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Pillai's problem with the Fibonacci and Padovan sequences

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Abstract

Let $(F_m)_{m\geqslant 0}$ and $(P_n)_{n\geqslant 0}$ be the Fibonacci and Padovan sequences given by the initial conditions $F_0 = 0, F_1 = 1, P_0 = 0, P_1 = P_2 = 1$ and the recurrence formulas $F_{m+2} = F_{m+1} + F_m, P_{n+3} = P_{n+1} + P_n$ for all $m, n \geqslant 0$, respectively. In this note we study and completely solve the Diophantine

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equation

$$P_n - F_m = P_{n_1} - F_{m_1}$$

in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$.

Keywords: Fibonacci, Padovan sequences, Pillai's type problem, Linear form in logarithms.

MSC: 11B39, 11D45, 11D61, 11J86.

1. Introduction

Let a, b be fixed positive integers and consider the Diophatine equation

$$a^n - b^m = a^{n_1} - b^{m_1} \tag{1.1}$$

in positive integers n, m, n_1, m_1 with $(n, m) \neq (n_1, m_1)$. In particular, we look for the integers which can be written as a difference of a power of a and a power of b in at least two distinct ways. In [11], Herschfeld proved that in the case $(a, b) = (2, 3)$ equation (1.1) has only finitely many solutions. In [15], Pillai extended this result to the case a, $b \geqslant 2$ being coprime integers. Both results are ineffective. In [16], Pillai conjectured that in the case $(a, b) = (2, 3)$ the only solutions of equation (1.1) are $(3, 2, 1, 1)$, $(5, 3, 3, 1)$ and $(8, 5, 4, 1)$. This conjecture remained open for about 37 years and was confirmed in [20] by Stroeker and Tijdeman by using Baker's theory on linear forms in logarithms.

Recently, the above problem now known as the *Pillai problem*, was posed in the context of linear recurrence sequences. Namely, let $\mathbf{U} := (U_n)_{n\geqslant 0}$ and $\mathbf{V} := (V_m)_{m\geqslant 0}$ be two linearly recurrence sequences of integers and look at the diophantine equation

$$U_n - V_m = U_{n_1} - V_{n_1} \tag{1.2}$$

in positive integers n, m, n_1, m_1 with $(n, m) \neq (n_1, m_1)$. This reduces to determining the integers which can be written as a difference of an element of $\mathbf{U}$ and an element of $\mathbf{V}$ in at least two distinct ways. This version was started by Ddamulira, Luca and Rakotomalala in [8] where they considered $\mathbf{U}$ as being the Fibonacci sequence and $\mathbf{V}$ as being the sequence of powers of 2. Many other cases have been studied, see for example [3, 6, 7, 10, 12, 13]. In [5], there is a general result, namely that if $\mathbf{U}$ and $\mathbf{V}$ satisfy some natural conditions, then equation (1.2) has only finitely many solutions which furthermore are all effectively computable. We recall that the *Fibonacci sequence* $(F_m)_{m\geqslant 0}$ is given by $F_0 = 0$, $F_1 = 1$ and the recurrence formula

$$F_{m+2} = F_{m+1} + F_m \quad \text{for all} \quad m \geqslant 0.$$

Its first few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

Now, let $(P_n)_{n\geqslant 0}$ be the *Padovan sequence*, named after the architect R. Padovan, given by $P_0 = 0$, $P_1 = P_2 = 1$ and the recurrence formula

$$P_{n+3} = P_{n+1} + P_n \quad \text{for all} \quad n \geqslant 0.$$

This is the sequence A000931 in [18]. Its first few terms are

$$0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16,\ 21,\ 28,\ 37,\ 49,\ 65,\ 86,\ 114,\ 151,\ \ldots$$

In this note, we study another case of equation (1.2) namely with the Fibonacci and the Padovan sequences. More precisely, we solve the equation

$$P_n - F_m = P_{n_1} - F_{m_1} \tag{1.3}$$

in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$. To avoid numerical repeated solutions we assume that $n \neq 1, 2, 4$ and $n_1 \neq 1, 2, 4$. That is whenever we think of 1 and 2 as members of the Padovan sequence que think of them as being P_3 and P_5, respectively. In the same way, $m \neq 1$ and $m_1 \neq 1$. With this conventions, our result is the following:

