Existence of limit cycles for some generalisation of the Liénard equations: the relativistic and the prescribed curvature cases

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Abstract. We study the problem of existence of periodic solutions for some generalisations of the relativistic Liénard equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0\,,$$

and the prescribed curvature Liénard equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0\,,$$

where the damping function depends both on the position and the velocity. In the associated phase-plane this corresponds to a term of the form $f(x,y)$ instead of the standard dependence on x alone. By controlling the continuability of the solutions, we are able to prove the existence of at least a limit cycle in the associated phase-plane for both cases, moreover we provide results with a prefixed arbitrary number of limit cycles. Some examples are given to show the applicability of these results.

Keywords: periodic orbits, limit cycles, Liénard relativistic equation, Liénard curvature equation.

2010 Mathematics Subject Classification: 34C25, 34C07.

1 Introduction

The problem of the existence of periodic solutions for the Liénard differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0\,, \tag{1.1}$$

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has been widely investigated in the literature and there is an enormous number of papers originated from the pioneering work by Liénard [7] in 1928. The interested reader can consult for instance [5,15,16] and the references quoted therein.

Recently, particular attention has been given to the study of generalisations of Eq. (1.1) of the form

$$\frac{d}{dt}\varphi(\dot{x}) + f(x)\dot{x} + g(x) = 0,\tag{1.2}$$

where a nonlinear function φ is involved. Once we replace the Newton acceleration $\ddot{x}$ in Eq. (1.2) by the relativistic one, we get the so called *relativistic Liénard* equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + f(x)\dot{x} + g(x) = 0,\tag{1.3}$$

originally introduced and studied in [9,11]. Similarly, one can study [10] the *prescribed curvature* equation of Liénard type

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + f(x)\dot{x} + g(x) = 0.\tag{1.4}$$

The former models have been studied using a phase-plane analysis. Indeed Eq. (1.3) is equivalent to the system

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -f(x)\frac{y}{\sqrt{1+y^2}} - g(x), \end{cases}\tag{1.5}$$

while Eq. (1.4) can be rewritten as

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -f(x)\frac{y}{\sqrt{1-y^2}} - g(x). \end{cases}\tag{1.6}$$

Let us observe that, at a first glance, Eqs. (1.3) and (1.4) may look similar each other, however once we pass to the phase-plane it is clear that system (1.6) is defined only in the strip $|y| < 1$, while the former in the whole plane. This geometric feature makes the analysis completely different; indeed in order to obtain a winding trajectory and then apply the Poincaré–Bendixson Theorem, it is crucial to control the possible blow-up of trajectories when $|y| \to 1$. To tackle this issue, Eq. (1.4) was studied in [10] in the particular case

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda f(x)\dot{x} + g(x) = 0,\tag{1.7}$$

where the real positive parameter λ should be taken sufficiently small, exactly to control the trajectories.

The aim of this paper is to consider the more general case where the term $f(x)$ in Eqs. (1.3) and (1.4) has been replaced by $f(x,\dot{x})$. Let us observe that a particular case involving a term $f(x,\dot{x})$, polynomial in $\dot{x}$, has been already investigated in [4] in the classical Liénard problem. The interested reader can found other cases in [3,6,12,17]. The goal of the present paper is thus to deal with the following equations

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0,\tag{1.8}$$

and

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0. \tag{1.9}$$

Observe that the first equation can be rewritten in the phase-plane as follows

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -f(x,y)\frac{y}{\sqrt{1+y^2}} - g(x), \end{cases} \tag{1.10}$$

with $f(x,y) = \hat{f}(x,y/\sqrt{1+y^2})$. In the following we will impose some conditions on $f(x,y)$ to guaranteed the existence of a winding orbit (e.g. Lemma 3.5), let us however notice that the above relation between $f(x,y)$ and $\hat{f}(x,\dot{x})$ allows us to conclude that the assumptions on $\hat{f}$ can be very mild, because system (1.10), covers more general cases, as well as the following system (1.11). The second Eq. (1.9) becomes

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -f(x,y)\frac{y}{\sqrt{1-y^2}} - g(x), \end{cases} \tag{1.11}$$

using $f(x,y) = \hat{f}(x,y/\sqrt{1-y^2})$. In particular, the phase portrait of these two systems will be studied, and the problem of existence, uniqueness and number of limit cycles will be investigated.

The structure of the paper is the following. In Section 2 some basic properties of the above systems will be presented, in particular we will develop further the method presented in [9,10] to deal with a case where the number of limit cycles can be completely determined studying the curves where $f(x,y) = 0$. The main results will be presented and proved in Section 3, while in Section 4 some examples will be presented and discussed.

