

Proving skills in geometry of secondary grammar school leavers specialized in mathematics

Ákos Győry^a, Eszter Kónya^b

^aFöldes Ferenc Secondary School
gyoryakos@gmail.com

^bUniversity of Debrecen
eszter.konya@science.unideb.hu

Submitted: September 4, 2019

Accepted: November 19, 2019

Published online: December 17, 2019

Abstract

We examined the evolution of the van Hiele level of some study groups specialized in mathematics from 2015 to 2018, then selected two of these groups and measured the students' proof skills by Zalman Usiskin's proof test. We examined whether students were able to read from the text of the statement the given fact and the fact to be proved, whether they were able to draw a figure and, using the labels, whether they were able to perform a simple proof requiring 2-3 steps.

Keywords: van Hiele levels, reasoning and proving, specialized mathematics education, given fact, fact to be proved, role of figures.

MSC: D74, E54, G44.

1. Introduction

The elementary geometry is one of the most appropriate areas of mathematics for developing students' proving abilities, because it is complex and expressive. Constructing a reasoning chain consisting of 1-2 steps does not require hard abstraction; that is why this area can be studied and developed from grades 6-7.

Teaching geometry curriculum in secondary school includes many challenges for teachers and students alike, as solving geometric problems often goes beyond recalling well-practiced patterns. The first author's own experience shows that the most difficult part of geometry education is the development of students' reasoning and proving skills. He currently works as a secondary school mathematics teacher in Hungary and teaches students specialized in mathematics. The number of their mathematics lessons are more than average, and in addition to the normal requirements, they also acquire special topics. These students, who are particularly interested in mathematics, have to take an entrance exam in this form of education. Although they are talented in mathematics, they still have difficulty in solving problems that require proof. Their educational program will be discussed later.

Since 2005, there has been a two-tier graduation system in Hungary. This means that students can take higher or standard level school leaving examination. One of the main goals of the specialized mathematics education is to provide students with the adequate knowledge to pass an advanced school leaving examination. In this examination there have often appeared such tasks that require construction of a short, simple proof. Instructions of graduation and the framework of mathematics curriculum require from students to be able to produce an exact logical chain by means of their thoughts and acquired knowledge to solve some simple problems and to conceive and write the solution in a clear form. Therefore, it is of high priority to examine and analyze the difficulties that students face in solving tasks which require proof in order to be able to integrate experiences into the teaching process.

The present study is a part of a longitudinal research. In a previous study we followed up the evolution of van Hiele level of students specialized in mathematics from 2015 to 2018 (Győry & Kónya, 2018). Two of the study groups were selected for deeper examination, in which the mean of the students' van Hiele level reached 4, that is, according to the theory they were able to implement a few-step proof. Now we examine what proving ability they actually have, how they can formulate their thoughts in writing, and what typical mistakes they make. This article is about the obtained results which we will use in the future to accomplish a developmental teaching experiment.

2. Theoretical background

“The teaching of mathematical proof appears to be a failure in almost all countries, no matter how this teaching is organized. . . in USA mathematical proofs are taught only to students who take the geometry course. . .” (Balacheff, 2017, p. 1) Because of the importance of the topic, a number of studies has been made on the examination of students' reasoning-proving skills (Stylianides, 2008; G. Hanna & M. de Villiers, 2008; Balacheff, 2017, D. S. Hong & K. M. Choi, 2018). Perhaps the main difficulty is that most students consider the proofs construction only a necessary task required by the teacher. It is a long teaching process until the construction of proof becomes an intrinsic need of students, rather than just meeting

the teacher's expectation.

In our opinion one of the biggest mistakes a mathematics teacher can make is to teach concepts and statements without motivating which establishes them. Accordingly students need to be led to see definitions and statements not only as things prescribed by mathematicians, but to try to look for the root causes of them. To achieve this, the teacher has to educate the students on their own independent and critical thinking, which is essential for them to create a mathematical proof successfully. In connection with this, Lakatos wrote quite sharply in the '70s: "It has not yet been sufficiently realised that present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought." (Lakatos, 1976, p. 152)

Teaching of proofs is also a hard task because it requires, on the one hand, students to have a thorough knowledge of previous learning, additionally to be able to make a corollary based on some facts and to operate with abstract concepts. According to Ambrus (2004), the teaching of proof has three phases: (1) assumption of statements; (2) finding and executing an idea of proof based on previous proofs' strategies and methods; (3) description of the proof.