Theorem 1.1. *All non-negative integer solutions* (n, m, n_1, m_1) *of equation* (1.3) *belong to the set*

$$\left\{\begin{array}{ccccc}
(3,2,0,0), & (3,3,0,2), & (3,4,0,3), & (5,2,3,0), & (5,3,3,2), \\
(5,3,0,0), & (5,4,3,3), & (5,4,0,2), & (5,5,0,4), & (6,2,5,0), \\
(6,3,5,2), & (6,3,3,0), & (6,4,5,3), & (6,4,3,2), & (6,4,0,0), \\
(6,5,3,4), & (6,5,0,3), & (6,6,0,5), & (7,2,6,0), & (7,3,6,2), \\
(7,3,5,0), & (7,4,6,3), & (7,4,5,2), & (7,4,3,0), & (7,5,5,4), \\
(7,5,3,3), & (7,5,0,2), & (7,6,3,5), & (8,2,7,0), & (8,3,7,2), \\
(8,3,6,0), & (8,4,7,3), & (8,4,6,2), & (8,4,5,0), & (8,5,6,4), \\
(8,5,5,3), & (8,5,3,2), & (8,5,0,0), & (8,6,5,5), & (8,6,0,4), \\
(8,7,0,6), & (9,3,8,0), & (9,4,8,2), & (9,4.7,0), & (9,5,8,4), \\
(9,5,7,3), & (9,5,6,2), & (9,5,5,0), & (9,6,7,5), & (9,6,5,4), \\
(9,6,3,3), & (9,6,0,2), & (9,7,5,6), & (10,3,9,0), & (10,4,9,2), \\
(10,5,9,4), & (10,5,8,2), & (10,5,7,0), & (10,6,7,4), & (10,6,6,3), \\
(10,6,5,2), & (10,6,3,0), & (10,7,7,6), & (10,7,3,5), & (10,8,3,7), \\
(11,4,10,0), & (11,5,10,3), & (11,5,9,0), & (11,6,10,5), & (11,6,9,4), \\
(11,6,8,2), & (11,6,7,0), & (11,7,9,6), & (11,7,7,5), & (11,7,5,4), \\
(11,7,3,3), & (11,7,0,2), & (11,8,7,7), & (12,5,11,2), & (12,6,10,2), \\
(12,7,8,3), & (12,7,7,2), & (12,7,6,0), & (12,8,6,6), & (12,8,0,5), \\
(12,9,6,8), & (13,5,12,0), & (13,6,12,4), & (13,7,12,6), & (13,7,10,2), \\
(13,8,8,5), & (13,8,6,4), & (13,8,5,3), & (13,8,3,2), & (13,8,0,0), \\
(13,9,0,7), & (13,10,0,9), & (14,6,13,2), & (14,7,12,2), & (14,8,11,5), \\
(14,8,10,3), & (14,8,9,0), & (14,9,9,7), & (14,9,5,6), & (14,10,9,9), \\
(15,8,13,5), & (15,8,12,0), & (15,9,12,7), & (15,9,8,3), & (15,9,7,2), \\
(15,9,6,0), & (15,10,12,9), & (15,10,6,8), & (15,11,6,10), & (16,7,15,2), \\
(16,8,14,0), & (16,9,14,7), & (16,9,12,2), & (16,10,14,9), & (16,10,9,7), \\
(16,10,5,6), & (17,8,16,5), & (17,10,11,3), & (18,8,17,0), & (18,9,17,7),
\end{array}\right\}$$

$$\left\{\begin{array}{ccccc} (18,10,17,9), & (18,11,8,6), & (18,11,5,5), & (18,11,0,4), & (19,11,14,4), \\ (19,12,7,9), & (20,11,17,4), & (20,12,14,8), & (20,12,11,5), & (20,12,10,3), \\ (20,12,9,0), & (20,13,9,11), & (20,14,9,13), & (21,11,19,4), & (21,13,3,9), \\ (22,13,15,5), & (23,11,22,4), & (25,15,10,4), & (25,15,9,2) & \end{array}\right\}$$

The set of integers which can be written as the difference of a Padovan number and a Fibonacci number in at least two distinct ways is

$$\left\{\begin{array}{ccccccccc} -226, & -82, & -52, & -34, & -33, & -30, & -27, & -18, & -13, \\ -12, & -9, & -8, & -6, & -5, & -4, & -3, & -2, & -1, \\ 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, \\ 9, & 10, & 11, & 13, & 15, & 16, & 20, & 25, & 28, \\ 31, & 32, & 36, & 44, & 52, & 62, & 65, & 111, & 262. \end{array}\right\}.$$