2 Definitions and basic facts

Throughout the paper, we assume the functions $f(x,y)$ and $g(x)$ to be regular enough to ensure the existence and uniqueness of the associated Cauchy problem. Moreover we assume $xg(x) > 0$ for $x \neq 0$. Therefore $(0,0)$ is the unique equilibrium point for both systems (1.10) and (1.11).

The slope of the trajectories of latter systems is given by the following expressions, where y' denotes the derivative of y with respect to x:

$$y'(x) = -f(x,y) - g(x)\frac{\sqrt{1+y^2}}{y}, \tag{2.1}$$

and

$$y'(x) = -f(x,y) - g(x)\frac{\sqrt{1-y^2}}{y}. \tag{2.2}$$

Let us consider the phase-plane system equivalent to the classical Liénard system, Eq (1.1)

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x), \end{cases} \tag{2.3}$$

whose slope is given by

$$y' = -f(x) - \frac{g(x)}{y} . \tag{2.4}$$

Then one can remark that the main difference between the classical case and the former ones (1.10) and (1.11), lies on the fact that trajectories of system (2.3) cannot cannot escape to infinity (i.e. no vertical asymptote is allowed) in any finite interval $\alpha < x < \beta$; this is no longer true for systems (1.10) and (1.11) due to the presence of the function $f(x,y)$. Another difference, as already mentioned, is that trajectories of system (1.11) are constrained in the horizontal strip $|y| < 1$.

Therefore, in the study of Eq. (1.10) in order to avoid the possibility of the existence of vertical asymptotes some growth restriction on $f(x,y)$ is required, as for instance that in any finite interval $\alpha < x < \beta$ there exist two positive constants L and D such that $|f(x,y)| < L|y|$ for every x in $\alpha < x < \beta$ and $|y| > D$. This growth restriction will be easily improved in the Lemma 3.5. On the other hand when dealing with Eq. (1.11) this lemma is not necessary due to the fact that solutions are constrained in the horizontal strip $|y| < 1$.

Continuing our analysis of the phase-plane we can compare the slope (2.1) with the one of system (2.4), this will allow us to use the former ones to drive the trajectories of the second one. We eventually conclude that if $f(x,y) > f(x)$ in $xy > 0$, then trajectories of system (1.10) "enter" trajectories of system (1.5), while in $xy < 0$ we have the opposite situation. More precisely we mean that once a trajectory of system (1.10) transversally crosses a given orbit of system (1.5), then the former will never intersect the latter again and so it will remain constrained in a region bounded by the second orbit. This property can be used in order to prove the intersection of the trajectories with the x-axis for $|x|$ large enough, in fact orbits of the system (1.10) will be guided by the ones of system (1.5), and this will be used for finding suitable conditions in order to prove that trajectories of system (1.10) turn clockwise.

When $f(x,y)$ is identically to zero, the system (1.10) becomes the *relativistic Duffing* system, namely

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -g(x) , \end{cases} \tag{2.5}$$

while system (1.11) becomes the *prescribed curvature Duffing* system, namely

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -g(x) . \end{cases} \tag{2.6}$$

The phase-portraits of systems (2.5) and (2.6) has been previously studied in [9,10], however we decided to hereby present a short analysis because this will determine a crucial step in order to study the phase-portrait of systems (1.10) and (1.11).

The first observation is that both systems (2.5) and (2.6) have a Hamiltonian structure for a suitable function H

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x,y) \\ \dot{y} = -\frac{\partial H}{\partial x}(x,y) ; \end{cases} \tag{2.7}$$

for system (2.5) we should use

$$H_r(x,y) = \sqrt{1+y^2} - 1 + G(x) , \tag{2.8}$$

while for system (2.6)

$$H_c(x,y) = -\sqrt{1-y^2} + 1 + G(x),\tag{2.9}$$

where in both cases $G(x)$ is the integral of $g(x)$, $G(x) = \int_0^x g(s)\,ds$.

Notice that in the original case of the Liénard system we would have $H(x,y) = \frac{1}{2}y^2 + G(x)$; despite the difference, all these Hamiltonian functions will play the same role.

Moreover, it is possible to show that the origin is a global centre for both systems (2.5) and (2.6) if and only if $G(x)$ diverges to infinity for $|x| \to \infty$, exactly as in the classical case, while if $G(x)$ is bounded the origin is a local centre. We observe that $H_r(x,y)$ and $H_c(x,y)$ may be viewed as an energy first integral for both the modified Duffing systems (see [5] for a discussion of the energy function in the classical case).

