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On the X-coordinates of Pell equations which are rep-digits, II

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Abstract

For a positive integer d which is not a square, we show that there is at most one value of the positive integer X participating in the Pell equation $X^2 - dY^2 = \pm 4$ which is a rep-digit, that is all its base 10 digits are equal, except for $d = 2, 5, 13$.

Keywords: Pell equation, Rep-digit, Linear forms in complex logarithms.

MSC: 11A25 11B39, 11J86

1. Introduction

Let d be a positive integer which is not a perfect square. It is well-known that the Pell equation

$$X^2 - dY^2 = \pm 4 \tag{1.1}$$

has infinitely many positive integer solutions (X, Y). Furthermore, putting (X_1, Y_1) for the smallest such solution (solution with minimal value for X), all the positive integer solutions are of the form (X_n, Y_n) for some positive integer n where

$$\frac{X_n + \sqrt{d}Y_n}{2} = \left(\frac{X_1 + \sqrt{d}Y_1}{2}\right)^n.$$

There are many papers in the literature which solve Diophantine equations involving members of the sequences $\{X_n\}_{n\geq 1}$ or $\{Y_n\}_{n\geq 1}$ being squares, or perfect powers of larger exponents of some other integers, etc. (see, for example, [4, 5]).

Let $g \geq 2$ be an integer. A natural number N is called a *base g rep-digit* if all of its base g-digits are equal; that is, if

$$N = a\left(\frac{g^m - 1}{g - 1}\right), \quad \text{for some} \quad m \geq 1 \text{ and } a \in \{1, 2, \ldots, g-1\}.$$

When $g = 10$, we omit the base and simply say that N is a rep-digit. Diophantine equations involving rep-digits were also considered in several papers which found all rep-digits which are perfect powers, or Fibonacci numbers, or generalized Fibonacci numbers, and so on (see [1–3, 7, 9, 11–15, 17] for a sample of such results). In this paper, we study when can X_n be a rep-digit. This reduces to the Diophantine equation

$$X_n = a\left(\frac{10^m - 1}{9}\right), \quad m \geq 1 \text{ and } a \in \{1, \ldots, 9\}. \tag{1.2}$$

Of course, for every positive integer X, there is a unique square-free integer $d \geq 2$ such that

$$X^2 - dY^2 = -4.$$

Namely d is the product of all prime factors of X^2+4 which appear at odd exponents in its factorization. In particular, taking $X = a(10^m - 1)/9$, we get that any rep-digit is the X-coordinate of the Pell equation (1.1) corresponding to some specific square-free integer d. If $X > 2$, we can instead look at $X^2 - 4$ and write it as dY^2 for some positive integers d and Y with d squarefree, and then

$$X^2 - dY^2 = 4.$$

In particular, we can take $X = a(10^m - 1)/9$ with $a \in \{1, \ldots, 9\}$ and $m \geq 1$, where we ask in addition that $a \geq 3$ when $m = 1$. Here, we study the square-free integers d such that the sequence $\{X_n\}_{n\geq 1}$ contains at least two rep-digits. Our result is the following.

Theorem 1.1. *Let $d \geq 2$ be square-free. The Diophantine equation*

$$X_n = a\left(\frac{10^m - 1}{9}\right), \quad m \geq 1 \text{ and } a \in \{1, \ldots, 9\} \tag{1.3}$$

has at most one positive integer solution n except when $d = 2, 5, 13$ for which we have

$$2^2 - 2\cdot 2^2 = -4, \quad 6^2 - 2\cdot 4^2 = 4,$$

$$1^2 - 5\cdot 1^2 = -4, \quad 3^2 - 5\cdot 1^2 = 4, \quad 4^2 - 5\cdot 2^2 = -4 \quad 7^2 - 5\cdot 3^2 = 4, \quad 11^2 - 5\cdot 5^2 = -4,$$

and

$$3^2 - 13\cdot 1^2 = -4, \quad 11^2 - 13\cdot 3^2 = 4.$$

2. Linear forms in logarithms

We need some results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. We start by recalling Theorem 9.4 of [4], which is a modified version of a result of Matveev [16]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_l|, 3\},$$

and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots l,$$

where for an algebraic number η of minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\right).$$

The following consequence of Matveev's theorem is Theorem 9.4 in [4].