In Hungarian mathematics education the last two phases are the most emphasized, but in the special mathematics educational form the first phase is also expected. In this paper we deal with the last two phases in more detail. In elementary school, mathematical statements are often considered by students as absolute truth, which later makes it difficult to understand the need for mathematical proof. Only a minority of the students are then concerned with the question "Why?". It aggravates the difficulty of the mathematics teacher that he/she needs use statements without proof in the teaching process, as students' understanding of mathematical concepts and their thinking are often not ready for the proper execution of a proof. The use of these so-called "school axioms" is essential for the proper development of thinking. (Szendrei, 2005) It is an additional problem for students to read clearly from the text of the statement the conditions and the fact to be proved. This is especially problematic when the statement is not given in the form "If ... then ...". (Ambrus, 2004)

Geometry tasks often include figures, but in most cases students have to draw their own figures. In the latter case, the student must correctly represent the concepts used in the task. This can be a pitfall in solving the problem, because a poorly drawn figure can steer the process of thinking in the wrong direction. A frequent mistake is that the student is able to spell out the definition correctly, but is unable to apply it in the solution. (Fishbein, 2012) If a figure is attached to the task, it serves to understand the task in some cases, that is to say it is an integral part of the task, but there are also cases where it serves merely to illustrate the task by reason of better understanding. However, even in the latter case, it is possible that the figure may lead to an unestablished or inaccurate conclusion for students. (Dvora, Dreyfus, 2004) We did not intend to investigate these aspects in the current research, but we will briefly touch on it in relation to one of the problems that suggests a false conclusion.

Balacheff distinguishes three categories of proof that are built on each other (Balacheff, 2017). These are the following:

- explanation: an individual intends to establish for somebody else the validity of a statement;
- proof: an explanation which is accepted by a community at a given time;
- mathematical proof: a proof which is accepted by mathematicians.

In secondary school, generally accepted proof is the second type described by Balacheff.

Stylianides categorized reasoning and proving in a so-called analytic framework in which he describes two types of proofs (Stylianides, 2008):

- generic example: it is a proof that uses a particular case seen as representative of the general case;
- demonstration: it is a proof that does not rely on the “representativeness” of a particular case.

In secondary school we use both methods of proving.

We relied on the van Hiele theory to examine proof skills. According to van Hiele theory, students’ geometric thinking is evolved through sequential and hierarchical levels. Five different levels are distinguished and somebody can only reach the level n if he or she has achieved the criteria of all the levels below level n . In many papers, van Hiele levels are scaled from 0 to 4, but we will scale from 1 to 5. Achieving level 4 is a requirement for proving in secondary school. This is the level of formal deduction. At this level students understand the meaning of deduction. (Usiskin, 1982) According to the theory, at this level students are able to formulate causality, construct simpler proofs, and realise the need of proofs. They are aware of the concept of generalization, and they know and use different methods of proof (constructive, contrapositive, induction etc.). They are able to distinguish between necessary and sufficient condition. They are not yet able to provide a full proof and tend to treat statements requiring proof as fundamental truths. Based on our previous research (Győry & Kónya, 2018) the mean of van Hiele level of the examined students was around 4, so grounded in the theory and Usiskin’s results we assumed that they were able to perform a 2-3 step proof.

3. Research question

In our research, we were curious about the reasoning ability of secondary school students talented in mathematics, how they express themselves in writing, and what typical mistakes they make.

4. Research methodology

4.1. The circumstances of the survey

The students took part in special mathematics program. This is an unusual form of education progresses in ten secondary schools in Hungary at the present time. Following the entrance examination, students study mathematics in small groups (12-20 students per group). A further feature of this program is that students study mathematics on average 6-8 hours per week, usually with two subject teachers per study group. One of the teachers teaches geometry and the other teaches algebra. One of the main aims of this form of education is to teach solving mathematical problems on their own. The students could be said to be motivated, talented, and many of them extremely outstanding. Students in the specialized mathematics program, on the one hand, acquire deeper mathematics skills and, on the other hand, learn certain topics faster than students in normal mathematics training. This form of education thus contains elements of the so-called acceleration and enrichment programs designed specifically for talented students. (Poli, 2018) We mention as an important difference that the Hungarian education system does not provide an opportunity for a specially talented student to learn subject of higher grade.

In a previous study we kept track of the Hiele levels of several study groups from 2015 to 2018. Two of the study groups were selected for deeper analysis. One group took part in a four-grade (Grade 9-12) and the other a six-grade (Grade 7-12) system of education. Hereinafter we will denote the four-class group with N, while the six-class group will be denoted by H. The number of examined students was 27, 14 of them from the group N and 13 from the group H. Initially, the members of the groups were more, but by reason of the longitudinal monitoring we only considered the performance of the students who wrote each test. The van Hiele levels of the two groups were already measured in grade 9, in 2015. The results achieved by the students were averaged. The mean for the group N was 3.80 and the mean for the group H was 4.25. (We scaled the van Hiele levels from 1 to 5.)