Taking $H_r(x,y)$, respectively $H_c(x,y)$, as a Lyapunov function for systems (1.10), respectively (1.11), we obtain, for its time-derivative along the trajectories the relations

$$\frac{dH_r}{dt}(x,y) = \frac{y}{\sqrt{1+y^2}}\left(-f(x,y)\frac{y}{\sqrt{1+y^2}} - g(x)\right) + g(x)\frac{y}{\sqrt{1+y^2}} = -f(x,y)\frac{y^2}{1+y^2},\tag{2.10}$$

for the system (1.10), while

$$\frac{dH_c}{dt} = \frac{y}{\sqrt{1-y^2}}\left(-f(x,y)\frac{y}{\sqrt{1-y^2}} - g(x)\right) + g(x)\frac{y}{\sqrt{1-y^2}} = -f(x,y)\frac{y^2}{1-y^2},\tag{2.11}$$

in the case of system (1.11). Therefore, when $f(x,y)$ is positive the trajectories of systems (1.10) and (1.11) enter trajectories of systems (2.5) and (2.6) respectively, while when $f(x,y)$ is negative trajectories of system of systems (1.10) and (1.11) exit trajectories of systems (2.5) and (2.6). This behaviour will be crucial in the following and as a first result we get that if $f(0,0) < 0$ then the origin is a source for both systems (1.10) and (1.11). Let us observe that the latter result holds true even if $f(0,0) = 0$ provided that $f(x,y) < 0$ sufficiently close to the origin.

3 Main results

At this point we are able to present a first result, holding for both systems (1.10) and (1.11), which provides examples of a system with a prescribed number of periodic solutions in the light of the classical Poincaré example (the interested reader can consult the § 3.3 of the survey [8]), moreover it shows the flexibility given by the fact that $f(x,y)$ depends on two variables.

Theorem 3.1. *Consider system* (1.10)

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -f(x,y)\frac{y}{\sqrt{1+y^2}} - g(x), \end{cases}$$

let us assume $G(x) \to \infty$ *for* $|x| \to \infty$*, and let for any fixed integer n and positive increasing sequence* c_k

$$f(x,y) = \prod_{k=1}^{n}(H_r(x,y) - c_k).\tag{3.1}$$

Then the system (1.10) *exhibits exactly n limit cycles. Moreover their sizes become arbitrarily large as n increases.*

   On the other hand if G is bounded, namely $G(x) \leq K$, *where* $K < \min\{G(-\infty), G(+\infty)\}$, *let*

$$f(x,y) = \prod_{k=1}^{n} (H_r(x,y) - Kd_k),\tag{3.2}$$

for any positive decreasing sequence d_k, $d_1 < 1$, *then the system* (1.10) *admits again exactly n limit cycles.*

Proof. We observe that for unbounded G, once $H_r(x,y) = c_k$ (or Kd_k in the case of bounded G), the system (1.10) reduces to the Duffing one (2.4) and it clearly exhibits closed trajectories. The rest of the proof follows straightforwardly using the signs discussion previously done.   □

Remark 3.2. A similar result holds for system (1.11). Namely if G is unbounded or bounded by some $K' > 1$ and one chooses

$$f(x,y) = \prod_{k=1}^{n} (H_c(x,y) - e_k),\tag{3.3}$$

for any positive decreasing sequence e_k, $e_1 < 1$ then the system has n limit cycles. Conversely if $G \leq K' \leq 1$, with $K' < \min\{G(-\infty), G(+\infty)\}$, one can take

$$f(x,y) = \prod_{k=1}^{n} (H_c(x,y) - K'e_k),\tag{3.4}$$

and again the system exhibits n limit cycles.

   The latter result looks more interesting, in fact it allows to prove the existence of limit cycles without the need for a small parameter as done in Eq. (1.7) to force the solutions to remain into the horizontal strip.

   At this point in order to get the existence of a limit cycle, it is necessary to create a winding trajectory, being the origin a source because of the assumptions made above. Therefore we consider again systems (1.10) and (1.11) and present our main results.

3.1   The relativistic case

Theorem 3.3. *Consider system* (1.10) *and let us assume the regularities conditions on* $f(x,y)$ *and* $g(x)$ *given in Section* 2. *If* $f(0,0) < 0$, $G(x) \to \infty$ *for* $|x| \to \infty$, *and there exist smooth functions* $\phi(x) > 0$ *and* $\psi(x) < 0$ *such that:*

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1 + [\phi(x)]^2} \qquad \forall x \leq \alpha < 0,\tag{3.5}$$

and

$$f(x,\psi(x))\psi(x) < -\psi'(x)\psi(x) - g(x)\sqrt{1 + [\psi(x)]^2} \qquad \forall x \geq \beta > 0.\tag{3.6}$$

There exists a positive function $T(x)$ *such that*

$$|f(x,y)| \leq LT(x)|y| \quad \forall x \in [\alpha,\beta] \text{ and } |y| \geq D > 0.\tag{3.7}$$

Assume then $f(x,y) > 0$ *for* $x > \beta$ *and* $y > 0$ *and* $x < \alpha$ *and* $y < 0$.
   Then the system (1.10) *exhibits at least one stable limit cycle.*

Let us present two preliminary lemmas, upon which the proof of our theorem will be based. We start with the following lemma which is based on previous results [1,2,13,14].