Theorem 2.1. *If* $\Gamma \neq 0$ *and* $\mathbb{L} \subseteq \mathbb{R}$, *then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

When $l = 2$ and $\eta_1,\ \eta_2$ are positive and multiplicatively independent, we can do better. Namely, let in this case $B_1,\ B_2$ be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\eta_i), \frac{|\log \eta_i|}{d_{\mathbb{L}}}, \frac{1}{d_{\mathbb{L}}}\right\} \quad i = 1, 2,$$

and put

$$b' := \frac{|d_1|}{d_{\mathbb{L}} \log B_2} + \frac{|d_2|}{d_{\mathbb{L}} \log B_1}.$$

Furthermore, let

$$\Lambda = d_1 \log \eta_1 + d_2 \log \eta_2.$$

Note that $\Lambda \neq 0$ when η_1 and η_2 are multiplicatively independent.

Theorem 2.2. *With the above notations, assuming that $\mathbb{L}$ is real, η_1, η_2 are positive and multiplicatively independent, then*

$$\log|\Lambda| > -24.34 d_{\mathbb{L}}^4 \left(\max\left\{\log b' + 0.14, \frac{21}{d_{\mathbb{L}}}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2.$$

Note that $e^{\Lambda} - 1 = \Gamma$, so Γ is close to zero if and only if Λ is close to zero, which explains the relation between Theorems 2.1 and 2.2.

3. The Baker-Davenport lemma

Here, we recall the Baker-Davenport reduction method (see [8, Lemma 5a]), which turns out to be useful in order to reduce the bounds arising from applying Theorems 2.1 and 2.2.

Lemma 3.1. *Let $\kappa \neq 0$ and μ be real numbers. Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and put*

$$\xi = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\xi > 0$, then there is no solution to the inequality

$$0 < |m\kappa - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$\frac{\log(AQ/\xi)}{\log B} \leq k \quad \text{and} \quad m \leq M.$$

4. Bounding the variables

We assume that (X_1, Y_1) is the minimal solution of the Pell equation (1.1). Set

$$X_1^2 - dY_1^2 =: \pm 4$$

and

$$x_n = \frac{X_n}{2}, \quad y_n = \frac{Y_n}{2} \quad \text{for all} \quad n \geq 1.$$

We have

$$x_n^2 - dy_n^2 =: \varepsilon_n, \quad \varepsilon_n \in \{\pm 1\}.$$

Put

$$\delta := x_1 + \sqrt{x_1^2 - \varepsilon_1} = x_1 + \sqrt{d} y_1, \quad \eta := x_1 - \sqrt{d} y_1 = \varepsilon_1 \delta^{-1}, \quad \text{with} \quad \delta \geq (1+\sqrt{5})/2.$$

Then, we get

$$x_n = \frac{1}{2}(\delta^n + \eta^n),$$

or, equivalently,

$$X_n = \delta^n + \eta^n.$$

We start with some general considerations concerning equation (1.2). From equation (1.2), we have

$$X_n = a\left(\frac{10^m - 1}{9}\right) > a(1 + 10 + \cdots + 10^{m-1}) > 10^{m-1}.$$

We get

$$10^{m-1} \leq X_n < 10^m. \tag{4.1}$$

Furthermore,

$$2\delta^n > \delta^n + \eta^n = X_n \geq \delta^n - \delta^{-n} \geq \frac{\delta^n}{2},$$

where the last inequality follows because $n \geq 1$ and $\delta \geq (1+\sqrt{5})/2 > \sqrt{2}$. So,

$$\frac{\delta^n}{2} \leq X_n < 2\delta^n \quad \text{holds for all} \quad n \geq 1. \tag{4.2}$$

Using now the equations (4.1) and (4.2), we have

$$10^{m-1} \leq X_n < 2\delta^n \quad \text{and} \quad \frac{\delta^n}{2} \leq X_n \leq 10^m.$$

Hence, we obtain

$$nc_1 \log\delta - c_2 \leq m \leq nc_1 \log\delta + c_2 + 1, \quad c_1 := 1/\log 10, \quad c_2 := c_1 \log 2. \tag{4.3}$$

From the left-hand side inequality of (4.3), we also deduce that

$$n \log\delta < m \log 10 + \log 2. \tag{4.4}$$

Since $\delta \geq (1+\sqrt{5})/2$, we get that

$$n \leq m\frac{\log 10}{\log((1+\sqrt{5})/2)} + \frac{\log 2}{\log((1+\sqrt{5})/2)} < 4.8m + 2.$$