The result obtained shows that the students are roughly at level 4 of the hierarchical theory, which is the level of formal deduction. We repeatedly measured the van Hiele level of the same students over the years, and each time we got a mean of 3.5-4 (Győry & Kónya, 2018). Since the mean of van Hiele levels of the two groups was quite similar, the results of the measurements will not be divided into groups, but will be aggregated.

To assess proof skills we took as a basis the proof test can be found in the paper van Hiele Levels and Achievement in Secondary School Geometry by Zalman Usiskin (1982). The proof test from this article was conducted in April 2018, when both groups were in 12th grade in some weeks before the final examination.

4.2. The proof test

Usiskin and his colleagues, who dealt extensively with the van Hiele theory, were curious about how this theory can describe and predict the geometric achievement of secondary school students, including proving, and at what level students are able to describe their proof. They found that at level 4 students are able to independently create simpler proofs, whereas in case of lower van Hiele levels they are not. That is, the van Hiele level is a good predictor. In our work, we do not aim to discover the relationship between the 4th van Hiele level and proof skills. Based on the results of Usiskin, we assumed that our students would be able to perform some simple proofs consisting of one or two steps at this level. After selecting one of the 3 proof tests in Usiskin's paper (Usiskin, 1982, pp. 173–177.), we examined systematic mistakes, proof ideas, how to draw figures and the written communication.

The test consists of 6 exercises, and requires the following prior knowledges from geometry curriculum in Hungary up to grade 10.

- Knowledge of angle pairs.
- Basic properties of triangles. Basic cases of congruence and similarity of triangles.
- Knowledge and use of Pythagorean theorem.
- Knowledge of properties of parallelograms.

It should be noted that the first task of the original test asked students to make solution in the two-column style prevalently used in the US. As this method is not well known in Hungary and is completely unknown to students, we did not ask for a two-column description of the proof in the Hungarian translation in contrast with the original version.¹

The writing conditions were very similar to those of the Usiskin. We differed only in one case from them: our working time was 45 minutes in contrary to the 35 minutes. We did this because, on the one hand, we did not give help steps in our first task, and on the other hand, we feared that students might run out of time, which would have significantly affected the in-depth analysis of the tasks.

5. Discussion

We will discuss in detail three of the six tasks: Task 2, Task 4 and Task 6. Why did we just choose these tasks?

1. The second of the six tasks is the only one that does not have to be proved, and no figure is attached to it. Furthermore, the formulation of this statement differs from all other tasks' one. The condition and the corollary are not given

¹The English version of the test we have written can be found in the Appendix.

in two separate sentences, but in a single “If ... then ...” type sentence. In case of all other tasks have been attached figures and the statement must be proved.

2. Task 4 is the only one where the attached figure is only for illustrative purposes and does not play a role in understanding and may even be omitted.
3. The structure of tasks 1, 3, 5 and 6 is completely the same. In these statements, the condition and the corollary are included in separate sentences, and there is a figure in all of them that is necessary to understand the text of the statement. We chose Task 6 of these because the figure may even suggest false information.

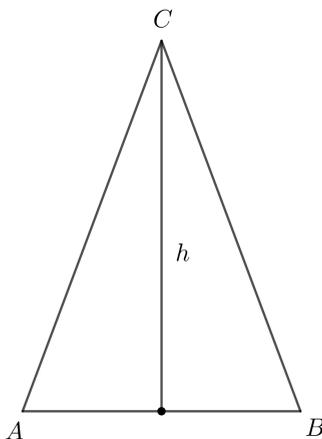
Task 2

Statement: If an altitude is drawn to the base of an isosceles triangle, then it bisects the vertex angle.

- a. Draw a figure and label it.
- b. Write, in terms of your figure, what is given and what is to be proved in this statement.

The solution

a.



- b. Given: $CA = CB$; $CT \perp AB$.
To prove: $\angle ACT = \angle BCT$.

What were we able to examine by means of this task?

- (1) Is the student able to draw a figure for the task?
- (2) Is the student able to read from the text what is given and what is to be proved, i.e. what fact do we infer from which fact (height \implies bisector)?
- (3) Is the student able to use symbols properly?

We can see that Task 2 was given in the form “If ... then ...”, so the condition and the corollary were detached well. This made the students’ job a little easier.

Results

About a quarter of the students could only answer correctly (Table 1).