All such representations of each of these numbers are

$$\begin{aligned}
-226 &= P_{20} - F_{14} = P_9 - F_{13}; \\
-82 &= P_{20} - F_{13} = P_9 - F_{11}; \\
-52 &= P_{15} - F_{11} = P_6 - F_{10}; \\
-34 &= P_{13} - F_{10} = P_0 - F_9; \\
-33 &= P_{21} - F_{13} = P_3 - F_9; \\
-30 &= P_{19} - F_{12} = P_7 - F_9; \\
-27 &= P_{14} - F_{10} = P_9 - F_9; \\
-18 &= P_{12} - F_9 = P_6 - F_8 = P_{15} - F_{10}; \\
-13 &= P_{13} - F_9 = P_0 - F_7; \\
-12 &= P_{10} - F_8 = P_3 - F_7; \\
-9 &= P_{11} - F_8 = P_7 - F_7; \\
-8 &= P_8 - F_7 = P_0 - F_6; \\
-6 &= P_{16} - F_{10} = P_{14} - F_9 = P_9 - F_7 = P_5 - F_6; \\
-5 &= P_{12} - F_8 = P_6 - F_6 = P_0 - F_5; \\
-4 &= P_{10} - F_7 = P_7 - F_6 = P_3 - F_5; \\
-3 &= P_{18} - F_{11} = P_8 - F_6 = P_5 - F_5 = P_0 - F_4; \\
-2 &= P_6 - F_5 = P_3 - F_4 = P_0 - F_3; \\
-1 &= P_{11} - F_7 = P_9 - F_6 = P_7 - F_5 = P_5 - F_4 = P_3 - F_3 = P_0 - F_2; \\
0 &= P_{13} - F_8 = P_8 - F_5 = P_6 - F_4 = P_5 - F_3 = P_3 - F_2 = P_0 - F_0; \\
1 &= P_{10} - F_6 = P_7 - F_4 = P_6 - F_3 = P_5 - F_2 = P_3 - F_0; \\
2 &= P_9 - F_5 = P_8 - F_4 = P_7 - F_3 = P_6 - F_2 = P_5 - F_0; \\
3 &= P_{15} - F_9 = P_{12} - F_7 = P_8 - F_3 = P_7 - F_2 = P_6 - F_0; \\
4 &= P_{11} - F_6 = P_{10} - F_5 = P_9 - F_4 = P_8 - F_2 = P_7 - F_0; \\
5 &= P_9 - F_3 = P_8 - F_0; \\
6 &= P_{25} - F_{15} = P_{10} - F_4 = P_9 - F_2;
\end{aligned}$$

$$
\begin{aligned}
7 &= P_{20} - F_{12} = P_{14} - F_8 = P_{11} - F_5 = P_{10} - F_3 = P_9 - F_0;\\
8 &= P_{13} - F_7 = P_{12} - F_6 = P_{10} - F_2;\\
9 &= P_{11} - F_4 = P_{10} - F_0;\\
10 &= P_{17} - F_{10} = P_{11} - F_3;\\
11 &= P_{12} - F_5 = P_{11} - F_2;\\
13 &= P_{13} - F_6 = P_{12} - F_4;\\
15 &= P_{16} - F_9 = P_{14} - F_7 = P_{12} - F_2;\\
16 &= P_{15} - F_8 = P_{13} - F_5 = P_{12} - F_0;\\
20 &= P_{14} - F_6 = P_{13} - F_2;\\
25 &= P_{19} - F_{11} = P_{14} - F_4;\\
28 &= P_{16} - F_8 = P_{14} - F_0;\\
31 &= P_{18} - F_{10} = P_{17} - F_9;\\
32 &= P_{22} - F_{13} = P_{15} - F_5;\\
36 &= P_{16} - F_7 = P_{15} - F_2;\\
44 &= P_{17} - F_8 = P_{16} - F_5;\\
52 &= P_{18} - F_9 = P_{17} - F_7;\\
62 &= P_{20} - F_{11} = P_{17} - F_4;\\
65 &= P_{18} - F_8 = P_{17} - F_0;\\
111 &= P_{21} - F_{11} = P_{19} - F_4;\\
262 &= P_{23} - F_{11} = P_{22} - F_4\,.
\end{aligned}
$$

In [19], Stewart notes that 3, 5 and 21 are both Fibonacci and Padovan numbers and asks whether there are any others. This problem was solved by De Weger in [21], where he proves that all integers which are both Fibonacci and Padovan numbers are 0, 1, 2, 3, 5, 21. Actually, he proves that the distance between Fibonacci and Padovan numbers growths exponentially. We remark that as a particular case of our result, we also have a solution of Stewart problem.

2. Tools

In this section, we gather the tools we need to prove Theorem 1.1. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha^{(1)}, \ldots, \alpha^{(d)}$ denote its conjugates. The *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log \max\left\{|\alpha^{(i)}|, 1\right\}\right).$$

This height satisfies the following basic properties. For α, β algebraic numbers and $m \in \mathbb{Z}$ we have

- $h(\alpha+\beta) \leqslant h(\alpha)+h(\beta)+\log(2)$,
- $h(\alpha\beta) \leqslant h(\alpha)+h(\beta)$,
- $h(\alpha^m) = |m|h(\alpha)$.

Now, let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\alpha_1,\dots,\alpha_\ell$ positive elements of $\mathbb{L}$ and $b_1,\dots,b_\ell \in \mathbb{Z}\setminus\{0\}$. Let $B \geqslant \max\{|b_1|,\dots,|b_\ell|\}$ and

$$\Lambda = \alpha_1^{b_1}\cdots\alpha_\ell^{b_\ell} - 1.$$

Let $A_1,\dots,A_\ell$ be real numbers with

$$A_i \geqslant \max\{d_{\mathbb{L}}\, h(\alpha_i), |\log\alpha_i|, 0.16\}, \qquad i=1,2,\dots,\ell.$$

The first tool we need is the following result due to Matveev in [14] (see also Theorem 9.4 in [4]).