Lemma 3.4. *Let us assume there exist $\alpha < 0$ and $\beta > 0$ and two smooth functions $\phi(x)$ and $\psi(x)$ such that*

$$\phi(x) > 0 \quad \forall x \leq \alpha \text{ and } \psi(x) < 0 \quad \forall x \geq \beta.$$

Assume moreover

$$f(x, \phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1 + [\phi(x)]^2} \qquad \forall x \leq \alpha, \tag{3.8}$$

and

$$f(x, \psi(x))\psi(x) < -\psi'(x)\psi(x) - g(x)\sqrt{1 + [\psi(x)]^2} \qquad \forall x \geq \beta. \tag{3.9}$$

Then the orbits enter the regions bounded by $y = \phi(x)$ for $x \leq \alpha$ and $y = \psi(x)$ for $x \geq \beta$ (see Fig. 3.1).

Proof. Let us consider the case involving $\phi(x)$ being the one for $\psi(x)$ similar. The slope of system (1.10) is given by

$$\frac{dy}{dx}(x) = -f(x, y) - g(x)\frac{\sqrt{1 + y^2}}{y}, \tag{3.10}$$

and thus evaluated on the graph of the function $\phi(x)$ gives:

$$
\begin{aligned}
\frac{dy}{dx}(x)\big|_{y=\phi(x)} &= -f(x, \phi(x)) - g(x)\frac{\sqrt{1 + [\phi(x)]^2}}{\phi(x)} \\
&= \frac{1}{\phi(x)}\left[-f(x, \phi(x))\phi(x) - g(x)\sqrt{1 + [\phi(x)]^2}\right],
\end{aligned}
\tag{3.11}
$$

where in the rightmost equality we factorised $\phi(x)$.

Recalling the assumption (3.8) and the positiveness of $\phi(x)$ for $x \leq \alpha$ we get:

$$\frac{dy}{dx}(x)\big|_{y=\phi(x)} < \phi'(x) \qquad \forall x \leq \alpha,$$

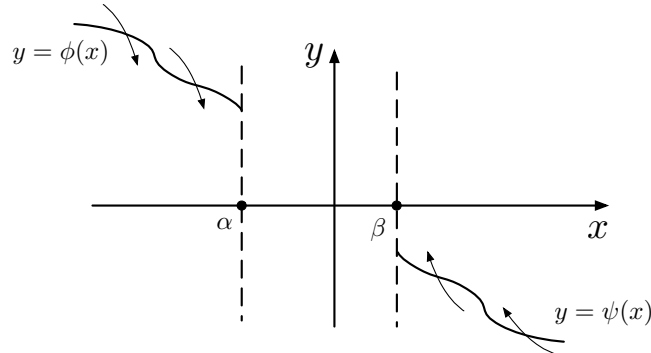that is orbits starting on $y = \phi(x)$ will enter the region bounded by such curve if $x \leq \alpha$.

Figure 3.1: The curves $y = \phi(x)$ and $y = \psi(x)$ and their relation with the trajectories of the system (1.10).

□

Let us now emphasise some relevant cases obtained by specialising the functions ϕ and ψ:

- $\phi(x) = \psi(x) = -g(x)$, then the condition of the lemma becomes

$$f(x, -g(x)) > g'(x) + \sqrt{1 + [g(x)]^2} \qquad \forall x \le \alpha < 0,$$

  and similarly for $x \ge \beta > 0$.

- $\phi(x) = \psi(x) = -x$, hence

$$f(x, -x) > 1 + \sqrt{1 + x^2} \qquad \forall x \le \alpha,$$

  and similarly for $x \ge \beta > 0$.

- $\phi(x) = k > 0$, $\psi(x) = -k$, then

$$f(x, k) > -g(x)\sqrt{1 + k^2}/k \qquad \forall x \le \alpha < 0,$$

  and

$$f(x, -k) > g(x)\sqrt{1 + k^2}/k \qquad \forall x \ge \beta > 0.$$

We are now presenting a lemma allowing us to avoid the presence of vertical asymptotes for the orbits.

Lemma 3.5. *Let us assume there exists a positive continuous function $T(x)$ and two positive constants L and D such that*

$$|f(x, y)| \le LT(x)|y| \qquad \forall x \in [-M, M] \text{ and } |y| \ge D. \tag{3.12}$$

Then any orbit starting on the lines $x = \pm M$ cannot escape to infinity (i.e. no vertical asymptote is allowed).

Proof. Let us notice that using the bound (3.12) we can show that the orbit can be continued in the future starting from any $x_0 = -M$ and $y_0 \ge D$ up to $x = M$.   □

We observe that this lemma is necessary once dealing with general $f(x, y)$ as in Eq. (1.10), on the contrary when such function is actually obtained by $\hat{f}(x, y/\sqrt{1 + y^2})$ the lemma is no longer necessary because the latter function is bounded for $|y| \to \infty$.