	Good solution	Wrong solution
Number of students	7	20

Table 1: Results (Task 2)

Let’s look at the statistics for the parts of the task one after the other.

- (1) The students did not have a problem with drawing the figures and introducing the labels.
- (2) Reading the given fact (condition) from the text (Table 2).

	Good solution	Wrong solution
Number of students	10	17

Table 2: What is given? (Task 2)

Only about one-third of the students were able to solve this part. We found a typical mistake: more than two-thirds of the students who gave the wrong solution could not separate the condition, the corollary and the statement itself, and confused them (Table 3).

	To be written what to be proved	To be written the whole statement	Other
Number of students	6	6	5

Table 3: Wrong solutions (Task 2)

(3) Reading the fact must be proved from the text (Table 4).

	Good solution	Wrong solution
Number of students	18	9

Table 4: What to prove? (Task 2)

Already two thirds of the students have succeeded in this section.

As a typical mistake we could note that most students confuse the concept of the given fact, the fact to be proved and the statement, as to be shown in *Ádám's* solution below.

Given: "If the triangle ABC is isosceles, then m_c bisects the side " c " (splits it into 2 pieces of " x " parts); The legs are of equal length."²

To be proved: "The above assumption is true only if the angle at C is bisected by altitude (2 pieces of α angle are created)."

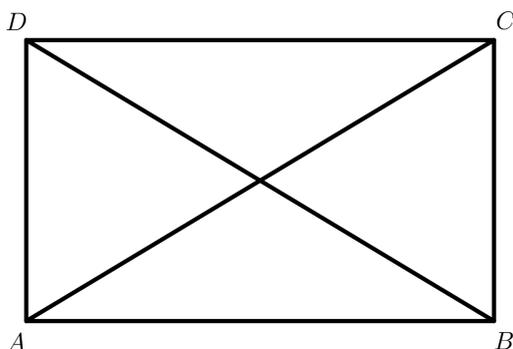
Conclusion

1. There is a need to clarify the distinction between the condition and what is to be proved from each other and from the statement itself.
2. Emphasis should be placed on transforming a statement into an "If ... then ..." sentence throughout the teaching process.

Task 4

We know about the following figure: $ABCD$ is a rectangle.

Prove that the diagonals are congruent.



Use the labels to describe what is given.

Use the labels to describe what is to be proved.

²In Hungary m_c denotes the altitude to the side c .

A possible solution:

Given: $AB = CD$ and $\angle DAB = 90^\circ$.

To be proved: $AC = BD$.

Proof:

The DAB and CBA triangles are congruent because two pairs of sides of these triangles and the included angles are equal ($AB = BA$, $AD = BC$, $\angle DAB = \angle CBA$).

So the third sides of the two triangles are equal in length: $DB = AC$, which we just wanted to prove.

What were we able to examine by means of this task?

1. Is the student able to read from the text of the statement what is given and what is to be proved, i.e. what fact do we infer from which fact and
2. is he/she able to describe these with his/her own notions?
3. Is the student able to create a simple reasoning?

Results

1. Reading the given fact from the text (Table 5).

	Good solution	Incompleted solution
Number of students	17	10

Table 5: What is given? (Task 4)

Students who did not solve this part flawlessly, without exception, forgot about the angles (that is, each angle of the quadrilateral ABCD is a right angle), but they demanded the equality of the opposite sides. Probably this little mistake was made due to the figure attached to the problem, because of it they considered the equality of angles as obvious fact.

It is important to note that this task is very different in its formulation from Task 2, as in this case the condition and the corollary are given in separate sentences. Based on our findings, it can be claimed that if the task is set in this style, the students are able to read the condition from the text of the task.

2. Reading the fact must be proved from the text. (Table 6)

	Good solution	Wrong solution
Number of students	24	3

Table 6: What to prove? (Task 4)

Overall, this part did well for the students probably due to the style of the formulation.

3. Execution of proof (Table 7).

	Good solution with Pythagorean theorem	Good solution with elementary geometry way	Nearly good solution	Wrong solution
Number of students	17	4	2	4

Table 7: The way of proving (Task 4)

We can say that the students passed this subtask successfully, because 21 out of 27 students reasoned correctly (Table 7). It is interesting to note that only 4 of the 21 students chose purely geometric proofs, while the other 17 reasoned with Pythagorean theorem, suggesting an algebraic approach. We conceive it is due to the fact that students associate the Pythagoras theorem with a right triangle immediately because of its central role in secondary school calculations. Kitti's solution demonstrates this way of proving:

$$"BD^2 = BC^2 + DC^2, AC^2 = AB^2 + BC^2.$$

$$\text{Since } DC = AB \implies AC^2 = DC^2 + BC^2 \implies BD^2 = AC^2.$$

Since BD and AC are also positive, hence $BD = AC$."