Theorem 2.1. *Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4\cdot 30^{\ell+3}\cdot \ell^{4.5}\cdot d_{\mathbb{L}}^2\cdot(1+\log d_{\mathbb{L}})\cdot(1+\log B)A_1\cdots A_\ell.$$

In this note we always use $\ell=3$. Further, $\mathbb{L}=\mathbb{Q}(\gamma,\alpha)$ has degree $d_{\mathbb{L}}=6$, where γ and α are defined at the beginning of Section 3. Thus, once and for all we fix the constant

$$C := 1.43908\times 10^{13} \;>\; 1.4\cdot 30^{3+3}\cdot 3^{4.5}\cdot 6^2\cdot(1+\log 6)$$

The second one, is a version of the reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [2] (See also Dujella and Pethő [9]). For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 2.2. *Let M be a positive integer. Let $\tau,\mu,A>0, B>1$ be given real numbers. Assume that p/q is a convergent of τ such that $q>6M$ and that $\varepsilon := \|q\,\mu\| - M\|q\,\tau\| > 0$. Then there is no solution to the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

in positive integers n, m and w satisfying

$$n \leqslant M \quad \textit{and} \quad w \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

Finally, the following result will be very useful. This is Lemma 7 in [17].

Lemma 2.3. *If $m\geqslant 1$, $T>(4m^2)^m$ and $T > x/(\log x)^m$. Then*

$$x < 2^m T(\log T)^m.$$

3. Proof of Theorem 1.1

We start with some basic properties of our sequences. For a complex number z we write $\overline{z}$ for its complex conjugate. Let $\omega \neq 1$ be a cubic root of 1. Put

$$\gamma := \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}, \quad \delta := \omega\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\frac{9-\sqrt{69}}{18}},$$

and

$$\alpha := \frac{1+\sqrt{5}}{2}, \quad \beta := \frac{1-\sqrt{5}}{2}.$$

It is clear that $\gamma, \delta, \overline{\delta}$ are the roots of the $\mathbb{Q}$-irreducible polynomial $X^3 - X - 1$. It can be proved, by induction for example, that the Binet formulas

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad P_n = c_1\gamma^n + c_2\delta^n + c_3\overline{\delta}^n \quad \text{hold for all} \quad n \geqslant 0, \tag{3.1}$$

where

$$c_1 = \frac{\gamma(\gamma+1)}{2\gamma+3}, \quad c_2 = \frac{\delta(\delta+1)}{2\delta+3}, \quad c_3 = \overline{c_2}.$$

The first formula in (3.1) is well known. The second one follows from the general theorem on linear recurrence sequences since the above polynomial is the characteristic polynomial of the Padovan sequence. Further, the inequalities

$$\alpha^{n-2} \leqslant F_n \leqslant \alpha^{n-1}, \quad \gamma^{n-3} \leqslant P_n \leqslant \gamma^{n-1} \tag{3.2}$$

also hold for all $n \geqslant 1$. These can be proved by induction. We note that

$$\gamma = 1.32471\ldots, \quad |\delta| = 0.86883\ldots, \quad c_1 = 0.54511\ldots, \quad |c_2| = 0.28241\ldots,$$

and

$$\alpha = 1.61803\ldots, \quad |\beta| = 0.61803\ldots$$

Now we start with the study of our equation (1.3) in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$ where, as we have said, $n, n_1 \neq 1, 2, 4$, $m, m_1 \neq 1$. We note, if $m = m_1$ then $P_n = P_{n_1}$ which implies $n = n_1$, a contradiction. Thus, we assume that $m > m_1$. Rewriting equation (1.3) as

$$P_n - P_{n_1} = F_m - F_{m_1} \tag{3.3}$$

we observe the right-hand is positive. So, the left-hand side is also positive and therefore, $n > n_1$. Now, we compare both sides of (3.3) using (3.2). We have

$$\gamma^{n-8} \leqslant P_n - P_{n_1} = F_m - F_{m_1} \leqslant F_m \leqslant \alpha^{m-1}.$$

Indeed, the left-hand side inequality is clear if $n_1 = 0$. If $n_1 = 3, n \geqslant 5$. For $n = 5$ it is also clear and for $n \geqslant 6$ we have $P_n - P_{n_1} \geqslant P_n - P_{n-1} = P_{n-5} \geqslant \gamma^{n-8}$. Thus, $\gamma^{n-8} \leqslant \alpha^{m-1}$. In a similar way,

$$\gamma^{n-1} \geqslant P_n - P_{n_1} = F_m - F_{m_1} \geqslant \alpha^{m-4}.$$

where the inequality at the right-hand side is clear for both $m_1 = 0$ and $m_1 \neq 0$. Thus,

$$(n-8)\frac{\log\gamma}{\log\alpha} \leqslant m-1 \quad \text{and} \quad (n-1)\frac{\log\gamma}{\log\alpha} \geqslant m-4. \tag{3.4}$$

Since $\log\gamma/\log\alpha = 0.584357\ldots$ we have that if $n \leqslant 540$ then $m \leqslant 318$. A brute force search with *Mathematica* in the range $0 \leqslant n_1 < n \leqslant 540$, $0 \leqslant m_1 < m \leqslant 318$, with our conventions, we obtained all solutions listed in Theorem 1.1.