If $m \geq 2$, the last inequality above implies that $n < 6m$. If $m = 1$, then $X_n \leq 9$, so $\delta^n \leq 18$ by (4.2). Since $\delta \geq (1+\sqrt{5})/2$, we get that $n \leq 6$, so the inequality $n \leq 6m$ holds also when $m = 1$. We record this as

$$n \leq 6m. \tag{4.5}$$

Next, using (1.3), we get

$$\delta^n + \eta^n = a\left(\frac{10^m - 1}{9}\right).$$

Put $b := a/9$. We have

$$\delta^n b^{-1} 10^{-m} - 1 = -b^{-1} 10^{-m} \eta^n - 10^{-m}.$$

Thus,

$$\begin{aligned} \left|\delta^n b^{-1} 10^{-m} - 1\right| &\le \frac{1}{b 10^m \delta^n} + \frac{1}{10^m} = \frac{1}{10^m}\left(1 + \frac{9}{a\delta^n}\right) \\ &< \frac{6}{10^m}, \end{aligned}$$

using that $a \ge 1$, $n \ge 1$ and $\delta \ge (1+\sqrt{5})/2$. Thus,

$$\left|\delta^n b^{-1} 10^{-m} - 1\right| < \frac{6}{10^m}. \tag{4.6}$$

We now assume that $m \ge 2$ and search for an upper bound on it. Since $m \ge 2$, it follows that the right-hand side in (4.6) above is $< 1/2$. Put

$$\Lambda := n \log \delta - \log b - m \log 10.$$

Since $|e^\Lambda - 1| < 1/2$, it follows that

$$|\Lambda| < 2|e^\Lambda - 1| < \frac{12}{10^m}.$$

Let us return to (4.6) and put

$$\Gamma := e^\Lambda - 1 = \delta^n b^{-1} 10^{-m} - 1.$$

Note that Γ is nonzero. Indeed, if it were zero, then $\delta^n = b 10^m$. Hence, $\delta^n \in \mathbb{Q}$. Since δ is an algebraic integer and $n \ge 1$, it follows that $\delta^n \in \mathbb{Z}$. Since δ is a unit, we get that $\delta^n = 1$, so $n = 0$, which is a contradiction. Thus, $\Gamma \ne 0$. We apply Matveev's theorem. If $a \ne 9$ (so, $b \ne 1$), we then take

$$l = 3,\ \eta_1 = \delta,\ \eta_2 = b,\ \eta_3 = 10,\ d_1 = n,\ d_2 = -1,\ d_3 = -m,\ D = \max\{n, m\}.$$

Clearly, $\mathbb{L} = \mathbb{Q}[\sqrt{d}]$ contains all the numbers η_1, η_2, η_3 and has degree $d_{\mathbb{L}} = 2$. We have

$$h(\eta_1) = (1/2)\log \delta, \quad h(\eta_2) \le \log 9 \quad \text{and} \quad h(\eta_3) = \log 10.$$

Thus, we can take

$$A_1 = \log \delta, \quad A_2 = 2\log 9 \quad \text{and} \quad A_3 = 2\log 10.$$

Now, Theorem 2.1 tells us that

$$\log|\Gamma| > -1.4 \times 30^6 \times 3^{4.5} \times 2^2 (1 + \log 2)(1 + \log D)(\log \delta)(2\log 9)(2\log 10).$$

Comparing the above inequality with (4.6), we get

$$m\log 10-\log 6<1.4\times 30^6\times 3^{4.5}\times 2^4(1+\log 2)(1+\log D)(\log\delta)(\log 9)(\log 10).$$

Thus,

$$m<1.4\times 30^6\times 3^{4.5}\times 2^4\times(\log 9)(1+\log 2)\times(\log\delta)\cdot(1+\log D)$$

or

$$m<8.6\cdot 10^{12}(1+\log D)\log\delta.$$

Since $D\le 6m$ (see (4.5)), we get

$$m<8.6\cdot 10^{12}(1+\log(6m))\log\delta. \tag{4.7}$$

This was when $b\neq 1$. In case $b=1$, we take $l=2$ and apply the same inequality (except that now $\eta_2:=1$ is no longer present) getting a better result. Finally, this was under the assumption that $m\ge 2$ but if $m=1$ then inequality (4.7) also holds. Let us record what we have proved so far.