The last ingredient needed to show that orbits turn clockwise is the following lemmas.

Lemma 3.6. *Let us assume $G(x)$ is unbounded and $f(x, y) > 0$ for $x > b$, for some $b > 0$, then any trajectory starting on the vertical line (b, y_b), $y_b > 0$, will intersect the positive x-axis.*

Proof. The proof is straightforward; it is based on Eq. (2.10) and the properties of the Duffing system (2.5) which exhibits a global centre because of the assumption on $g(x)$. See Fig. 3.2 for a geometrical representation of such result.   □

Let us note that assuming $f(x, y) > 0$ for $x < a$, for some $a < 0$. A similar result can be used to prove that any trajectory starting from (a, y_a), $y_a < 0$, will intersect the negative x-axis. We can now prove the Theorem 3.3
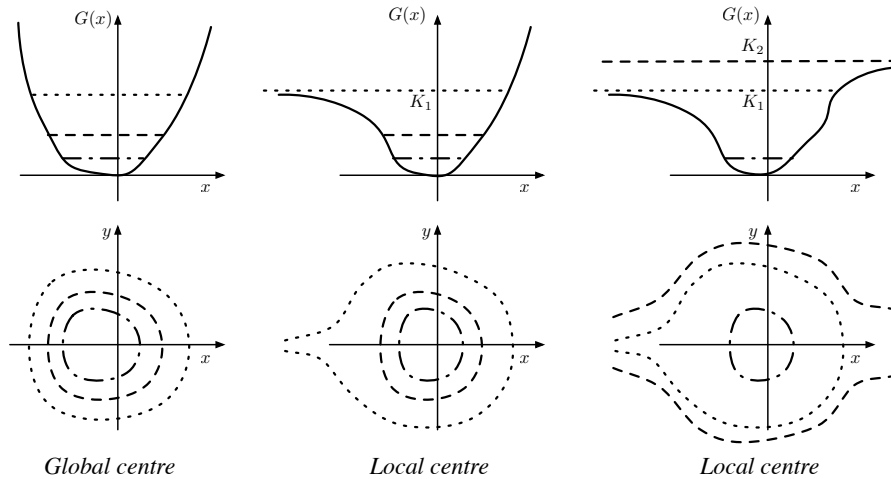
Global centre   Local centre   Local centre

Figure 3.2: The global and local centre cases according to the behaviour of $G(x)$ for $|x| \to \infty$.

Proof. The reader can consult Fig. 3.3 to follow the progress of the proof. Let us consider a point $P(x_0, y_0)$, $x_0 < \alpha$ and $y_0 = \phi(x_0) > 0$, in virtue of Lemma 3.4 the trajectory originating from this point it is bounded away from the x-axis for negative t and it enters the graph of the function $\phi(x)$ and therefore it will intercept the line $x = \alpha$. From this point on, the trajectory can intersect the x-axis at some point in between $(0, \beta)$; on the contrary by virtue of Lemma 3.5 the trajectory will reach the line $x = \beta$ at some $y_1 > 0$. By Lemma 3.6 the trajectory will be guided by the trajectories of the Duffing system (2.5) and thus will reach the x-axis at some point $x > \beta$.

Using condition (3.9), again from Lemma 3.4, the trajectory will intersect the line $x = \beta$ at the point $\psi(\beta) < y_2 < 0$. As before, such trajectory can intersect either the negative x-axis in $(\alpha, 0)$, or it reaches twice the line $x = \alpha$ once for some negative y_3 and eventually for $0 < y_4 < \phi(\alpha)$. In both cases we produced a winding trajectory, which completes the proof by recalling that the origin is a source and using the Poincaré–Bendixson Theorem. $\square$
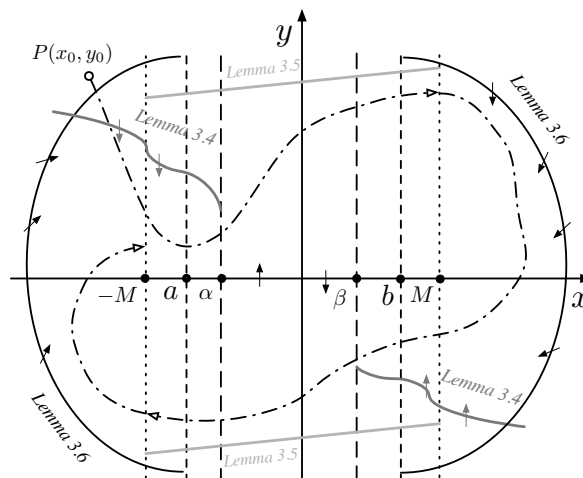

Figure 3.3: Scheme of the proof of the Theorem 3.3.