We considered two solutions as "nearly good", because the students stopped, though there was only one step left to complete the proof.

Of the 3 students who were not able to read the corollary from the statement (Table 6), 2 students were not able to successfully complete the proof either. The additional student proved also well, but he described his proof to the part to be proved. So he understood what the task demanded from him as proof, but he was probably careless or the formulation was perhaps unusual to him. Of the 4 students, who could not prove the statement (Table 7), 2 students were able to separate the fact to be proved from the statement well. We have thus got that it is essential for the correct implementation of the proof that the individual can sharply separate the part to be proved from the statement.

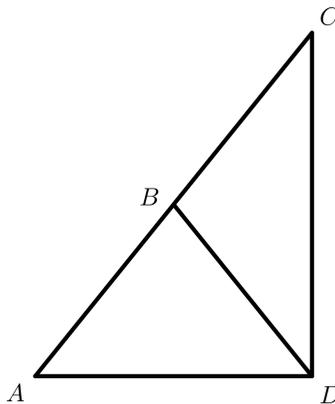
Of the students who were unable to prove the statement (4 in total, Table 7), two students abandoned the solution after one step, one student initially started from the fact to be proved, and one student tried to write trigonometric relations, no avail.

Conclusion

- If we give a statement for the student in “separate” style, that is, the given fact and the fact to be proven are expressed in separate sentences, they will be able to read out the condition and the corollary from the statement. This means that the formulation “If ... then ...” is also worth splitting at first into two sentences.
- The students use his labels well in the formulation of the condition and the corollary.
- A simple reasoning of no more than 2-3 steps was not a problem for the students. This ascertaining is consistent with the result of Usiskin, who obtained similar results for students at van Hiele level 4.

Task 6

We know about the following figure: B is the midpoint of the AC section. $AB = BD$. Prove that the angle CDA is a right angle.



A possible solution:

According to the conditions: $AB = BD = BC$.

This means that the triangles ABD and DCB are isosceles.

Since in an isosceles triangle the angles opposite the legs are equal, accordingly $\angle BAD = \angle BDA$ and $\angle BDC = \angle BCD$.

From this we can conclude that

$$\begin{aligned} \angle CDA &= \angle BDA + \angle BDC = \frac{2 \cdot \angle BDA + 2 \cdot \angle BDC}{2} = \\ &= \frac{\angle BAD + \angle BDA + \angle BDC + \angle BCD}{2} = \frac{\angle CAD + \angle ADC + \angle ACD}{2} = \end{aligned}$$

$$= \frac{180^\circ}{2} = 90^\circ.$$

The structure of the text is similar to that of Task 4, that is, the given fact and the fact to be proved are clearly separated in two sentences.

What were we able to examine by means of this task?

- Is the student able to interpret the figure?
- Is the student able to construct a proof which consists more than 3 simple steps?
- How do the students reason? What do they refer to?
- What level of detail do the students reason? How do they use the language of mathematics?

Results

How did they manage to create the proof? (Table 8)

	Good solution	Wrong solution
Number of students	13	14

Table 8: Creating the proof (Task 6)

Overall, about half of the students succeeded in making the proof. In order to draw conclusions, we analyzed the good and the wrong solutions.

Good solutions (13 students)

The proof was correctly described by 10 students. It is important to note that none of these students referred to the fact that “there are equal angles opposite congruent sides”. This is the point, where the problem already mentioned in the introduction arises, which is one of the dilemmas of mathematics education in secondary school. Namely, how to distinguish between statements requiring proof and so-called basic truths to be treated as fact.

One of the 13 students described the proof in a fairly short way: he added to the given figure a semicircle with centre B and with radius BC (which lies on the points A, D , and C due to the condition), and then wrote that “the angle CDA is right angle by reason of Thales’ theorem.”

There were 2 students who worked on the figure. In such cases the train of thought of a proof is difficult to follow, not to mention that in the case of a false reasoning it is difficult to determine where a failure was committed.