From now on, we assume that $n > 540$. Thus, from (3.4), we have that $m > 311$ and also that $n > m$. From Binet's formula (3.1), we rewrite our equation as

$$\left|c_1\gamma^n - \frac{\alpha^m}{\sqrt{5}}\right| \leqslant 2|c_2||\delta|^n + \frac{1}{\sqrt{5}} + \gamma^{n_1-1} + \alpha^{m_1-1} < \max\{\gamma^{n_1+6}, \alpha^{m_1+4}\}.$$

Dividing through by $\alpha^m/\sqrt{5}$ we get

$$\left|\sqrt{5}c_1\gamma^n\alpha^{-m} - 1\right| < \max\{\gamma^{n_1-n+16}, \alpha^{m_1-m+6}\}, \tag{3.5}$$

where we have used $\gamma^{n-8} \leqslant \alpha^{m-1}$, $\sqrt{5} < \alpha\gamma^2$ and $\sqrt{5} < \alpha^2$. Let Λ be the expression inside the absolute value in the left-hand side of (3.5). Observe that $\Lambda \neq 0$. To see this, we consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{K} := \mathbb{Q}(\alpha, \gamma, \delta)$ over $\mathbb{Q}$ defined by $\sigma(\gamma) := \delta, \sigma(\delta) := \gamma$ and $\sigma(\alpha) := \alpha$. We note that $\sigma(\overline{\delta}) = \overline{\delta}$ and $\sigma(\beta) = \beta$. If $\Lambda = 0$ then $\sigma(\Lambda) = 0$ and we get

$$\frac{\alpha^m}{\sqrt{5}} = \sigma(c_1\gamma^n) = c_2\delta^n.$$

Thus,

$$\frac{\alpha^m}{\sqrt{5}} = |c_2||\delta|^n < 1,$$

which is absurd since $m > 311$. So, $\Lambda \neq 0$. We apply Matveev's inequality to Λ by taking

$$\alpha_1 = \sqrt{5}c_1,\ \alpha_2 = \gamma,\ \alpha_3 = \alpha, \quad b_1 = 1,\ b_2 = n,\ b_3 = -m.$$

Thus, $B = n$. Further, $h(\alpha_2) = \log\gamma/3$, $h(\alpha_3) = \log\alpha/2$. For α_1 we use the properties of the height to conclude

$$h(\alpha_1) \leqslant \log\gamma + 7\log 2.$$

So we take $A_1 = 30.8$, $A_2 = 0.57$, $A_3 = 1.45$. From Matveev's inequality we obtain

$$\log|\Lambda| > -C(1+\log n)\cdot 30.8\cdot 0.57\cdot 1.45 > -3.66336\times 10^{14}(1+\log n),$$

which, compared with (3.5) we obtain

$$\min\{(n-n_1)\log\gamma, (m-m_1)\log\alpha\} \leqslant 3.66337\times 10^{14}(1+\log n).$$

Now we study each one of these two possibilities.

Case 1. $\min\{(n-n_1)\log\gamma, (m-m_1)\log\alpha\} = (n-n_1)\log\gamma$.

In this case, using Binet's formulas (3.1), we rewrite our equation as

$$\left|c_1(\gamma^{n-n_1}-1)\gamma^{n_1} - \frac{\alpha^m}{\sqrt{5}}\right| \leqslant 4|c_2||\delta|^{n_1} + 1 + \alpha^{m_1-1} < 2\cdot\alpha^{m_1+2} \leqslant \alpha^{m_1+4}.$$

Thus,

$$\left|c_1\sqrt{5}(\gamma^{n-n_1}-1)\gamma^{n_1}\alpha^{-m} - 1\right| < \frac{1}{\alpha^{m-m_1-6}}. \tag{3.6}$$

Let Λ_1 be the expression inside the absolute value in the left-hand side of (3.6). We note that $\Lambda_1 \neq 0$. For if not, we apply the above σ to it and we have $\sigma(\Lambda_1) = 0$. Thus,

$$\frac{\alpha^m}{\sqrt{5}} = |\sigma(c_1)(\delta^n - \delta^{n_1})| \leqslant 2|c_2| < 1,$$

which is absurd since $m > 311$. We apply Matveev's inequality to Λ_1 and for this we take

$$\alpha_1 = \sqrt{5}c_1(\gamma^{n-n_1}-1),\ \alpha_2 = \gamma,\ \alpha_3 = \alpha, \quad b_1 = 1,\ b_2 = n_1,\ b_3 = -m.$$

