INSTITUTE OF ECONOMICS
Centre for Economic and Regional Studies

KRTK KTI

Közgazdaság- és Regionális Tudományi Kutatóközpont
KÖZGAZDASÁG-TUDOMÁNYI INTÉZET

CERS-IE WORKING PAPERS | KRTK-KTI MŰHELYTANULMÁNYOK

Trading Networks with Frictions

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CERS-IE WP – 2020/8

February 2020

https://www.mtakti.hu/wp-content/uploads/2020/01/CERSIEWP202008.pdf

ABSTRACT

We show how frictions and continuous transfers jointly affect equilibria in a model of matching in trading networks. Our model incorporates distortionary frictions such as transaction taxes, bargaining costs, and incomplete markets. When contracts are fully substitutable for firms, competitive equilibria exist and coincide with outcomes that satisfy a cooperative stability property called *trail stabity*. In the presence of frictions, competitive equilibria might be neither stable nor (constrained) Pareto-efficient. In the absence of frictions, on the other hand, competitive equilibria are stable and in the core, even if utility is imperfectly transferable.

JEL codes: C62, C78, D47, D51, D52, L14

Keywords: Trading networks; frictions; competitive equilibrium; matching with contracts; trail stabity; transaction taxes; commission

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Kereskedelmi hálózatok súrlódásokkal

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ÖSSZEFOGLALÓ

Megmutatjuk, hogy a kereskedelmi hálózat modellben mind a súrlódás, mind a folytonos átválthatóság megléte hogyan befolyásolja a közgazdasági egyensúlyt. Modellünkben a torzítást eredményező súrlódási tényezők lehetnek tranzakciók után fizetendő illetékek, az alkufolyamathoz kapcsolódó költségek vagy hiányos piacok. Amennyiben a modellben szereplő cégek számára az egyes szerződések korlátlanul helyettesíthetők, úgy mindig létezik közgazdasági egyensúly, és megegyezik a trail-stabilitásnak elnevezett kooperatív stabilitási tulajdonságot teljesítő végeredményekkel. Súrlódás megléte esetén azonban a közgazdasági egyensúlyi sem nem feltétlenül stabil, sem pedig nem feltétlenül Pareto-optimális. Súrlódás hiányában azonban a közgazdasági egyensúly akkor is stabil és mag-tulajdonságú, ha a hasznosság nem tökéletesen átváltható.

JEL: C62, C78, D47, D51, D52, L14
Kulcsszavak: kereskedelmi hálózatok, súrlódás, versenyző egyensúly, tranzakciós adó

Trading networks with frictions

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We show how frictions and continuous transfers jointly affect equilibria in a model of matching in trading networks. Our model incorporates distortionary frictions such as transaction taxes, bargaining costs, and incomplete markets. When contracts are fully substitutable for firms, competitive equilibria exist and coincide with outcomes that satisfy a cooperative stability property called *trail stability*. In the presence of frictions, competitive equilibria might be neither stable nor (constrained) Pareto-efficient. In the absence of frictions, on the other hand, competitive equilibria are stable and in the core, even if utility is imperfectly transferable.

1 INTRODUCTION

Interdependence and specialization of production are central features of the modern economy. Many firms have complex, bilateral relationships with dozens of buyers and suppliers. The terms of these relationships are typically encoded in complex *contracts* that specify goods traded or services rendered, delivery dates, penalties for non-completion, and, of course, prices. Markets that involve heterogeneous and highly specialized contracts, talented workers, or sophisticated machines can often be concentrated and thin. In such markets, it is *à priori* implausible to assume that agents act as price-takers.

Models of matching with contracts, inspired by the work of Gale and Shapley [1962], elegantly capture interaction in thin markets [Crawford and Knoer, 1981, Hatfield and Milgrom, 2005, Kelso and Crawford, 1982, Roth, 1984]. Matching models do not typically assume that agents are price-takers: instead, agents are free to engage in highly specific contracts and rely on the consent of counterparties to maintain contractual relationships. The equilibrium concepts employed in the matching literature, such as *stability*, require that recontracting should not be profitable. Unlike typical general equilibrium models, matching models can also incorporate indivisibilities, which are often present in thin markets. Finally, matching models capture frictions, such as transaction taxes [Dupuy et al., 2017], bargaining costs [Galichon et al., 2018], and the incompleteness of the financial market [Jagadeesan, 2017].[1]

While cooperative solution concepts are well-founded thin markets, competitive solution concepts are often more natural in thick markets [Edgeworth, 1881, Kelso and Crawford, 1982]. Nevertheless, competitive and cooperative solution concepts are both appealing to some extent in markets of all sizes. For example, competitive equilibrium could be a reasonable solution concept even in thin markets because it does not require firms to coordinate directly with one another. Cooperative solutions, on the other hand, offer a credible foundation for the analysis of thick markets that cannot clear—for example, due to price controls.[2]

This paper establishes an equivalence between competitive equilibrium and an intuitive stability concept in markets with frictions. As we will argue, our equivalence result provides new cooperative foundations for competitive equilibrium and competitive foundations for our stability concept. We also show how frictions matter for the connection between competitive and cooperative solution concepts.