Lemma 4.1. *Denoting by* $\delta:=x_1+\sqrt{d}y_1$, *all positive integer solutions* (m,n) *of equation* (1.2) *satisfy*

$$m<8.6\cdot 10^{12}(1+\log(6m))\log\delta.$$

All this is for the equation $X_n=a(10^m-1)/9$. Now we assume that

$$X_{n_1}=a_1\left(\frac{10^{m_1}-1}{9}\right)\quad\text{and}\quad X_{n_2}=a_2\left(\frac{10^{m_2}-1}{9}\right).$$

where $a_1,\ a_2\in\{1,\dots,9\}$.

To fix ideas, we assume that $n_1<n_2$, so $m_1\le m_2$. We put as before $b_i:=a_i/9$ for $i=1,2$. From the above analysis, assuming that $m_1\ge 2$, we have that

$$|n_i\log\delta-\log b_i-m_i\log 10|<\frac{12}{10^{m_i}}\quad\text{holds for}\quad i\in\{1,2\}. \tag{4.8}$$

The argument proceeds in two steps according to whether $b_1b_2<1$ or $b_1b_2=1$.

Suppose now that $b_1b_2<1$**.**

We multiply the equation (4.8) for $i=1$ with n_2 and the one for $i=2$ with n_1, subtract them and apply the absolute value inequality to get

$$\begin{aligned}
&|n_2\log b_1-n_1\log b_2+(n_2m_1-n_1m_2)\log 10| \qquad (4.9)\\
&=|n_1(n_2\log\delta-\log b_2-m_2\log 10)-n_2(n_1\log\delta-\log b_1-m_1\log 10)|\\
&\le n_1|n_2\log\delta-\log b_1-m_2\log 10|+n_2|n_1\log\delta-\log b_1-m_1\log 10|\\
&\le\frac{12n_1}{10^{m_2}}+\frac{12n_2}{10^{m_1}}\le\frac{24n_2}{10^{m_1}}.
\end{aligned}$$

If the right-hand side above is at least 1/2, we then get

$$10^{m_1} \leq 48n_2 < 300m_2,$$

giving

$$m_1 < c_1 \log(300m_2). \tag{4.10}$$

Assume now that the right-hand side in (4.9) is smaller than 1/2. Putting,

$$\Lambda_0 := n_2 \log b_1 - n_1 \log b_2 + (n_2m_1 - n_1m_2) \log 10,$$

we get $|\Lambda_0| < 1/2$. Putting

$$\Gamma_0 := b_1^{n_2} b_2^{-n_1} 10^{n_2m_1 - n_1m_2} - 1,$$

we get that

$$|\Gamma_0| = |e^{\Lambda_0} - 1| < 2|\Lambda_0| < \frac{48n_2}{10^{m_1}}, \tag{4.11}$$

where the middle inequality above follows from the fact that $|\Lambda_0| < 1/2$. We apply Matveev's theorem to estimate a lower bound on Γ_0. But first, let us see that it is nonzero. Assuming $\Gamma_0 = 0$, we get

$$b_1^{n_2} b_2^{-n_1} = 10^{n_2m_1 - n_1m_2}. \tag{4.12}$$

Assume first that $n_2m_1 - n_1m_2 = 0$. Then $b_1^{n_2} = b_2^{n_1}$. Thus, b_1 and b_2 are multiplicatively independent and they belong to the set

$$\left\{\frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1\right\}.$$

They are not both 1 and n_1 and n_2 are both positive. So, the only possibilities are that $b_1 = b_2$, or

$$\{b_1, b_2\} = \left\{\frac{1}{9}, \frac{1}{3}\right\}, \left\{\frac{2}{3}, \frac{4}{9}\right\}. \tag{4.13}$$

If $b_1 = b_2$, then $b_1^{n_1} = b_2^{n_2}$ implies $n_1 = n_2$, which together with $n_2m_1 = n_1m_2$ leads to $m_1 = m_2$. Thus, $(n_1, m_1) = (n_2, m_2)$ and $a_1 = a_2$ (because $b_1 = b_2$), and this is not convenient for us. If $\{b_1, b_2\}$ is one of the two sets from (4.13), then one of $b_1,\ b_2$ is the square of the other one. Thus, since $b_1^{n_1} = b_2^{n_2}$ and $n_2 > n_1$, we get $n_2 = 2n_1$. Since also $n_2m_1 = n_1m_2$, we have $m_2 = 2m_1$. Hence, also $b_2 = b_1^2$ and $b_1 \in \{1/3, 2/3\}$. So, we get the pair of equations

$$X_{n_1} = b_1 10^{m_1} - b_1 \quad \text{and} \quad X_{2n_1} = b_1^2 10^{2m_1} - b_1^2.$$