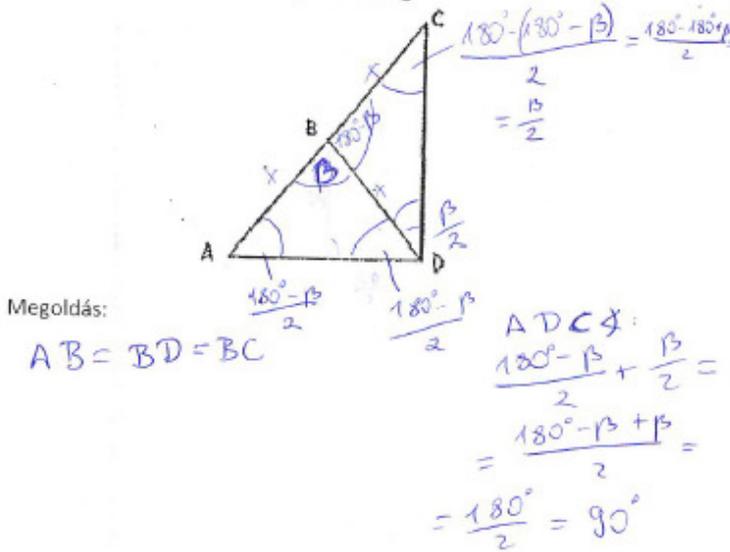
Figure 1 shows István’s solution, where the chain of reasoning is relatively easy to follow.

6. Az alábbi ábráról a következőket tudjuk:

B az AC szakasz felezőpontja.

$AB = BD$.

Bizonyítsd be, hogy a CDA szög derékszög!



D

Figure 1: Solution on the figure

The student's steps could have been as follows (repeatedly using the sum of the interior angles of a triangle is 180°):

1. He described the given fact: $AB = BD = BC$, then denoted the length of these segments in the figure by x .
2. He denoted the measure of the angle ABD by β .
3. Since the triangle ADB is isosceles, its angles at A and D are equal: $\frac{180^\circ - \beta}{2}$.
4. Because the angles at B are adjacent angles, hence $\angle DBC = 180^\circ - \beta$.
5. The triangle DCB is isosceles, so its angles at D and C are equal: $\frac{180^\circ - (180^\circ - \beta)}{2} = \frac{\beta}{2}$.
6. Since he had already determined the two angles at D , he wrote that $\angle ADC = \frac{180^\circ - \beta}{2} + \frac{\beta}{2} = \frac{180^\circ - \beta + \beta}{2} = \frac{180^\circ}{2} = 90^\circ$.

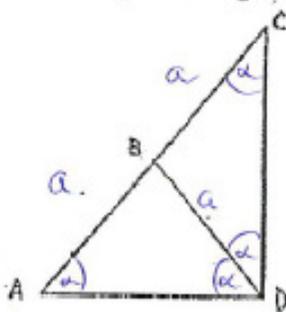
Although he did not write it, he proved the claim.

Wrong solutions (14 students)

Three of the 14 students did not begin the solution, left the sheet blank. There was one student who began the solution in a good way, but after a certain point, he stopped reasoning. In the case of two students, the train of thought could not be followed and the reasoning described was chaotic. However, for the remaining eight students, we observed a typical, repeating mistake. These students drew a false inference grounded in the figure, namely equality of certain angles ($\angle BAD = \angle BDA = \angle BDC = \angle BCD$). This is illustrated by Éva's solution in Figure 2.

6. Az alábbi ábráról a következőket tudjuk:
 B az AC szakasz felezőpontja.
 $AB = BD$.

Bizonyítsd be, hogy a $\angle CDA$ szög derékszög!



Megoldás:

ABD_{Δ} és DBC_{Δ} egyenlőszögű

$$180^\circ = 4\alpha$$

$$90^\circ = 2\alpha$$

$$\angle ADC = 2\alpha = 90^\circ$$

Figure 2: A typical wrong solution³

³ ABD and DBC triangles are isosceles.

Conclusions

- During the teaching process, the teacher must make sure that the students do not just write the proof in the figure, but use the appropriate notations to describe their reasoning in adequate detail.
- It is necessary to clarify with the students, what facts require reasoning and what facts do not in a proof. Anyway this is a very problematic element of the teaching process.
- It should be emphasized that information suggested by the figure should not be accepted without doubt. This is well illustrated by the typical mistake of solutions of Task 6 (Figure 2).

6. Summary

The research described in this article is a part of a comprehensive series of experiments. As a first phase, we examined the evolution of the Hiele level of several study groups specialized in mathematics from 2015 to 2018. Two of the study groups were selected and we measured the students' proof skills, in 2018, by Usiskin's proof test.

During teaching proofs, strong emphasis must be placed on the ability of students to pick out the given fact and the fact to be proven from the statement. To do this, it seems to be to follow the sequence below.