We focus on trading networks to capture complex production linkages. Following Ostrovsky [2008], Hatfield and Kominers [2012], Hatfield et al. [2013], and Fleiner et al. [2018b], we assume that agents interact via an exogenously

[1]The financial market is *incomplete* if agents suffer from uninsurable risk—that is, if there is some Arrow [1953] security that is absent or cannot be traded without transaction costs.
[2]See Drèze [1975], Hatfield et al. [2012, 2016], Andersson and Svensson [2014], and Herings [2015].

specified set of bilateral *trades*—which specify who is trading, what good or service is being traded, and any non-pecuniary parameters of exchange. Trades have directions that correspond to the flow of goods: upstream trades represent purchases and downstream trades represent sales. In a market outcome, transfers are made for every realized trade, encapsulating the role of money in the economy [Hatfield et al., 2013]. We summarize outcomes as a set of realized *contracts*, each of which specifies a trade and a price.

Our model can capture distortionary frictions in reduced form. Formally, we allow agents to place different values on transfers associated to different trades. Intuitively, when frictions are present, receiving one unit of transfer may not fully offset the cost of paying one unit of transfer. For example, transaction taxes and bargaining costs cause there to be a wedge between payment and receipt. There might also be wedges between forms of transfer when financial markets are incomplete. For example, if transfers are in trade credit that is subject to imperfectly-insurable default risk, then creditors value payments less than debtors. Similarly, if currency markets are imperfect, then firms may value local currency more than foreign currency. However, like in general equilibrium models, we assume that transfers associated to trades are one-dimensional, so that each realized trade has a well-defined price. This uni-dimensionality condition rules out partial financing of purchases with trade credit and requires that each trade is priced in a single currency [Jagadeesan, 2017].

Our first main result provides sufficient conditions for the existence of competitive equilibria. The key assumption is that preferences over contracts are *fully substitutable* [Hatfield and Kominers, 2012, Hatfield et al., 2013, Ostrovsky, 2008]—that is, that upstream (resp. downstream) trades are grossly substitutable for each other, and that upstream and downstream trades are grossly complementary to one another. Full substitutability can be regarded as the requirement that the goods that flow in trades are grossly substitutable [Baldwin and Klemperer, 2018, Hatfield et al., 2019]. In our model, full substitutability and a mild regularity condition together ensure that competitive equilibria exist.[3]

To relate the competitive and cooperative approaches to the analysis of markets with frictions, we first explore cooperative interpretations of competitive equilibria. We show that competitive equilibrium outcomes are always *trail-stable*—i.e., immune to sequential deviations in which a firm that receives an upstream (resp. downstream) contract offer can either accept the offer outright or make an additional downstream (resp. upstream) contract offer [Fleiner et al., 2018b]. Trail stability is a natural extension of Gale and Shapley's (1962) pairwise stability property to trading networks. Other solution concepts in matching theory are *stability* (in the sense of Hatfield et al. [2013])—which requires that there is no group of firms that can commit to recontracting among themselves (possibly while dropping some existing contracts)—and the *core*. However, in the presence of frictions, competitive equilibrium outcomes are typically neither stable nor in the core.

Stable and trail-stable outcomes, on the other hand, have competitive interpretations. We say that an outcome *lifts* to a competitive equilibrium if the outcome can be supported by competitive equilibrium prices—as an outcome already specifies the prices of realized trades, showing that an outcome lifts to a competitive equilibrium amounts to specifying equilibrium prices for unrealized trades. We show that trail-stable and stable outcomes lift to competitive equilibria under full substitutability and regularity conditions.[4] In the presence of frictions, therefore, the trail stability and competitive equilibrium solution concepts are essentially equivalent, but they both differ from stability.

[3]As Hatfield and Kominers [2012] and Hatfield et al. [2013] show, full substitutability is necessary (in the maximal domain sense) for the existence of equilibria in trading networks.

[4]Hatfield et al. [2013] show that stable outcomes lift to competitive equilibria under full substitutability in transferable utility economies. Our results apply even in the presence of frictions and income effects.

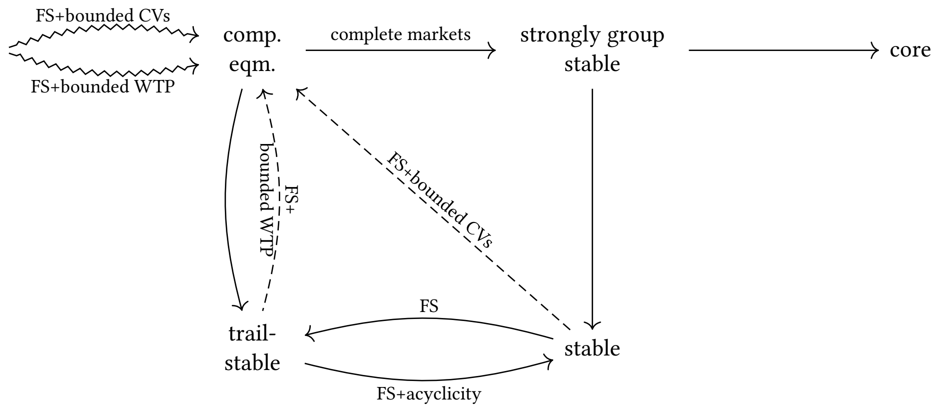

Fig. 1. **Summary of our results.** The squiggly arrows represent existence results, the ordinary arrows represent relationships between solution concepts, and the dashed arrows shows lifting results. Arrows are labeled by the hypotheses of the corresponding results. FS stands for full substitutability (see Assumption 1), "Bounded CVs" stands for "bounded compensating variations" (see Assumptions 2 and 2′), and "bounded WTP" stands for "bounded willingness to pay" (see Assumption 3).