We have $B = n$. The heights of α_2 and α_3 are already calculated. For α_1 we use the height properties and we get

$$h(\alpha_1) \leqslant \frac{3.66338\times 10^{14}(1+\log n)}{3}.$$

Thus, we can take $A_1 = 7.32676\times 10^{14}(1+\log n)$ and A_2, A_3 as above. From Matveev's inequality we obtain

$$\log|\Lambda_1| > -C(1+\log n)\cdot(7.32676\times 10^{14}(1+\log n))\cdot 0.57\cdot 1.45,$$

which compared with (3.6) gives

$$(m-m_1)\log\alpha < 8.71446\times 10^{27}(1+\log n)^2.$$

Case 2. $\min\{(n-n_1)\log\gamma, (m-m_1)\log\alpha\} = (m-m_1)\log\alpha$.

To this case, we rewrite our equation as

$$\left|c_1\gamma^n - \frac{(\alpha^{m-m_1}-1)\alpha^{m_1}}{\sqrt{5}}\right| < \gamma^{n_1-1} + 2|c_2| + 1 < \gamma^{n_1+4}.$$

Thus,

$$\left|1 - \left(\frac{\alpha^{m-m_1}-1}{\sqrt{5}c_1}\right)\gamma^{-n}\alpha^{m_1}\right| < \frac{1}{\gamma^{n-n_1-7}}, \tag{3.7}$$

where we have used $1 < c_1\gamma^3$. Let Λ_2 be the expression inside the absolute value in the left-hand side of (3.7). We note that $\Lambda_2 \neq 0$. Indeed, if it is not the case then by applying the above σ to it we obtain $\sigma(\Lambda_2) = 0$. Thus

$$1 < \frac{\alpha^{m-1}(\alpha-1)}{\sqrt{5}} \leqslant \frac{\alpha^m - \alpha^{m_1}}{\sqrt{5}} = \sqrt{5}|c_2||\delta|^n < \sqrt{5}|c_2| < 1,$$

where the left-hand side inequality holds since $m > 311$, which is absurd. So, $\Lambda_2 \neq 0$ and we apply Matveev's inequality to it. To do this, we take

$$\alpha_1 = \frac{\alpha^{m-m_1} - 1}{\sqrt{5}c_1}, \; \alpha_2 = \gamma, \; \alpha_3 = \alpha, \quad b_1 = 1, \; b_2 = -n, \; b_3 = m_1.$$

Thus, $B = n$. The heights of α_2 and α_3 are already calculated. From the properties of the height for α_1 we obtain

$$h(\alpha_1) \leqslant \frac{3.66338 \times 10^{14}(1 + \log n)}{2}.$$

Thus, we can take $A_1 = 1.09901 \times 10^{15}(1 + \log n)$ and A_2, A_3 as above. Hence, from Matveev's inequality we obtain

$$\log|\Lambda_2| > -C(1 + \log n) \cdot \big(1.09901 \times 10^{15}(1 + \log n)\big) \cdot 0.57 \cdot 1.45,$$

which compared with (3.7) we get

$$(n - n_1)\log\gamma < 1.30717 \times 10^{28}(1 + \log n)^2.$$

So, from the conclusion of the two cases we have that

$$\max\{(n - n_1)\log\gamma, (m - m_1)\log 2\} < 1.30717 \times 10^{28}(1 + \log n)^2.$$

Now we get a bound on n. To do this we rewrite our equation as

$$\left| c_1(\gamma^{n-n_1} - 1)\gamma^{n_1} - \frac{(\alpha^{m-m_1} - 1)\alpha^{m_1}}{\sqrt{5}} \right| < 4|c_2| + 1 < 2.2.$$

Thus,

$$\left| \left(\sqrt{5}c_1 \frac{\gamma^{n-n_1} - 1}{\alpha^{m-m_1} - 1} \right) \gamma^{n_1}\alpha^{-m_1} - 1 \right| < \frac{2.2 \cdot \sqrt{5}}{\alpha^m - \alpha^{m_1}} \leqslant \frac{6.6 \cdot \sqrt{5}}{\alpha^m} < \frac{1}{\gamma^{n-16}}, \tag{3.8}$$

where we have used $\gamma^{n-8} < \alpha^{m-1}$ and $6.6 \cdot \sqrt{5} < \alpha\gamma^8$. Let Λ_3 be the expression inside the absolute value in the left-hand side of (3.8). As above, if $\Lambda_3 = 0$ we apply the above σ and we obtain $\sigma(\Lambda_3) = 0$. Then

$$1 < \frac{\alpha^{m-1}(\alpha - 1)}{\sqrt{5}} \leqslant \frac{\alpha^m - \alpha^{m_1}}{\sqrt{5}} = |c_2(\delta^n - \delta^{n_1})| \leqslant 2|c_2| < \frac{2}{3},$$

and as above, we get a contradiction. Thus, $\Lambda_3 \neq 0$ and we apply Matveev's inequality to it. To do this, we take

$$\alpha_1 = \sqrt{5}c_1 \frac{\gamma^{n-n_1} - 1}{\alpha^{m-m_1} - 1}, \; \alpha_2 = \gamma, \; \alpha_3 = \alpha, \quad b_1 = 1, \; b_2 = n_1, \; b_3 = -m_1.$$