- (1) The teacher should give the statement to the students in such a form that the given fact and the fact to be proven are in two separate sentences. For example: "Let's consider a rhombus. Prove that its diagonals are perpendicular to each other."
- (2) The teacher should give the statements in the "If... then..." structure to the students and should clarify where the given fact and where the fact to be proved appear. It is worthwhile to reword the statements, which are given in such a way like we saw in (1), with the students themselves. "If a quadrilateral is a rhombus, then its diagonals are perpendicular to each other."

Of course, statements made with the help of quantifiers and other means also occur in lessons. ("Every rhombus' diagonals are perpendicular to each other." "The diagonals of a rhombus are perpendicular to each other." Etc.). The study of these cases will be a part of a later study.

Students in specialized mathematics education, reaching van Hiele level 4, are able to complete a simple proof, which requires at most 2 or 3 steps, but many of them have problems with multi-step proofs, so these need much more practice.

The teacher should pay special attention to the figures, too, in teaching of proofs, as well. It should be made clear to the students that they ought to make a clear figure with appropriate labels. The teacher should emphasize that students ought not to accept the facts suggested by the figure unless they have actually

reasoned the validity of these facts. The important thing is that the teacher should make sure that the students do not write the proof only in the figure, but that they formulate their thoughts based on the labels in the figure and write them down using mathematical language.

One of the most difficult parts of teaching of proofs is that the teacher clarifies with students what fact needs to be reasoned and what fact does not in secondary school. This is, of course, a long process, which demands a huge problem-solving routine that can only be achieved through lots of practice.

Based on the results, we also designed a series of developmental experiments in a study group, specialized in mathematics, at grade 9 (this group takes part in 4-form education, i.e. this is its first school year in the secondary school).

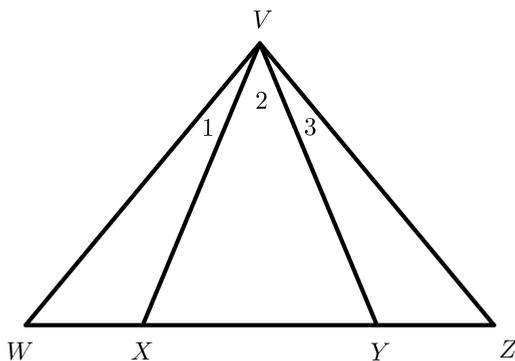
7. Appendix

Tasks of our version of the proof test.

1. From the figure below we know the following:

The angle at W is equal to the angle at Z and the length of the segment WX is equal to the length of the segment YZ .

Prove that the angle denoted by 1 is equal to the angle denoted by 3!



Solution:

2. *Statement:* If an altitude is drawn to the base of an isosceles triangle, then it bisects the vertex angle.
 - a. Draw a figure and label it.
 - b. Write, in terms of your figure, what is given and what is to be proved in this statement.

Figure:

Given:

To prove:

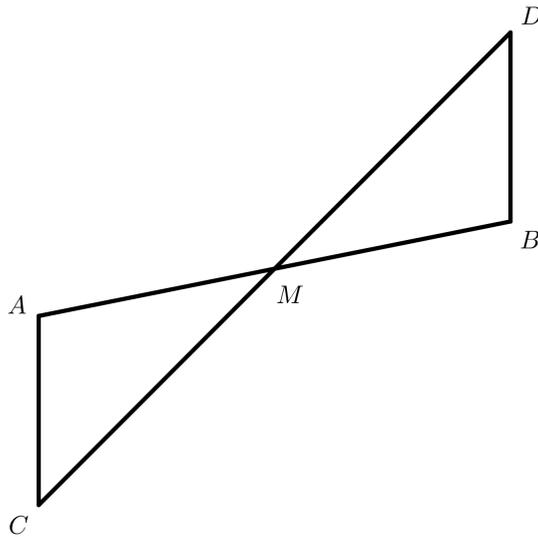
Do not prove the statement.

3. From the figure below we know the following:

M is the midpoint of the segment AB .

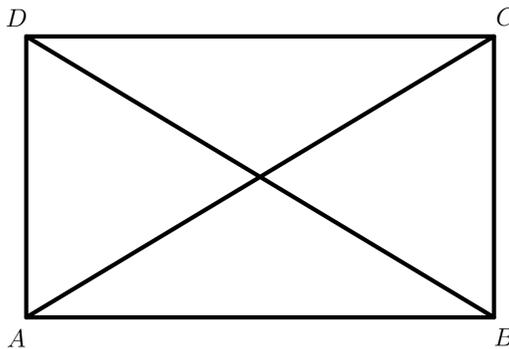
M is the midpoint of the segment CD .

Prove that the triangles ACM and BDM are similar!