Let us observe that assuming both Eq. (3.5) and (3.6) is perhaps too restrictive. Indeed one can prove a similar result relaxing one of the above as the following result shows:

Theorem 3.7. *Consider system* (1.10) *and let us assume the regularities conditions on* $f(x,y)$ *and* $g(x)$ *given in Section* 2. *If* $f(0,0) < 0$, $G(x) \to \infty$ *for* $|x| \to \infty$, *and there exists a smooth function* $\phi(x)$ *such that:*

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1 + [\phi(x)]^2} \qquad \forall x \leq \alpha, \tag{3.13}$$

moreover there exists a positive function $T(x)$ *such that*

$$|f(x,y)| \leq LT(x)|y| \qquad \forall x \in [\alpha, \beta] \text{ and } |y| \geq D > 0. \tag{3.14}$$

Assume then $f(x,y) > 0$ *for* $x > \beta$ *and every* y. *Then the system* (1.10) *exhibits a least* 1 *stable limit cycle*

The proof is similar to the previous one and therefore it will be omitted. A similar result clearly holds if one assume Eq. (3.6) and $f(x,y) > 0$ for $x < \alpha$ and every y.

3.2 The prescribed curvature case

If we consider system (1.11) a lemma similar to 3.5 can be proved with the additional assumption that the orbit should be constrained in the strip $|y| < 1$; for the same reason, as previously mentioned, a result similar to Lemma 3.4 will not be interesting in this contest. However a result similar to the one provided by Theorem 3.3 holds true.

Theorem 3.8. *Consider system* (1.11) *and let us assume the regularities conditions on* $f(x,y)$ *and* $g(x)$ *given in Section* 2. *If* $f(0,0) < 0$, $G(x) \to G_\infty \geq 1$ *for* $|x| \to \infty$ *and there exists a smooth function* $\phi(x)$ *such that:*

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1 + [\phi(x)]^2} \quad \forall x \leq \alpha, \tag{3.15}$$

and $\phi(\alpha) \leq 1$. $f(x,y) > 0$ *for* $x < \alpha$ *and all* y. *Then the system* (1.11) *exhibits at least one stable limit cycle provided* $f(x,y)$ *is sufficiently small for* $x > \alpha$.

The proof is similar to the one of Theorem 3.7 and thus it will be omitted. A similar result can be stated using the function $\psi(x)$ with $\psi(\beta) \geq -1$. A slight different version of this result together with its dual version, will require a growth assumption on $f(x,y)$ only in the strip $(\alpha, \beta) \times \mathbb{R}$ and requiring $f(x,y) > 0$ also for $x > \beta$. However this result, even if supported by numerical evidence similar to the one of Fig. 7 [10], is less appealing for the heavy assumption on the growth of $f(x,y)$. Moreover, some restrictions on the growth of $G(x)$ seem necessary, as for instance $G(x) < 1$ for all x. A detailed discussion on this assumption may be found in [10].

4 Some examples

The aim of this section is to present some examples of application of the theory developed so far. The numerical results, in particular the 0-isocline and the orbits, have been realised using the *MATLAB* software [18], while the phase-space portraits using the open source *Field-Play* [19].

4.1 Using the 0-isocline for the functions $\phi(x)$ and $\psi(x)$

Let us consider $f(x, y) = |y| \cos^2 y(x^2 - 1)$ and $g(x) = x$, then the system (1.10) rewrites

$$
\begin{cases}
\dot{x} = \frac{y}{\sqrt{1+y^2}} \\
\dot{y} = -\frac{y|y|(\cos y)^2(x^2-1)}{\sqrt{1+y^2}} - x,
\end{cases}
\tag{4.1}
$$

whose slope is thus (see Eq. (2.1)) given by

$$
y'(x) = -|y|(\cos y)^2(x^2 - 1) - x\frac{\sqrt{1+y^2}}{y}.
$$

Finally the 0-isocline, $y_0(x)$, is implicitly given by:

$$
(\cos y_0(x))^2 \frac{|y_0(x)|y_0(x)}{\sqrt{1 + [y_0(x)]^2}} = -\frac{x}{x^2 - 1}.
\tag{4.2}
$$

This function is invariant with respect to the transformation $(x, y) \to (-x, -y)$ and thus it would be enough to study it for $y \geq 0$. For a sake of clarity let us define the functions

$$
A(y) = (\cos y_0(x))^2 \frac{|y_0(x)|y_0(x)}{\sqrt{1 + [y_0(x)]^2}} \quad \text{and} \quad B(x) = -\frac{x}{x^2 - 1},
$$

In the limit $x \to 0^+$, we have $B(x) \to 0^+$, hence to satisfy the equation $A(y) = B(x)$ we have $y_0 \to 0^+$ and $(\cos y_0)^2 \to 0$, that is $y_0 \to \pi/2 + k\pi$, for $k \in \mathbb{N} \cup \{0\}$. On the other hand once $x \to 0^-$, we get $B(x) \to 0^-$ and thus the only root of $A(y)$ is given by $y_0 \to 0^-$.