Hence, $B = n$. The height of α_2 and α_3 have already been calculated. For α_1 we use the properties of the height to conclude that

$$\begin{aligned} h(\alpha_1) &\leqslant \log\gamma + (n-n_1)\frac{\log\gamma}{3} + (m-m_1)\frac{\log\alpha}{2} + 9\log 2 \\ &< \frac{6.53586\times 10^{28}(1+\log n)^2}{6}. \end{aligned}$$

Thus, we can take $A_1 = 6.53586 \times 10^{28}(1+\log n)^2$ and A_2, A_3 as above. From Matveev's inequality we get

$$\log|\Lambda_3| > -C\cdot\left((1+\log n)\cdot 6.53586\times 10^{28}(1+\log n)^2\right)\cdot 0.57\cdot 1.45,$$

which compared with (3.8) yields $n < 2.2116\times 10^{43}(\log n)^3$. Thus, from Lemma 2.3 we obtain

$$n < 1.75894 \times 10^{50}. \tag{3.9}$$

Now we reduce this upper bound on n. To do this, let Γ be defined as

$$\Gamma = n\log\gamma - m\log\alpha + \log\left(\sqrt{5}\,c_1\right),$$

and we go to (3.5). Assume that $\min\{n-n_1, m-m_1\} \geqslant 20$. Observe that $e^\Gamma - 1 = \Lambda \neq 0$. Therefore $\Gamma \neq 0$. If $\Gamma > 0$, then

$$0 < \Gamma < e^\Gamma - 1 = |\Lambda| < \max\{\gamma^{n_1-n+16}, \alpha^{m_1-m+6}\}.$$

If $\Gamma < 0$, we then have $1 - e^\Gamma = |e^\Gamma - 1| = |\Lambda| < 1/2$. Thus, $e^{|\Gamma|} < 2$ and we get

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < 2\max\{\gamma^{n_1-n+16}, \alpha^{m_1-m+6}\}.$$

So, in both cases we have

$$0 < |\Gamma| < 2\max\{\gamma^{n_1-n+16}, \alpha^{m_1-m+6}\}.$$

Dividing through $\log\alpha$ we get

$$0 < |n\tau - m + \mu| < \max\left\{\frac{374}{\gamma^{n-n_1}}, \frac{75}{\alpha^{m-m_1}}\right\},$$

where

$$\tau := \frac{\log\gamma}{\log\alpha}, \quad \mu := \frac{\log\left(\sqrt{5}\,c_1\right)}{\log\alpha}.$$

We apply Lemma 2.2. To do this we take $M := 1.75894 \times 10^{50}$ which is the upper bound on n by (3.9). With the help of *Mathematica* we found that the convergent

$$\frac{p_{111}}{q_{111}} = \frac{1055018110290384419279582749015021525092270854503951799 7}{1805433708589770760526539129691547197889880925836949175 4}$$

of τ satisfies that $q_{111} > 6M$ and that $\varepsilon := \|q_{111}\mu\| - M\|q_{111}\tau\| = 0.450294 > 0$. Thus, by Lemma 2.2 with $A := 374$, $B := \gamma$ or $A := 75$, $B := \alpha$, we get that either

$$n - n_1 \leqslant 476 \quad \text{or} \quad m - m_1 \leqslant 275.$$

Now we study each one of these two cases. We first assume that $n - n_1 \leqslant 476$ and $m - m_1 \geqslant 20$. In this case, we consider

$$\Gamma_1 = n_1 \log\gamma - m\log\alpha + \log(\sqrt{5}c_1(\gamma^{n-n_1} - 1))$$

and we go to (3.6). We see that $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$. Thus, $\Gamma_1 \neq 0$ and, with a similar argument as the previous one we obtain

$$0 < |\Gamma_1| < \frac{2\alpha^6}{\alpha^{m-m_1}}.$$

Dividing through $\log\alpha$ we get

$$0 < |n_1\tau - m + \mu| < \frac{75}{\alpha^{m-m_1}},$$

where τ is the same one as above and

$$\mu := \frac{\log(\sqrt{5}c_1(\gamma^{n-n_1} - 1))}{\log\alpha}.$$

We note that $n_1 > 0$, since otherwise we would have $n \leqslant 476$ which contradicts $n > 540$. Thus, we can apply Lemma 2.2. Consider

$$\mu_k := \frac{\log(\sqrt{5}c_1(\gamma^k - 1))}{\log\alpha}, \quad k = 1, 2, \ldots, 476.$$

With the help of *Mathematica* we found that the denominator of the 111-th convergent above of τ is such that $q_{111} > 6M$ and $\varepsilon_k \geqslant 0.00129842 > 0$ for all $k = 1, 2, \ldots, 476$. Thus, by Lemma 2.2 with $A := 75$, $B := \alpha$ we obtain that the maximum value of $\log(q_{111} \cdot 75/\varepsilon_k)/\log\alpha$, $k = 1, 2, \ldots, 476$, is less than 287. Therefore $m - m_1 \leqslant 287$.