The relationship between stability and competitive equilibria changes dramatically in the absence of distortionary frictions. In this case, there are no wedges between payments and receipts, and we say that the market is *complete*.[5] Completeness ensures that competitive equilibrium outcomes are *strongly group stable* (in the sense ofHatfield et al. [2013]), hence in particular stable, in the core, and Pareto-efficient. As a result, the (strong group) stability, trail stability, and competitive equilibrium solution concepts are all essentially equivalent in complete markets. Figure 1 summarizes our results.

Taken as a whole, our results provide new foundations for competitive equilibrium and trail stability in thin and thick markets. Our competitive interpretation of trail stability guarantees that, as long firms coordinate on a trail-stable outcome, they act *as if* they take prices as given. Hence, even though price-taking may not be a reasonable assumption *per se* in thin markets, it is actually a consequence of cooperative behavior. On the other hand, our cooperative interpretation of competitive equilibrium guarantees that firms cannot improve upon equilibrium outcomes even by deviations along trails. Therefore, while it may be difficult for firms to coordinate with each other in thick markets, any equilibrium will yield a trail-stable outcome as long as firms take prices as given.

From an applied perspective, our model may be of interest to structural econometricians. Recent work on estimation in matching markets with transfers has focused on frictionless trading networks [Fox, 2017, 2018, Fox et al., 2018] and two-sided markets with frictions [Cherchye et al., 2017, Galichon et al., 2018].[6] Since our model allows for both frictions and interconnectedness, it opens up new applications. Consider, for example, the housing market. Houses are highly differentiated and agents might act as both buyers and sellers, making the housing market an interconnected trading network. There is no vertical supply chain structure. Interactions in the housing market suffer from bargaining

[5]Our completeness condition is analogous to the requirement in general equilibrium theory that the financial market is complete. Indeed, when the financial market is rich enough (i.e., all Arrow [1953] securities are present), agents' marginal rates of substitution between forms of transfer are equalized in equilibrium. By renormalizing the currency units of each form of transfer, we can assume that all agents are indifferent between all forms of transfer—see Section 6.

[6]Other papers have focused on structural estimation in two-sided matching markets with transferable utility. See, for example, Choo and Siow [2006], Fox [2010], Chiappori, Oreffice, and Quintana-Domeque [2012], Fox and Bajari [2013], Dupuy and Galichon [2014], Galichon and Salanié [2014], and Chiappori, Salanié, and Weiss [2017].

frictions and other transaction costs—such as real estate agent fees and stamp duty land taxes [Hilber and Lyytikäinen, 2017]—making utility imperfectly transferable.[7] Structural methods based on our model would allow the econometrician to partially identify agents' preferences by assuming that the observed market outcome is trail-stable—or, equivalently, associated to a competitive equilibrium.

Most previous models of matching in trading networks impose significant additional conditions on the structure of the trading network, the space of contracts, or preferences. Ostrovsky [2008], Westkamp [2010], and Hatfield and Kominers [2012] derive existence and structural results for acyclic networks, which cannot contain "horizontal" trade between intermediaries.[8] Hatfield et al. [2018] and Fleiner et al. [2018b] extend the analysis of Ostrovsky [2008] to general trading networks. However, Ostrovsky [2008], Westkamp [2010], Hatfield and Kominers [2012], and Fleiner et al. [2018b] all assume that there are finitely many contracts, ruling out continuous or unbounded prices and precluding comparisons between the matching and general equilibrium approaches. Hatfield et al. [2013] consider general trading networks with continuous prices and technological constraints, but assume that utility is perfectly transferable, ruling out distortionary frictions and income effects.[9] In a recent paper, Hatfield et al. [2018] introduce continuous prices into discrete models of matching in trading networks [Fleiner et al., 2018b, Hatfield and Kominers, 2012, Ostrovsky, 2008, Westkamp, 2010] while allowing for technological constraints [Hatfield et al., 2013]. Our model specializes that of Hatfield et al. [2018] to accommodate general equilibrium analysis. Hatfield et al. [2018] show when *chain stable* outcomes and stable outcomes—neither of which exist in our model—coincide. In contrast, we prove existence results and relate competitive equilibrium to trail stability and stability.

This paper proceeds as follows. Section 2 introduces the model. Section 3 explains how our model captures frictions and describes leading examples. Section 4 presents sufficient conditions for the existence of competitive equilibrium. Section 5 defines trail stability and stability and relates these concepts to competitive equilibrium. Section 6 analyzes complete markets. Section 7 concludes. Appendix A specializes to the case of acyclic networks. Appendix B formulates an equivalent definition of full substitutability. The Supplementary Appendices present the omitted proofs and additional examples.