We then consider the case of large x. For $x \to +\infty$, $B(x) \to 0^-$ and thus the unique zero of $A(y)$ is $y_0 \to 0^-$. Finally, if $x \to -\infty$, $B(x) \to 0^+$, hence beside the zero $y_0 \to 0^+$ we have also the positive zeros of $(\cos y_0)^2$, that is $y_0 \to \pi/2 + k\pi$, for $k \in \mathbb{N} \cup \{0\}$.

We observe that $f(0, 0) = 0$ but $f(x, y) < 0$ for $0 < x^2 < 1$ and thus one can prove that the origin is an unstable equilibrium. The function $G(x) = x^2/2$ and thus it is unbounded for $|x| \to \infty$. Taking $L = 1$ and $T(x) = 1 - x^2$ one can obtain $|f(x, y)| \leq LT(x)|y|$ for all $x \in [-1, 1]$ and $y > 0$. Finally $f(x, y)$ is positive for $|x| > 1$. Observe that $\dot{x} > 0$ for $y > 0$ and thus the 0-isocline is traversed from left to right by the orbits, the latter can thus play the role of the function $\phi(x)$ because it prevents the orbits to grow.

The hypotheses of Theorem 3.3 are met and thus the system will exhibit at least one stable limit cycle. A numerical integration of (4.1) shows that indeed the limit cycle is unique (see Fig. 4.1). However we are not able to prove the uniqueness of the limit cycle, and at this stage this remains a conjecture.

4.2 Using a constant function for $\phi(x)$ and $\psi(x)$

Let us consider $f(x, y) = (k^2 + x^2/2)(\sin y - 1/2) + x^2$, for some $k \in \mathbb{R}$, and $g(x) = x$, then the system (1.10) rewrites

$$
\begin{cases}
\dot{x} = \frac{y}{\sqrt{1+y^2}} \\
\dot{y} = -\frac{y}{\sqrt{1+y^2}}\left[\left(k^2 + \frac{x^2}{2}\right)\left(\sin y - \frac{1}{2}\right) + x^2\right] - x.
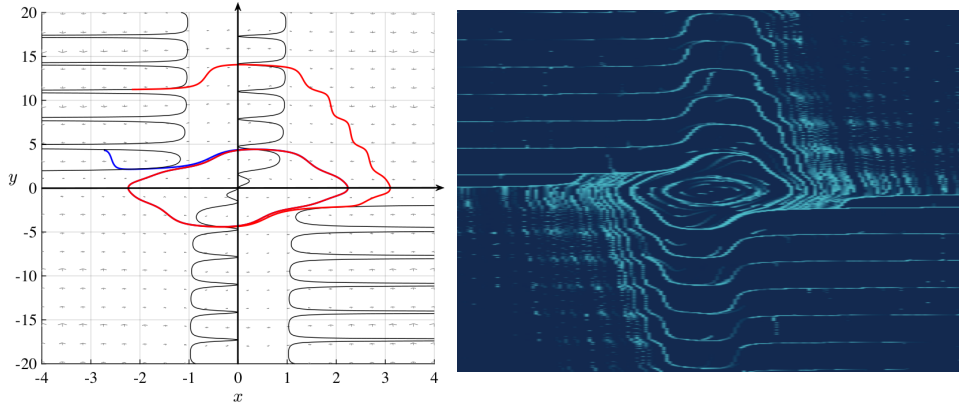\end{cases}
\tag{4.3}
$$

Figure 4.1: The phase-portrait of the system (1.10) with $f(x,y) = |y|\cos^2 y(x^2 - 1)$ and $g(x) = x$. On the left panel [18]: the black curves denote the different branches of the 0-isocline while the blue and red curves are two generic orbits of the system which accumulate on the (unique) stable limit cycle as provided by Theorem 3.3. On the right panel [19]: several orbits are traced to better appreciate a larger view of the phase-portrait; let us observe that the scales of the two figures are slightly different, this is the reason why the two limit cycles look different.

One has $f(0,0) = -k^2/2 < 0$, $|f(x,y)| \le 3k^2/2 + 7x^2/4$ and thus Lemma 3.5 holds true, and finally $f(x,y) > 0$ for $|x|$ large enough, say $x^2 > 6k^2$. Once again $G(x) \to \infty$ for $|x| \to \infty$.