Since in fact

$$X_{2n} = \delta^{2n} + \eta^{2n} = (\delta^n + \eta^n)^2 - 2(\delta\eta)^n = X_n^2 \pm 2,$$

we get that

$$b_1^2 10^{2m_1} - b_1^2 = X_{2n_1} = X_{n_1}^2 \pm 2 = (b_1 10^{m_1} - b_1)^2 \pm 2 = b_1^2 10^{2m_1} - 2b_1^2 10^{m_1} + b_1^2 \pm 2,$$

which leads to

$$2b_1^2 10^{m_1} = 2b_1^2 \pm 2,$$

so

$$10^{m_1} = 1 \pm b_1^{-2}.$$

The last equation above is impossible for $m_1 \geq 2$. For $m_1 = 1$ we get $10 = 1 \pm b_1^{-2}$, which gives $b_1 = 1/3$. Hence,

$$X_{n_1} = \frac{10-1}{3} = 3, \quad \text{and} \quad X_{2n_1} = \frac{10^2-1}{9} = 11.$$

Since $X_{2n_1} = X_{n_1}^2 \pm 2$, it follows that the sign is $+$, so $X_{n_1}^2 - dY_{n_1}^2 = -4$, giving $dY_{n_1}^2 = 13$, so $d = 13$, $Y_1 = 1$, $n_1 = 1$. These solutions are among the ones mentioned in the statement of the main theorem.

This deals with the case when $n_2m_1 - n_1m_2 = 0$. Assume next that $n_2m_1 - n_1m_2 \neq 0$. Then in the right-hand side of (4.12), both primes 2 and 5 are involved at a nonzero exponent. Thus, they should be also involved with nonzero exponents in the left-hand side of (4.12). Thus, one of b_1, b_2 is $5/9$ and the other is in $\{2/9,\ 4/9,\ 2/3,\ 8/9\}$. A minute of reflection shows that in all cases the exponents of 2 and 5 in the left-hand side of (4.12) have opposite signs, whereas in the right they have the same sign, and this is impossible.

Thus, $\Gamma_0 \neq 0$. Hence, we are entitled to apply Matveev's theorem in order to find a lower bound on Γ_0. In case $b_1 \neq 1$ and $b_2 \neq 1$, we take

$$l = 3,\ \eta_1 = b_1,\ \eta_2 = b_2,\ \eta_3 = 10,\ d_1 = n_2,\ d_2 = -n_1,\ d_3 = n_2m_1 - n_1m_2.$$

Clearly, $\mathbb{L} = \mathbb{Q}$ contains all the numbers η_1, η_2, η_3 and has degree $d_{\mathbb{L}} = 1$. Further, $D = \max\{|d_1|, |d_2|, |d_3|\} \leq n_2m_2 \leq 6m_2^2$. We have

$$h(\eta_1) \leq \log 9, \quad h(\eta_2) \leq \log 9 \quad \text{and} \quad h(\eta_3) = \log 10.$$

Thus, we can take

$$A_1 = \log 9, \quad A_2 = \log 9, \quad A_3 = \log 10.$$

Now, Theorem 2.1 tells us that

$$\log|\Gamma_0| > -1.4 \times 30^6 \times 3^{4.5}(1 + \log D)(\log 9)^2(\log 10).$$

Combining this with estimate (4.11) and using the fact that $48n_2 < 300m_2$ (see inequality (4.5)) we get

$$m_1 \log 10 \leq \log 300 + \log m_2 + 1.6 \times 10^{12}(1 + \log(6m_2^2)),$$

giving

$$m_1 < 7 \times 10^{11}(1 + \log(6m_2^2)). \tag{4.14}$$

The right-hand side of inequality (4.14) is larger than the right-hand side of inequality (4.10). So, regardless whether $24n_2/10^{m_1}$ is at least $1/2$ or smaller than $1/2$, estimate (4.14) holds. From equation (4.4), we get

$$\log \delta < (m_1 + 1) \log 10 < 1.7 \times 10^{12}(1 + \log(6m_2^2)),$$

which together with Lemma 4.1 gives

$$m_2 < \left(8.6 \times 10^{12}(1 + \log(6m_2))\right) \left(1.7 \times 10^{12}(1 + \log(6m_2^2))\right),$$

so

$$m_2 < 1.5 \times 10^{25}(1 + \log(6m_2))(1 + \log(6m_2^2)).$$

This gives $m_2 < 1.5 \times 10^{29}$. This was if both b_1 and b_2 are different than 1. If one of them is 1, we simply apply Matveev's theorem with $l = 2$ getting an even better bound for m_2.