2 MODEL

Our model is based on that of Hatfield et al. [2018] but requires that prices be continuous and unbounded.

2.1 Firms and contracts

There is a finite set F of firms and a finite set Ω of *trades*. Each trade $\omega \in \Omega$ is associated to a buyer $\mathrm{b}(\omega) \in F$ and a seller $\mathrm{s}(\omega) \in F$. Trades specify what is being exchanged as well as any non-pecuniary contract terms [Hatfield et al., 2013].

A *contract* is a pair (ω, p_ω) that consists of a trade ω and a price $p_\omega \in \mathbb{R}$. Thus, the set of contracts is $X = \Omega \times \mathbb{R}$. Let $\tau : X \to \Omega$ be the projection that recovers the trade associated with a contract. An *outcome* is a set $Y \subseteq X$ such that each trade is associated with at most one price in Y—formally, $|\tau(Y)| = |Y|$.

Given a set $\Xi \subseteq \Omega$ of trades and a firm $f \in F$, let $\Xi_{\to f}$ denote the set of trades in Ξ in which f acts as a buyer, let $\Xi_{f \to}$ denote the set of trades in Ξ in which f acts as a seller, and let $\Xi_f = \Xi_{\to f} \cup \Xi_{f \to}$ denote the set of trades in

[7] In contrast, Shapley and Shubik [1972] and Hatfield et al. [2013] assume that utility is perfectly transferable, while Shapley and Scarf [1974] and Abdulkadiroğlu and Sönmez [1999] assume that utility is non-transferable.
[8] In Appendix A, we impose acyclicity and show that trail-stable, stable, and competitive equilibrium outcomes coincide under full substitutability and a regularity condition.
[9] Hatfield et al. [2013] allow for fixed transaction costs, such as shipping costs and lump-sum transaction taxes, but not variable transaction taxes and the other frictions considered in this paper.

Ξ in which f is involved (either as a buyer or as a seller). For a set $Y \subseteq X$ of contracts, we define $Y_{\to f}$, $Y_{f\to}$, and Y_f analogously.

An *arrangement* is a pair $[\Xi; p]$ of a set of trades $\Xi \subseteq \Omega$ and a price vector $p \in \mathbb{R}^{\Omega}$. Given an arrangement $[\Xi; p]$, define an associated outcome $\kappa([\Xi; p]) \subseteq X$ by

$$\kappa([\Xi; p]) = \{(\omega, p_\omega) \mid \omega \in \Xi\}.$$

That is, $\kappa([\Xi; p])$ is the outcome at which the trades in Ξ are realized at prices given by p. Note that arrangements specify prices even for unrealized trades.

2.2 Utility functions and transfers

Each firm's utility depends only on the trades that involve it and on the transfers that it pays and receives. Formally, firm f has a utility function $u^f : \mathcal{P}(\Omega_f) \times \mathbb{R}^{\Omega_f} \to \mathbb{R} \cup \{-\infty\}$.[10] We assume that u^f is continuous and that

$$t \le t' \implies u^f(\Xi, t) \le u^f(\Xi, t')$$

with equality only if $u^f(\Xi, t) = -\infty$, so that monetary transfers are relevant to firms whenever their utility is finite. We also assume that $u^f(\varnothing, 0) \in \mathbb{R}$, so that money is relevant to firms at any outcome that they prefer to autarky. The transferable utility trading network model of Hatfield et al. [2013] is recovered when

$$u^f(\Xi, t) = v^f(\Xi) + \sum_{\omega \in \Omega_f} t_\omega$$

for some *valuation* function $v^f : \mathcal{P}(X_f) \to \mathbb{R} \cup \{-\infty\}$.

To analyze competitive equilibria, we need to consider firms' demands at any given price vector. Prices give rise to transfers in the following manner. Firms receive no transfer for a trade if they do not agree to the trade. Firms receive transfers equal to the prices of any realized sales (downstream trades) and pays transfers equal to the prices of any realized purchases (upstream trades). Maximizing utility at a price vector $p \in \mathbb{R}^{\Omega_f}$ gives rise to a collection of sets of demanded trades

$$D^f(p) = \arg\max_{\Xi \subseteq \Omega_f} u^f\left(\Xi, \left(p_{\Xi_{f\to}}, (-p)_{\Xi_{\to f}}, 0_{\Omega_f \smallsetminus \Xi}\right)\right).$$

Thus, D^f is the *demand correspondence* of firm f.

As is typical in matching theory (see Aygün and Sönmez [2013]), we also need to consider firms' choices from sets of available contracts. Given an outcome $Y \subseteq X_f$, define $U^f(Y) = u^f(\tau(Y), t)$, where t_ω is the transfer associated with trade ω.[11] Since prices are continuous, firms might be indifferent between certain outcomes. We therefore define the *choice correspondence* $C^f : \mathcal{P}(X_f) \rightrightarrows \mathcal{P}(X_f)$ by

$$C^f(Y) = \arg\max_{\text{outcomes } Z \subseteq Y} U^f(Z).$$

[10] We write $\mathcal{P}(Z)$ for the power set of a set Z.