Let us finally assume $\phi(x) = a$ and $\psi(x) = -a$, where $a = \pi/2$, then conditions (3.5) and (3.6) are satisfied if

$$f(x,a) > -x\frac{\sqrt{1+a^2}}{a} \quad \text{and} \quad f(x,-a) > x\frac{\sqrt{1+a^2}}{a}.$$

A direct computation shows that if $k^2 > \frac{4}{5\pi}\sqrt{1 + \frac{\pi^2}{4}}$, then the previous conditions are satisfied, indeed

$$f(x,a) = \frac{k^2}{2} + \frac{5}{4}x^2,$$

and with the chosen bound on k^2 we are sure that such parabola never intersects the line $y = -x\frac{\sqrt{1+a^2}}{a}$.

4.3　The curvature case

The last example concerns the prescribed curvature case. We will use that same function $f(x,y)$ of the first example, that is $f(x,y) = |y|(\cos 3y)^2(x^2 - 1)$. On the other hand for the $g(x)$ function we will use $g(x) = \mu x e^{-|x|}$, where $\mu < 1$, in such a way $\lim_{|x|\to\infty} G(x) = \mu < 1$ (see section 6 of [10]). Finally to constraint the orbits into the strip $|y| < 1$ we multiply $f(x,y)$ by a sufficiently small parameter λ, then the system (1.11) rewrites

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -\lambda\frac{y|y|(\cos 3y)^2(x^2-1)}{\sqrt{1-y^2}} - \mu x e^{-|x|}, \end{cases} \tag{4.4}$$
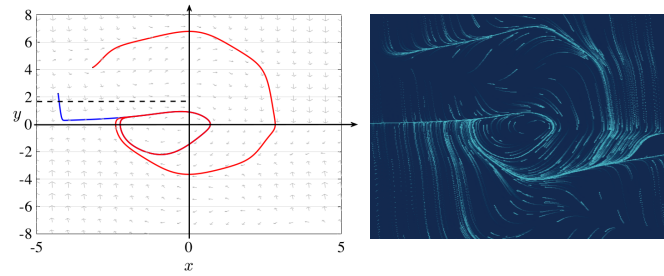
Figure 4.2: The phase-portrait of the system (1.10) with $f(x,y) = (1 + x^2/2)(\sin y - 1/2) + x^2$ and $g(x) = x$. On the left panel [18]: the dashed black line denotes the function $\phi(x) = \pi/2$, while the blue and red curves are two generic orbits of the system which accumulate on the stable limit cycle as provided by Theorem 3.3. On the right panel [19]: several orbits are traced to better appreciate a larger view of the phase-portrait.

whose slope is (see Eq. (2.2)):

$$y'(x) = -\lambda|y|(\cos 3y)^2(x^2 - 1) - \mu x e^{-|x|}\frac{\sqrt{1 - y^2}}{y}. \tag{4.5}$$

Finally the 0-isocline, $y_0(x)$, is implicitly given by:

$$\lambda(\cos 3y_0(x))^2\frac{|y_0(x)|y_0(x)}{\sqrt{1 - [y_0(x)]^2}} = -\frac{\mu x e^{-|x|}}{x^2 - 1}. \tag{4.6}$$

The study of the 0-isocline is very similar to the one done in Section 4.1 the main differences being now that not all the zeros of $(\cos 3y)^2$ should be taken into account because, only those falling inside the strip $|y| < 1$ do matter; secondly the positive-y branch stops at $(x, y) = (-1, 1)$ and the negative one at $(x, y) = (1, -1)$ (see Fig. 4.3 for a numerical example). One can use again the 0-isocline as function $\phi(x)$ and $\psi(x)$ to apply the Theorem 3.8.
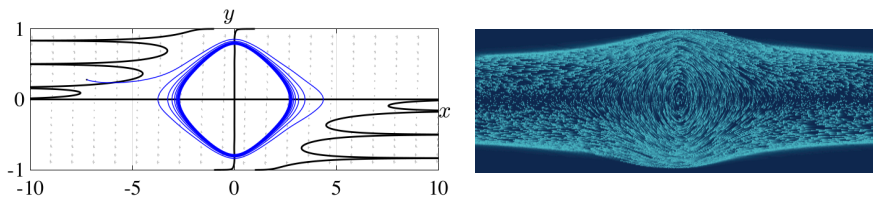

Figure 4.3: The phase-portrait of the system (4.4) with $f(x,y) = |y|(\cos(3y))^2(x^2 - 1)$, $g(x) = \mu x e^{-|x|}$, $\lambda = 0.01$ and $\mu = 1/2$. On the left panel [18]: the solid black lines denote the branches of the 0-isocline while the blue curve a generic orbit of the system which accumulate on the (unique) stable limit cycle as provided by Theorem 3.8. On the right panel [19]: several orbits are traced to better appreciate a larger view of the phase-portrait; let us observe that because of the small parameter λ the system is very close to the Duffing one which exhibits a global centre (in the strip $|y| < 1$), therefore it is hard to visualise the limit cycle for the system we are considering.

Acknowledgement

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