Suppose now that $b_1 = b_2 = 1$.

We return to (4.11) getting that $8/9 \le 24n_2/10^{m_1}$, which leads to (4.10), unless $n_1m_2 = n_2m_1$. In this last case, we get that $n_2/m_2 = n_1/m_1$. Thus, writing $n_1/m_1 = r/s$ in reduced terms, we get that $(n_1, m_1) = (\ell_1 r, \ell_1 s)$ and that $(n_2, m_2) = (\ell_2 r, \ell_2 s)$ for some positive integers $\ell_1 < \ell_2$. Hence, we have

$$X_{r\ell_1} = 10^{s\ell_1} - 1, \quad X_{r\ell_2} = 10^{s\ell_2} - 1.$$

The greatest common divisor of the right hand sides above is $10^s - 1 \ge 9$. The greatest common divisor of the left-hand sides above is X_r if $\ell_1\ell_2$ is odd and 1 or 2 otherwise. Thus, $\ell_1\ell_2$ must be odd and

$$X_r = 10^s - 1.$$

Consequently,

$$\delta^r - 10^s = -\eta^r - 1 \quad \text{and} \quad \delta^{\ell_2 r} - 10^{\ell_2 s} = -\eta^{\ell_2 r} - 1.$$

From the two equations above we get

$$\delta^{(\ell_2-1)r} + \delta^{(\ell_2-2)}10^s + \cdots + 10^{(\ell_2-1)s} = \frac{-\eta^{\ell_2 r} - 1}{-\eta^r - 1}.$$

The last relation above is impossible since its left-hand side is > 10 and its right hand side is

$$\le \frac{2}{1 - \frac{2}{1+\sqrt{5}}} < 10,$$

a contradiction.

In conclusion, (4.10) holds, which is stronger than (4.14), and the above arguments imply that $m_2 < 1.5 \times 10^{29}$. Hence, we have the following result.

Lemma 4.2. *The inequality*

$$m_2 < 1.5 \times 10^{29}$$

holds.

Now one needs to apply LLL to the bound

$$|\Lambda_0| < \frac{24n_2}{10^{m_1}} < \frac{24 \times 6 \times 1.5 \times 10^{29}}{10^{m_1}} < \frac{1}{10^{m_1-32}}$$

to get a reasonably small bound on m_1.

- First, we will consider the case $b_1 = b_2 := b$; i.e., $a_1 = a_2 := a$ or

$$\{b_1, b_2\} \in \left\{\frac{1}{9}, \frac{1}{3}\right\}, \left\{\frac{2}{3}, \frac{4}{9}\right\}.$$

In

$$\Lambda_0 := n_2 \log b_1 - n_1 \log b_2 + (n_2 m_1 - n_1 m_2) \log 10, \tag{4.15}$$

we set $X := n_1 - n_2$ or $X := 2n_2 - n_1$, and $Y := n_2 m_1 - n_1 m_2$ and divide both sides by $Y \log b$ (with $b = b_1 = b_2 \in \{1/9, 2/9, 3/9, 4/9, 5/9, 6/9, 7/9, 8/9\}$) to get

$$\left| \frac{\log 10}{\log b} - \frac{X}{Y} \right| < \frac{1}{Y(\log(1/b))10^{m_1-32}}. \tag{4.16}$$

