of the Sard–Smale Theorem, see [23]. By applying the Euler–Poincaré characteristic on each manifold of (4.1), we obtain under the condition (i) of Theorem 1.1

$$1 - \sum_{\tau_p \in \mathcal{K}^\infty, i(\tau_p) \leq k_0} (-1)^{i(\tau_p)} + \sum_{\partial J(w)=0, i(w) \leq k_0+1} (-1)^{i(w)},$$

where $i(w)$ is the Morse index of J at w. Thus,

$$N_{k_0+1} \geq \left| 1 - \sum_{\tau_p \in \mathcal{K}^\infty, i(\tau_p) \leq k_0} (-1)^{i(\tau_p)} \right|.$$

This finishes the proof of Theorem 1.1 and therefore the proof of Theorem 1.3.

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