In a similar way we study the other case. Assume that $m - m_1 \leqslant 275$ and $n - n_1 \geqslant 20$. In this case we consider

$$\Gamma_2 = n\log\gamma - m_1\log\alpha + \log\left(\frac{\sqrt{5}c_1}{\alpha^{m-m_1} - 1}\right)$$

and we go to (3.7). Observe that $1 - e^{-\Gamma_2} = \Lambda_2 \neq 0$. Hence, $\Gamma_2 \neq 0$ and, with an argument as above we conclude that

$$0 < |\Gamma_2| < \frac{2\gamma^7}{\gamma^{n-n_1}},$$

Dividing through by $\log\alpha$ we get

$$0 < |n\tau - m_1 + \mu| < \frac{30}{\gamma^{n-n_1}}.$$

where τ is as above and

$$\mu := \frac{\log\left(\sqrt{5}c_1/(\alpha^{m-m_1}-1)\right)}{\log\alpha}.$$

We note that $m_1 > 0$. Indeed, for if not, we get $m \leqslant 275$ which contradicts $m > 311$. Thus, we can apply Lemma 2.2 again. Consider

$$\mu_\ell := \frac{\log\left(\sqrt{5}c_1/(\alpha^{\ell}-1)\right)}{\log\alpha}, \quad \ell = 1,\dots,275.$$

Again, with *Mathematica* we quickly found that the same 111-th convergent of τ satisfies $q_{111} > 6M$ and $\varepsilon_\ell > 0.000693865 > 0$ for all $\ell = 1,\dots,257$. Thus, from Lemma 2.2 with $A := 30$, $B := \gamma$ we obtain that the maximum value of $\log(q_{111}\cdot 30/\epsilon_\ell)/\log\gamma$, $\ell = 1,\dots,257$ is $\leqslant 490$. Hence, $n - n_1 \leqslant 490$.

Summarizing what we have done, we first got that either $n - n_1 \leqslant 476$ or $m - m_1 \leqslant 257$. Assuming the first one we obtained that $m - m_1 \leqslant 287$, and assuming the second one we obtained $n - n_1 \leqslant 490$. So, altogether we have that $n - n_1 \leqslant 490$, $m - m_1 \leqslant 287$. It remains to study this case.

Consider

$$\Gamma_3 = n_1\log\gamma - m_1\log\alpha + \log\left(\sqrt{5}c_1\frac{\gamma^{n-n_1}-1}{\alpha^{m-m_1}-1}\right),$$

and we go to (3.8). Note that $e^{\Gamma_3} - 1 = \Lambda_3 \neq 0$. Thus, $\Gamma_3 \neq 0$ and since $n > 540$ with an argument as before we get

$$0 < |\Gamma_3| < \frac{2\gamma^{16}}{\gamma^n}.$$

Dividing through by $\log\alpha$ we obtain

$$o < |n_1\tau - m_1 - \mu| < \frac{374}{\gamma^n},$$

where τ is as above and

$$\mu := \frac{\log\left(\sqrt{5}c_1\left(\gamma^{n-n_1} - 1/\alpha^{m-m_1}-1\right)\right)}{\log\alpha}.$$

As above we note that n_1 and m_1 are positives. We apply Lemma 2.2 again. Consider

$$\mu_{k,l} := \frac{\log\left(\sqrt{5}c_1\left(\gamma^{k} - 1/\alpha^{\ell}-1\right)\right)}{\log\alpha}, \quad k = 1,\dots,490 \quad \ell = 1,\dots,287.$$

With *Mathematica* we find that the same 111-th convergent above of τ works again. That is, $q_{111} > 6M$ and $\varepsilon_{k,\ell} \geq 5.28933^{-8} > 0$ for all $k = 1, \ldots, 490$ and $\ell = 1, \ldots, 287$. Thus, by Lemma 2.2 with $A := 374$ and $B := \gamma$ we obtain that the maximum value of $\log(q_{111}374/\varepsilon_{k,\ell})/\log\gamma$, $k = 1, \ldots, 490$ and $\ell = 1, \ldots, 287$, is $\leqslant 533$. Thus, $n \leqslant 533$ which contradicts our assumption on n. This completes the proof of Theorem 1.1.

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