We assume that m_1 is so large that the right-hand side in (4.16) is smaller than $1/(2Y^2)$. This certainly holds if

$$10^{m_1-32} > 2(\log(1/b))^{-1} Y. \tag{4.17}$$

Since $|Y| < 1.5 \times 10^{59}$, it follows that the last inequality (4.17) holds provided that $m_1 \geq 92$ in all cases, which we now assume. In this case, X/Y is a convergent of the continued fraction of $\eta := \log 10 / \log b$ and $X < 1.5 \times 10^{59}$. Writing

$$a = 1, \quad \eta := [-2, 1, 19, 1, 5, 1, 6, 2, 5, 15, 3, \ldots, 7, 2, 121, 1, \ldots, 2, 569, 1, 2, 27, 7, \ldots]$$

$$a = 2, \quad \eta := [-2, 2, 7, 1, 1, 2, 4, 2, 99, \ldots]1, 292, 1, 6, 1, 3, 3, 2, 2, 5, \ldots, 1, 1, 1, 42, \ldots]$$

$$a = 3, \quad \eta := [-3, 1, 9, 2, 2, 1, 13, 1, 7, 18, \ldots, 2, 10, 3, 1, 1, 1, 1, 1, 6, \ldots, 1, 284, 2, \ldots]$$

$$a = 4, \quad \eta := [-3, 6, 4, 2, 1, 1, 1, 1, 45, 89, 1, 6, 1, 9, 1, 2, 625, \ldots, 2, 2, 1, 1716, 1, 1, \ldots]$$

$$a = 5, \quad \eta := [-4, 12, 9, 1, 1, 1, 1, 1, 2, 1, \ldots, 10, 1, 1, 12, 8860, 4, 13, 1, 1, 5, 3, 9, 1, \ldots]$$

$$a = 6, \quad \eta := [-6, 3, 8, 1, 3, 3, 22, 1, 1, 44, \ldots, 1, 1, 38, 1, 5, 1, 857, 1, 3, 1, 3, 1, 2, 1, \ldots]$$

$$a = 7, \quad \eta := [-10, 1, 5, 6, 118, 2, 8, 1, 2, 1, \ldots, 8, 23, 1, 30, 2, 2, 8, 1, 4, 2, 1, 1, 255, \ldots]$$

$$a = 8, \quad \eta := [-20, 2, 4, 1, 1, 3, 2, 7, 1, 2, 1, 9, 2, 6, \ldots, 1, 2, 1332, 1, 12, 1, 5, 1, 1, 2, \ldots]$$

for the continued fraction of η and p_k/q_k for the kth convergent, we get that $X/Y = p_j/q_j$ for some $j \leq 122$ in all cases. Furthermore, putting $M := \max\{a_j : 0 \leq j \leq 122\}$, we get $M = 8860$ (for $a = 5$). From the known properties of the continued fractions, we then get that

$$\frac{1}{8862Y^2} = \frac{1}{(M+2)Y^2} \leq \left|\eta - \frac{X}{Y}\right| < \frac{1}{Y(\log b)10^{m_1-32}},$$

giving

$$10^{m_1-32} < 8862(\log b)^{-1}Y < 8862(\log b)^{-1}(1.5 \times 10^{59}),$$

leading to $m_1 \leq 96$.

• We now consider the remaining cases. We transform the linear form (4.15) into one of the following forms:

$$\begin{aligned}\Lambda_1 = \quad & (m_1n_2 - m_2n_1 + \delta_1n_1 + \delta_2n_2)\log 2 + (\lambda_1n_1 + \lambda_2n_2)\log 3 \\ & +(m_1n_2 - m_2n_1 + \mu_1n_1 + \mu_2n_2)\log 5,\end{aligned}$$

$$\Lambda_2 := (\lambda_1n_1 + \lambda_2n_2)\log 3 + (\nu_1n_1 + \nu_2n_2)\log 7 + (m_1n_2 - m_2n_1)\log 10,$$

$$\begin{aligned}\Lambda_3 = \quad & (m_1n_2 - m_2n_1 + \delta_1n_1 + \delta_2n_2)\log 2 + (\lambda_1n_1 + \lambda_2n_2)\log 3 \\ & +(m_1n_2 - m_2n_1 + \mu_1n_1 + \mu_2n_2)\log 5 + (\nu_1n_1 + \nu_2n_2)\log 7,\end{aligned}$$

where $|\delta_i| \leq 3$, $|\lambda_i| \leq 2$, $|\mu_i| \leq 1$, $|\nu_i| \leq 1$, for $i = 1, 2$.

