An iterative method based on fractional derivatives for solving nonlinear equations

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1 Introduction

The theory of fractional order derivatives are almost as old as the integer-order [5]. There are many applications, for example in physics [1], [2], [6], finance [8], [9] or biology [3]. Our aim is to prove theoretical mathematical statements.

In this work our goal is to find a solution numerically for the equation $A(u) = f$. If we assume that u is time-dependent, then one can do this by finding a stationary solution of the equation $\partial_t u(t) = -(A(u(t)) - f)$. The numerical solution of this problem can be highly inaccurate. To avoid this we propose to replace the time derivative with a fractional one. Since the fractional order time derivative is a non-local operator, we expect that this stabilizes the time integration in the numerical solutions. Since the fractional order derivative here is defined as a limit of linear combination of past values, the time discretization will be simple. We also tested our method numerically in a fluid dynamical problem [10].

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2 Mathematical preliminaries

The following theorem is well known, see [11].

Theorem 1. *Let H real Hilbert-space, $A : H \to H$ nonlinear operator, which satisfies the conditions below with some positive constants $M \geq m$:*

1. $\langle A(u) - A(v), u - v\rangle \geq m\|u - v\|^2$,
2. $\|A(u) - A(v)\| \leq M\|u - v\|$.

Then for any $f, u_0 \in H$ there exist a unique solution u^* of the equation $A(u) = f$. If $t \in \mathbb{R}^+$ is small enough the following iteration converges to u^*.

$$u_{n+1} = u_n - t\big[A(u_n) - f\big]. \tag{1}$$

There exist many different definitions of the fractional derivative [4], [7] we will use here the one below which is based on finite differences.

Definition 1. For the exponent $\beta \in (0,1)$ the fractional order derivative for a given function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\frac{\partial^\beta f(t)}{\partial t^\beta} := \lim_{N\to\infty} \Big\{ \sum_{k=0}^{N} \binom{\beta}{k} (-1)^k \frac{f(t-kh)}{h^\beta} \Big\},$$

provided that the limit exists.

3 Results

Shortly, our objective is to find a solution for the equation $A(u) = f$ for a given nonlinear operator A, and for a given function f. The solution u is also time-dependent, our goal is to find a stationary solution for

$$-(A(u(t)) - f) = \partial_t u(t). \tag{2}$$

The method in Theorem 1 is one approach to this. Our idea was that to replace the time derivative in (2) with $\frac{\partial^\beta}{\partial t^\beta}$ for some $\beta \in (0,1)$, according to Definition 1, and discretise the equation in time by a natural way.

We need an additional statement before we prove.

Lemma 1. *(Pachpatte) Let $(\alpha_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$, $(h_n)_{n\in\mathbb{N}}$ nonnegative real sequences with the conditions below:*

$$\alpha_n \leq f_n + g_n \sum_{s=0}^{n-1} h_s \alpha_s. \tag{3}$$

Then the following inequality holds

$$\alpha_n \leq f_n + g_n \sum_{s=0}^{n-1} h_s f_s \prod_{\tau=s+1}^{n-1} \left(h_\tau g_\tau + 1\right). \tag{4}$$

The main result is a generalisation of Theorem 1. For simplicity, we will not prove the existence of the solution.

Theorem 2. *Let H be real Hilbert-space, $A : H \to H$ a nonlinear operator, which satisfies the conditions below with some positive constants $M \geq m$:*

1. $\langle A(u) - A(v), u - v\rangle \geq m\|u - v\|^2$,
2. $\|A(u) - A(v)\| \leq M\|u - v\|$.

Let u^* denote the solution of the equation $A(u) = f$. For any $f, u_0 \in H$ $\alpha \in (0,1)$, and $t \in \mathbb{R}^+$ small enough the following iteration converges to u^*.

$$u_{n+1} = \sum_{j=1}^{n+1} \binom{\alpha}{j} (-1)^{j+1} u_{n+1-j} - t\left[A(u_{n+1}) - f\right]. \tag{5}$$

Proof. We first add $t\left[A(u_{n+1}) - f\right] - u^*$ both sides of the equation (5) and taking their norms, we have that

$$\left\|u_{n+1} - u^* + t\left[A(u_{n+1}) - A(u^*)\right]\right\| = \left\|\sum_{j=1}^{n+1} \binom{\alpha}{j} (-1)^{j+1} u_{n+1-j} - u^*\right\|. \tag{6}$$

Using the first assumption, we get the lower estimation

$$\begin{aligned} &\|u_{n+1} - u^* + t\left[A(u_{n+1}) - A(u^*)\right]\|^2 \\ = &\|u_{n+1} - u^*\|^2 + t^2\|A(u_{n+1}) - A(u^*)\|^2 + 2t\langle A(u_{n+1}) - A(u^*), u_{n+1} - u^*\rangle \\ \geq &\|u_{n+1} - u^*\|^2 + 2tm\|u_{n+1} - u^*\|^2 \geq \|u_{n+1} - u^*\|^2. \end{aligned} \tag{7}$$