[11] Formally, we write

$$t_\omega = \begin{cases} 0 & \text{if } \omega \notin \tau(Y) \\ p_\omega & \text{if } (\omega, p_\omega) \in Y_{f\to} \\ -p_\omega & \text{if } (\omega, p_\omega) \in Y_{\to f} \end{cases}.$$

2.3 Competitive equilibrium

In a competitive equilibrium, firms act as price-takers and the market for each trade clears—either a trade is demanded (at the specified price) by both the buyer and the seller or it is demanded by neither. As in Hatfield et al. [2013], in order to fully specify a competitive equilibrium, we need to assign prices to all trades, including ones that are not realized.

Definition 1. An arrangement $[\Xi; p]$ is a *competitive equilibrium* if $\Xi_f \in D^f(p_{\Omega_f})$ for all f.

As interchangeable trades with different counterparties can be priced differently, our competitive equilibria have personalized prices (as in Hatfield et al. [2013]).[12] We call an outcome A a *competitive equilibrium outcome* if $A = \kappa([\Xi; p])$ for some competitive equilibrium $[\Xi; p]$.

3 DISTORTIONARY FRICTIONS

In our model, firms may value transfers from different trades differently, so that a unit of t_ω might be worth less to the firm than a unit of $t_{\omega'}$.[13] This feature allows our model to capture (in a reduced form) distortionary frictions, such as variable transaction taxes, bargaining costs, and certain forms of financial market incompleteness. This section illustrates exactly how our model can capture these distortionary frictions and how they in turn affect competitive equilibria.

3.1 Transaction taxes

Suppose, for example, that λ proportion of any transfer must be paid to the government. We assume that the recipient of the transfer pays the proportional transaction tax—this assumption is without loss of generality. Thus, the net transfer received or paid by a firm for a trade ω is

$$\widetilde{t}_\omega = \begin{cases} (1-\lambda)t_\omega & \text{if } t_\omega \geq 0 \\ t_\omega & \text{if } t_\omega < 0 \end{cases},$$

where t_ω is the gross transfer. Hence, when $t_\omega \geq 0$, the firm is a recipient of the transfer and receives $(1-\lambda)t_\omega$; when $t_\omega < 0$, the firm is a payer and pays t_ω in full. As a result, if firm f has quasilinear preferences and valuation $v^f : \mathcal{P}(X_f) \to \mathbb{R} \cup \{-\infty\}$, then the utility function u^f is

$$u^f(\Xi) = v^f(\Xi) + \sum_{\omega \in \Omega_f} \widetilde{t}_\omega.$$

When $\lambda < 1$ and $v^f(\varnothing) \in \mathbb{R}$, the utility function u^f satisfies our conditions on preferences (i.e., it is continuous and satisfies the requisite monotonicity conditions). Note that transaction taxes make utility imperfectly transferable even if preferences are quasilinear.

We can model transaction taxes similarly even in the presence of income effects. If firm f has utility function $\widehat{u}^f$ before taxes, then the net-of-tax utility function is

$$u^f(\Xi, t) = \widehat{u}^f\left(\Xi, \widetilde{t}\right).$$

More generally, our framework can capture non-linear transaction taxes and subsidies. Suppose that $\Lambda_\omega(|p_\omega|)$ tax must be paid on a transfer of size $|p_\omega|$ for trade ω. If firm f has utility function $\widehat{u}^f$ before taxes, then the net-of-tax

[12] For example, trades of the same good with different counterparties can have different prices in a competitive equilibrium.
[13] That is, firms could have different marginal rates of substitution between transfers associated to different trades.

utility function is

$$u^f(\Xi, t) = \widehat{u}^f\left(\Xi, \widetilde{t}\right),$$

where

$$\widetilde{t}_\omega = \begin{cases} t_\omega - \Lambda_\omega(t_\omega) & \text{if } t_\omega \geq 0 \\ t_\omega & \text{if } t_\omega < 0 \end{cases}.$$

The case of $\Lambda_\omega(|p_\omega|) = \lambda|p_\omega|$ recovers the proportional transaction tax discussed above. When marginal tax rates are strictly less than one[14] and $\widehat{u}^f$ is continuous and satisfies the requisite monotonicity properties, u^f is continuous and satisfies the requisite monotonicity properties as well. It is straightforward to extend the definition of $\widetilde{t}$ to capture transaction taxes that depend on the directions of transfers.

3.2 Bargaining costs and incomplete financial markets

There are at least two more interesting distortionary frictions that can sometimes be modeled as transaction taxes.

First, surplus might be lost during negotiation. In a reduced form, bargaining costs can be modeled as transaction taxes [Galichon et al., 2018], and hence fit neatly into the framework described in Section 3.1.

Second, financial markets might be imperfect or otherwise incomplete. For example, suppose that firms pay for goods in trade credit, which is paid off in cash after goods are exchanged. In the absence of risk aversion, uninsurable idiosyncratic default risk can also be modeled as a transaction tax.[15] Formally, the possibility that firm f defaults with (subjective) probability ρ can be modeled as losing ρ proportion of any payment made by f. Our model can still capture uninsurable idiosyncratic default risk in the presence of risk aversion, but not using the transaction tax framework. More generally, our model can capture settings in which firms disagree about the relative values of different forms of transfer due to the incompleteness of the financial market.[16]

3.3 Leading examples

We now illustrate how distortionary frictions can affect competitive equilibria. We focus on proportional transaction taxes (with $\lambda = 10\%$) for the sake of simplicity, but in light of the discussion of Section 3.2, we could instead incorporate bargaining costs or incomplete markets.