Solution:

4. We know about the following figure: $ABCD$ is a rectangle
Prove that the diagonals are congruent.



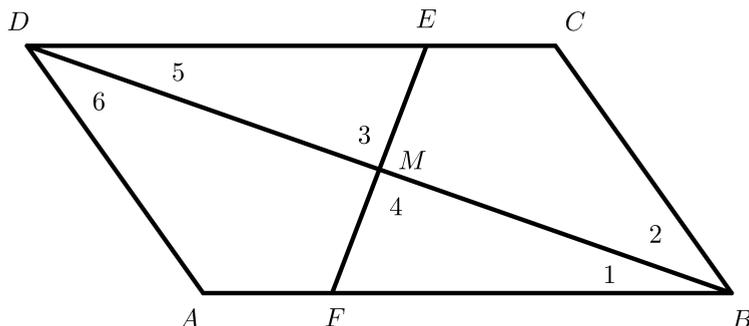
Solution:

Use the labels to describe what is given.

Use the labels to describe what is to be proved.

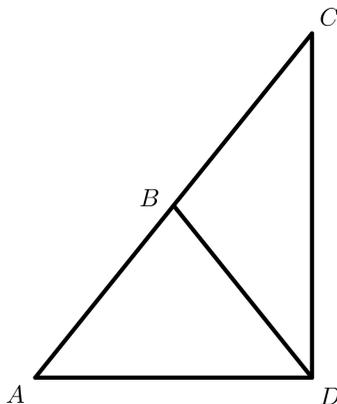
Proof:

5. From the following figure we know: $AB = DC$, $AD = BC$. M is the midpoint of the segment DB . M lies on the segment EF . Prove that $FM = ME$!



Solution:

6. We know about the following figure: B is the midpoint of the AC section. $AB = BD$. Prove that the angle CDA is a right angle.



Solution:

References

- [1] A. AMBRUS: *Introduction to Didactics of Mathematics (In Hungarian: Bevezetés a matematikadidaktikába)*, Budapest: ELTE Eötvös Kiadó, 1995.

- [2] N. BALACHEFF: *A Study of Students' Proving Processes At The Junior High School Level*, 66th NCTM Annual Meeting, 2017.
- [3] T. DVORA, T. DREYFUS: *Unjustified Assumptions Based On Diagrams In Geometry*, Proceedings of the 28th Conference of the International, Group for the Psychology of Mathematics Education 2 (2004), pp. 311–318.
- [4] E. FISCHBEIN: *The Theory of Figural Concepts*, Educational Studies in Mathematics 24.2 (1993), pp. 139–162.
- [5] E. GYŐRY Á. HERENDINÉ KÓNYA: *Development of high school students' geometric thinking with particular emphasis on mathematically talented students*, Teaching Mathematics and Computer Science 16.1 (2018), pp. 93–110.
- [6] G. HANNA, M. DE VILLIERS: *ICMI Study 19: Proof and Proving In Mathematics Education*, ZDM Mathematics Education 40.2 (2008), pp. 329–336.
- [7] E. HERENDINÉ KÓNYA: *The characteristics of the geometric thinking of teacher trainees (In Hungarian: A tanítójelöltek geometriai gondolkodásának jellegzetességei)*, Iskolakultúra 12 (2003), pp. 51–61.
- [8] D. S. HONG, K. M. CHOI: *Reasoning and Proving Opportunities in Textbooks: A Comparative Analysis Reasoning and proving opportunities in textbooks: A comparative analysis*, International Journal of Research in Education and Science (IJRES) 4.1 (2018), pp. 82–97.
- [9] T. KETTLER, M. CURLISS: *Mathematical Acceleration In A Mixed-Ability Classroom*, Gifted Child Today 26.1 (Winter 2003).
- [10] I. LAKATOS: *Proofs and refutations. The logic of mathematical discovery*, Cambridge: Cambridge University Press, 1976.
- [11] M. D. POLI: *Successful Programs and Strategies for Secondary Students Who Are Gifted and in Mathematics Classes: A Qualitative Study*, Doctoral dissertation, Indiana University of Pennsylvania, USA, 2018.
- [12] G. J. STYLIANIDES: *An Analytic Framework of Reasonin-and-Proving*, For the Learning of Mathematics, FLM Publishing Association 28.1 (2008), pp. 9–16.
- [13] L. SURÁNYI: *"I have experienced few such a motivating atmosphere" The 50th anniversary of special mathematics education (In Hungarian: "Kevés ilyen inspiráló légkört tapasztaltam" Ötvenéves a speciális matematika tagozat.)* Természet Világa 143.6 (2012).
- [14] Z. USISKIN: *Van Hiele Levels and Achievement in Secondary School Geometry*, University of Chicago (1982).