It is also known that $\sum_{j=1}^{\infty} \binom{\alpha}{j} (-1)^{j+1} = 1$ and $\binom{\alpha}{j}(-1)^{j+1} > 0$. Using this, the triangle inequality and (6) for the inequality in (7) we get

$$\begin{aligned} \|u_{n+1} - u^*\| &\leq \left\|\sum_{j=1}^{n+1} \binom{\alpha}{j} (-1)^{j+1} u_{n+1-j} - u^*\right\| \\ &= \left\|\sum_{j=1}^{n+1} \binom{\alpha}{j} (-1)^{j+1} u_{n+1-j} - \sum_{j=1}^{\infty} \binom{\alpha}{j} (-1)^{j+1} u^*\right\| \\ &\leq \sum_{j=1}^{n+1} \binom{\alpha}{j} (-1)^{j+1} \|u_{n+1-j} - u^*\| + \sum_{j=n+2}^{\infty} \binom{\alpha}{j} (-1)^{j+1} \|u^*\|. \end{aligned} \tag{8}$$

Let $\alpha_n := \|u_n - u^*\|$, $f_n := \sum_{j=n+1}^{\infty} \binom{\alpha}{j} (-1)^{j+1} \|u^*\|$ and $\beta_n = \binom{\alpha}{n}(-1)^{n+1}$. With these, we can rewrite (8) as

$$\alpha_{n+1} \leq f_{n+1} + \sum_{j=1}^{n+1} \beta_j \alpha_{n+1-j}. \tag{9}$$

Also using the notation h_j instead of β_{n+1-j}, (9) can be recognised as

$$\alpha_{n+1} \leq f_{n+1} + \sum_{j=0}^{n} h_j \alpha_j. \tag{10}$$

Therefore, with $g_n := 1$ we can apply Lemma 1.

$$\alpha_{n+1} \leq f_{n+1} + \sum_{s=0}^{n} h_s f_s \prod_{\tau=s+1}^{n} (h_\tau + 1). \tag{11}$$

Estimate $\prod_{\tau=s+1}^{n} (h_\tau + 1)$ as

$$\prod_{\tau=s+1}^{n} (h_\tau + 1) = \prod_{\tau=s+1}^{n} (\beta_{n+1-\tau} + 1)$$
$$\leq \prod_{\tau=1}^{n} (\beta_{n+1-\tau} + 1) \leq \left(\frac{n + \sum_{j=1}^{n} \beta_j}{n}\right)^n \leq \left(1 + \frac{1}{n}\right)^n \leq e.$$

Consequently, for (11) the following holds.

$$\alpha_{n+1} \leq f_{n+1} + \sum_{s=0}^{n} h_s f_s \prod_{\tau=s+1}^{n} (h_\tau + 1) \leq f_{n+1} + e \sum_{s=0}^{n} h_s f_s.$$

It is clear that if $n \to \infty$ then $f_{n+1} \to 0$. We prove that $\sum_{s=0}^{n} h_s f_s \to 0$.

$$\begin{aligned}\sum_{s=0}^{n} h_s f_s &= \|u^*\|\beta_{n+1} + \|u^*\| \sum_{s=1}^{n} \beta_{n+1-s} \sum_{j=s+1}^{\infty} \beta_j \\ &= \|u^*\|\beta_{n+1} + \|u^*\| \sum_{s=1}^{n} \beta_{n+1-s} \Big(1 - \sum_{j=1}^{s} \beta_j\Big) \\ &= \|u^*\|\beta_{n+1} + \|u^*\| \sum_{s=1}^{n} \beta_{n+1-s} - \|u^*\| \sum_{s=1}^{n} \sum_{j=1}^{n} \beta_{n+1-s}\beta_j.\end{aligned} \tag{12}$$

Observe first, that the last term in (12) is a Cauchy product.

$$\lim_{n\to\infty} \Big(\sum_{s=1}^{n} \sum_{j=1}^{n} \beta_{n+1-s}\beta_j\Big) = \Big(\sum_{j=1}^{\infty} \beta_j\Big)^2 = 1.$$

Therefore, the first term in (12) tends to zero, the second and the third term to $\|u^*\|$, since $\sum_{j=1}^{\infty} \beta_j = 1$. This means that $\alpha_{n+1} \to 0$ if $n \to \infty$, which has been stated. □

4 Discussion

In this work, we solved nonlinear time-independent equations of type $A(u) = f$, where the operator A is on a Hilbert space. We assumed that it is monotone and Lipschitz-continuous and we proved that the algorithm is convergent.

Our numerical experiences show that if we replace the time-derivative operator in the equation $\partial_t u = -[A(u) - f]$ with a fractional derivative, then it stabilizes the time integration in the numerical solutions. We have tested our method numerically in a fluid dynamical problem previously [10].

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