The first example considers a cyclic economy in which firms have quasilinear preferences and transaction taxes are incorporated using the framework described in Section 3.1. We show that equilibria can be Pareto-comparable.

Example 1 (Cyclic economy). There is a proportional transaction tax on all transfers of $\lambda = 10\%$. As depicted in Figure 2(a), there are two firms, f_1 and f_2, which interact via two trades. The firms share the same utility function

$$u^{f_i}(\Xi, t) = v(\Xi) + \sum_{\omega \in \Omega_{f_i}} \widetilde{t}_\omega,$$

[14] Formally, we require that Λ_ω is continuous, $\Lambda_\omega(0) = 0$, and $x_2 - \Lambda_\omega(x_2) < x_1 - \Lambda_\omega(x_1)$ for all $x_1 > x_2 > 0$.

[15] As Jagadeesan [2017] points out, our model cannot capture settings with imperfectly-insurable default risk in which firms partially finance purchases with trade credit and partially pay in cash.

[16] For example, firms might prefer one type of transfer over another if trades are priced in different currencies. The presence of multiple currencies with common exchange rates does not distort markets *per se*. On the other hand, uninsurable risk or transaction costs associated with currency conversion can be modeled as variable transaction costs.

(a) Trades in Examples 1 and 3. (b) Trades in Example 2.

Fig. 2. **Trades in Examples 1, 2, and 3.** Arrows point from sellers to buyers.

where the valuation v is defined by

$$v(\varnothing) = 0$$
$$v(\{\zeta, \psi\}) = 10$$
$$v(\{\zeta\}) = v(\{\psi\}) = -\infty.$$

There are two sets of trades that can be supported in competitive equilibria: $\varnothing$ and $\{\zeta, \psi\}$. For example, the arrangement $[\{\zeta, \psi\}; p]$ is a competitive equilibrium if $-100 \leq p_\zeta = p_\psi \leq 100$, and the arrangement $[\varnothing; p]$ is a competitive equilibrium if $p_\zeta = p_\psi \geq 100$ or $p_\zeta = p_\psi \leq -100$.[17]

Note that there are Pareto-comparable competitive equilibria: both f_1 and f_2 strictly prefer $[\{\zeta, \psi\}; (0, 0)]$ over any other competitive equilibrium with $p_\zeta = p_\psi$. As pointed out by Hart [1975], the existence of Pareto-comparable equilibria suggests that equilibria are constrained suboptimal.The competitive equilibria of the form $[\{\zeta, \psi\}; p]$ and $[\varnothing; p]$ with $p_\zeta = p_\psi \neq 0$ are constrained Pareto-inefficient.

In contrast, by the First Welfare Theorem, competitive equilibria cannot be Pareto-comparable in economies without transaction taxes (see Supplementary Appendix F).

The second example shows that adding an outside option for f_1 to Example 1 can shut down trade between f_1 and f_2. The fact that enlarging the market can harm all firms suggests that equilibria are constrained suboptimal in the enlarged market [Hart, 1975]. The constrained suboptimality is due to pecuniary externalities. In the context of Examples 1 and 2, adding an outside option can cause prices to become extreme, inducing heavy trading losses (due to taxes) that shut down the market. In contrast, in economies without transaction taxes, adding an outside option can only affect which other trades are realized if the outside option is used in equilibrium (see Supplementary Appendix F).

Example 2 (Cyclic economy with an outside trade). As depicted in Figure 2(b), there are three firms, f_1, f_2, and f_3, which interact via three trades. The firms' utility functions are

$$u^{f_i}(\Xi, t) = v^{f_i}(\Xi) + \sum_{\omega \in \Omega_{f_i}} \widetilde{t}_\omega,$$

[17] In general, $[\{\zeta, \psi\}; p]$ is a competitive equilibrium if and only if

$$\min\{p_\zeta, 0.9p_\zeta\} + \min\{-p_\psi, -0.9p_\psi\} \geq -10 \text{ and } \min\{-p_\zeta, -0.9p_\zeta\} + \min\{p_\psi, 0.9p_\psi\} \geq -10.$$

Similarly, $[\varnothing; p]$ is a competitive equilibrium if and only if

$$\min\{p_\zeta, 0.9p_\zeta\} + \min\{-p_\psi, -0.9p_\psi\} \leq -10 \text{ and } \min\{-p_\zeta, -0.9p_\zeta\} + \min\{p_\psi, 0.9p_\psi\} \leq -10.$$

where v^{f_i} is the valuation of firm f_i. We let $v^{f_i}(\varnothing) = 0$ for all firms. Extending Example 1, firm f_1's valuation is defined by

$$v^{f_1}(\{\zeta, \psi\}) = v^{f_1}(\{\zeta', \psi\}) = 10$$
$$v^{f_1}(\{\zeta\}) = v^{f_1}(\{\zeta'\}) = v^{f_1}(\{\psi\}) = -\infty$$
$$v^{f_1}(\{\zeta, \zeta'\}) = v^{f_1}(\{\zeta, \zeta', \psi\}) = -\infty.$$

As in Example 1, firm f_2's valuation is defined by

$$v^{f_2}(\{\zeta, \psi\}) = 10$$
$$v^{f_2}(\{\zeta\}) = v^{f_2}(\{\psi\}) = -\infty.$$

Firm f_3's valuation is defined by $v^{f_3}(\{\zeta'\}) = 300$.