Now, we will estimate lower bounds for Λ_i, $i = 1, 2, 3$ via the LLL algorithm (see Proposition 2.3.20 in [6]). One knows that $\Lambda_i \neq 0$, $i = 1, 2, 3$ by what is done above. We set $X_1 = X_3 := 10^{60}$ as upper bounds for $|m_1n_2 - m_2n_1 + \delta_1n_1 + \delta_2n_2|$, $|m_1n_2 - m_2n_1 + \mu_1n_1 + \mu_2n_2|$ and $X_2 = X_4 := 10^{31}$ as upper bounds for $|\lambda_1n_1 + \lambda_2n_2|$, $|\nu_1n_1 + \nu_2n_2|$. We take $C := (3X_1)^3$ for Λ_1, Λ_2 and $C := (4X_1)^4$ for Λ_3. Moreover, we consider the lattice Ω spanned by

$$v_1 := (1, 0, \lfloor C\log 2\rfloor), \;\; v_2 := (0, 1, \lfloor C\log 3\rfloor), \;\; v_3 := (0, 0, \lfloor C\log 5\rfloor),$$

for Λ_1

$$v_1 := (1, 0, \lfloor C\log 3\rfloor), \;\; v_2 := (0, 1, \lfloor C\log 7\rfloor), \;\; v_3 := (0, 0, \lfloor C\log 10\rfloor),$$

for Λ_2

$$\begin{aligned} v_1 &:= (1, 0, 0, \lfloor C\log 2\rfloor), \;\; v_2 := (0, 1, 0, \lfloor C\log 3\rfloor), \\ v_3 &:= (0, 0, 1, \lfloor C\log 5\rfloor), \;\; v_4 := (0, 0, 0, \lfloor C\log 7\rfloor),\end{aligned}$$

for Λ_3. Then, we compute Q, T, c_1, m according to Proposition 2.3.20 in [6] and we obtain:

$$5.5 \cdot 10^{-122} < |\Lambda_1| < \frac{1}{10^{m_1-32}} \quad \Rightarrow \quad m_1 \leq 153;$$

$$3.2 \cdot 10^{-122} < |\Lambda_2| < \frac{1}{10^{m_1-32}} \quad \Rightarrow \quad m_1 \leq 153;$$

$$8.1 \cdot 10^{-183} < |\Lambda_3| < \frac{1}{10^{m_1-32}} \quad \Rightarrow \quad m_1 \le 214.$$

Hence, we have the following numerical result.

Lemma 4.3. *The estimate* $m_1 \le 214$ *holds.*

For $a_1 \in \{1, 2, \dots, 9\}$, $1 \le n_1 \le 1284$, $1 \le m_1 \le 214$, we solve the equations

$$x_{n_1} = P_{n_1}(x_1) = a_1 \left(\frac{10^{m_1} - 1}{9} \right)$$

to see for which values of the triple (n_1, m_1) it has a solution $x_1 = X_1/2$ with positive integer X_1, where

$$x_n = P_n(X/2) = \left(\frac{X + \sqrt{X^2 \pm 4}}{2} \right)^n + \left(\frac{X - \sqrt{X^2 \pm 4}}{2} \right)^n.$$

We used a program written in Maple to see that $n_1 = 1$ in all cases. Here, $P_n(X)$ is one of the two polynomials giving x_n in terms of x_1 for the equation $x^2 - dy^2 = \pm 4$.

From equation (4.8), for $i = 2$ we get

$$\left| n_2 \frac{\log \delta}{\log 10} - \frac{\log b_2}{\log 10} - m_2 \right| < \frac{12}{(\log 10) 10^{m_2}}, \tag{4.18}$$

where $\delta = x_1 + y_1\sqrt{d} = x_1 + \sqrt{x_1^2 \pm 4}$, $x_1 = a_1(10^{m_1} - 1)/9$, and $b_2 = a_2/9$ with $a_1 \ne a_2$. To apply Lemma 3.1 to inequality (4.18), we put

$$\kappa = \frac{\log \delta}{\log 10}, \quad \mu = \frac{\log b_2}{\log 10}, \quad A = \frac{12}{\log 10}, \quad B = 10, \quad \text{and} \quad M = 1.5 \cdot 10^{29}.$$

The program was developed in PARI/GP running with 200 digits, for $1 \le m_1 \le 214$. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. In a few minutes, all the computations were done. In all cases, after the first run we obtained $m_2 \le 35$. We set $M = 35$ and the second run of the reduction method yields $m_2 \le 8$. In conclusion, we have

$$n_1 = 1, \quad 1 \le m_1 \le 8, \quad 1 \le m_2 \le 8, \quad 1 \le n_2 \le 48.$$

Now a verification by hand yields the final result.

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