Trade ζ' cannot be realized in equilibrium due to the technological constraints of f_1 and f_2. Thus, we must have $p_{\zeta'} \geq 300$ in any competitive equilibrium, as f_3 must weakly prefer $\varnothing$ over $\{\zeta'\}$ in equilibrium. For trade to occur, f_1 must prefer ζ over ζ', and so we must have $p_\zeta \geq 300$. With 10% taxation and $p_\zeta \geq 300$, at least \$30 in taxes must be paid if ζ is traded. But \$30 exceeds the gains from trade between f_1 and f_2, and so trade cannot occur in any competitive equilibrium. An example of a competitive equilibrium is $[\varnothing; p]$, where $p_\zeta = p_\psi = p_{\zeta'} = 350$. Thus, introducing an outside option that is not used can shut down a market when there are distortionary transaction taxes.[18]

4 EXISTENCE OF COMPETITIVE EQUILIBRIA

Due to the presence of indivisibilities, competitive equilibria need not exist in our model without further assumptions on preferences. Our key condition is full substitutability [Hatfield et al., 2013].[19] Intuitively, full substitutability requires that every firm views its upstream trades as gross substitutes for each other, its downstream trades as gross substitutes for each other, and its upstream and downstream trades as gross complements for one another.[20]

Assumption 1 (Full substitutability—FS, Hatfield et al., 2013)**.** For all $f \in F$ and all finite sets of contracts $Y, Y' \subseteq X_f$ with $Y_{f\to} \subseteq Y'_{f\to}$ and $Y_{\to f} \supseteq Y'_{\to f}$, we have

$$Z' \cap Y_{f\to} \subseteq Z \quad \text{and} \quad Z \cap Y'_{\to f} \subseteq Z'$$

if $C^f(Y) = \{Z\}$ and $C^f(Y') = \{Z'\}$.

Full substitutability requires that an expansion in the set of upstream (resp. downstream) options and a contraction in the set of downstream (resp. upstream) options only makes upstream (resp. downstream) contracts less attractive and downstream (resp. upstream) contracts more attractive for the firm. Technically, we impose this condition only on sets of contracts from which the firm's utility-maximizing choice is unique. In Appendix B, we show that full substitutability is equivalent to a substitutability property that deals with indifferences more explicitly.[21]

[18]However, firms f_1 and f_2 trade ζ and ψ in every core outcome, and the core is non-empty. Indeed, the outside option does not disrupt trade in the core because f_1 and f_3 cannot form a core block without breaking off *all* trade with f_2.
[19]Full substitutability generalizes gross substitutability [Gul and Stacchetti, 1999, Kelso and Crawford, 1982]. We use the *choice-language full substitutability* condition introduced by Hatfield et al. [2013], which extends the same-side substitutability and cross-side complementarity conditions of Ostrovsky [2008] to choice correspondences.
[20]Section IIB in Hatfield et al. [2013] provides a detailed discussion of the full substitutability condition in the context of trading networks with transferable utility. For example, full substitutability rules out complementarities between inputs.
[21]Several of our proofs use the equivalence between our two definitions of full substitutability.

Hatfield et al. [2013] also need to assume that firms' valuations of sets of trades are never $+\infty$ to ensure that competitive equilibria exist. We impose a similar condition that is adapted to settings in which utility is not perfectly transferable. Our condition requires that compensating variations of moving from autarky to trade are bounded below—i.e., that no set of trades is so desirable that it is preferred to autarky at any level of total transfers. This condition is satisfied in transferable utility economies when valuations are bounded above.

Assumption 2 (Bounded compensating variations—BCV)**.** For all $f \in F$, we have

$$\inf_{u^f(\Xi,t)\geq 0} \sum_{\omega\in\Omega_f} t_\omega > -\infty.$$

BCV requires that net transfers $\sum_{\omega\in\Omega_f} t_\omega$ are bounded below over all transfer vectors t that are acceptable alongside some set of trades Ξ. If a firm is willing to accept trades alongside arbitrary negative net transfers, then BCV fails. BCV is a weak assumption that is likely to be satisfied in any real-world economy. In particular, BCV is satisfied in Examples 1 and 2. Note that BCV allows for technological constraints, in that it permits sets of trades to be so undesirable to a firm that they remain worse than autarky regardless of how much the firm receives in transfers.

FS and BCV together ensure that competitive equilibria exist in trading networks. In Supplementary Appendix F, we show by example that competitive equilibria may not exist if BCV is not satisfied.

Theorem 1. *Under FS and BCV, competitive equilibria exist.*

To prove Theorem 1, we construct a modified economy by giving every firm options to execute all trades at a very undesirable price. Specifically, we give every firm the option to make any trade by paying a cost of

$$\Pi > -\sum_{f\in F} \inf_{u^f(\Xi,t)\geq 0} \sum_{\omega\in\Omega_f} t_\omega. \tag{1}$$

The penalty Π can be chosen to be finite due to BCV. Hence, firms have bounded willingness to pay for any contract in the modified economy, in a sense that we make precise in Section 5.2.[22] We discretize prices and use a generalized Deferred Acceptance algorithm [Fleiner et al., 2018b, Hatfield and Kominers, 2012, Ostrovsky, 2008] to show the existence of approximate equilibria in the modified economy. A limiting argument yields the existence of competitive equilibria in the modified economy, as in Crawford and Knoer [1981] and Kelso and Crawford [1982]. The fact that Π is sufficiently large (i.e., (1) is satisfied) ensures that we actually obtain competitive equilibria in the original economy.[23]

5 RELATIONSHIPS BETWEEN COMPETITIVE EQUILIBRIA AND COOPERATIVE SOLUTION CONCEPTS

We now study the relationships between competitive equilibria and cooperative solution concepts from matching theory. Instead of assuming that firms are price-takers, we allow firms to recontract while keeping or dropping existing contracts. We focus on two solution concepts: trail stability and stability.

A key ingredient of any reasonable stability property is individual rationality, which requires that no firm wants to drop any signed contract.

[22]Hatfield et al. [2013] apply a related, but not exactly analogous, transformation in the proof of their existence result (Theorem 1 in Hatfield et al. [2013]). Specifically, Hatfield et al. [2013] give firms both the option to make a trade by paying a cost of Π and the option to dispose of an undesired trade for a cost of Π (for a sufficiently large Π). The Hatfield et al. [2013] approach does not in general preserve full substitutability at the level of generality of our model.
[23]Theorem 1 generalizes Theorem 2 in Kelso and Crawford [1982] and Theorem 1 in Hatfield et al. [2013].

Definition 2 (Hatfield et al., 2013, Roth, 1984)**.** An outcome $A \subseteq X$ is *individually rational* if $A_f \in C^f(A_f)$ for all $f \in F$.

5.1 Trail stability

Trail stability [Fleiner et al., 2018b] is a natural extension of pairwise stability (in the sense of Gale and Shapley [1962]) to trading networks. A trail is a sequence of contracts such that a buyer in one contract is a seller in the next contract. A trail may involve a firm more than once and can begin and end with contracts that involve the same firm.

Definition 3. A sequence of contracts $(x_1, \ldots, x_n)$ is a *trail* if $\mathrm{b}(x_i) = \mathrm{s}(x_{i+1})$ for all $1 \leq i \leq n-1$.

Trail-stable outcomes are immune to sequential deviations called locally blocking trails. A locally blocking trail begins with a firm offering a sale that it wishes to sign given its existing contracts, possibly while dropping some existing contracts. The buyer may accept the offered contract while dropping some of his existing contracts, in which case a locally blocking trail is formed. The buyer may also hold the proposal and offer an additional sale to the original proposer or to another firm. This trail of linked offers continues until a firm accepts an offered contract without having to offer another sale, in which case a locally blocking trail is formed.[24]

Our formal definition of trail stability extends the definition given by Fleiner et al. [2018b] to settings with indifferences.

Definition 4. A trail $(z_1, \ldots, z_n)$ *locally blocks* an outcome A if:

- $A_{f_1} \notin C^{f_1}(A_{f_1} \cup \{z_1\})$, where $f_1 = \mathrm{s}(z_1)$;
- $A_{f_{i+1}} \notin C^{f_{i+1}}(A_{f_{i+1}} \cup \{z_i, z_{i+1}\})$ for $1 \leq i \leq n-1$, where $f_{i+1} = \mathrm{b}(z_i) = \mathrm{s}(z_{i+1})$; and
- $A_{f_{n+1}} \notin C^{f_{n+1}}(A_{f_{n+1}} \cup \{z_n\})$, where $f_{n+1} = \mathrm{b}(z_n)$.

Such a trail is called a *locally blocking trail.* An outcome is *trail-stable* if it is individually rational and there is no locally blocking trail.

A trail locally blocks an individually rational outcome if, at every point at which a trail passes through a firm, the firm would like some of the contracts that are available to it locally in the trail (when given access to the existing contracts). Intuitively, one should think of contracts in a locally blocking trail as being proposed by telephone by a manager at one firm to a manager at another [Fleiner et al., 2018b]. If the sequence of phone conversations returns to a firm, a different manager (e.g., one from another division) picks up the phone and considers the latest offer. Her decisions are independent of the offers received and made by the first manager. Any manager's unilateral decision to accept an offered contract completes a locally blocking trail.

5.2 A cooperative interpretation of competitive equilibria

The main result of this section provides a cooperative interpretation of competitive equilibrium that holds even in the presence of frictions.

Theorem 2. *Every competitive equilibrium outcome is trail-stable.*

Theorem 2 implies that price-taking firms cannot improve upon a market equilibrium by deviating along trails. In light of Theorem 2, any prediction of our model that holds in every trail-stable outcome must hold in every competitive equilibrium outcome.

[24]Note that locally blocking trails can also develop in the reverse direction, with firms offering to buy instead of to sell.

To see the intuition behind Theorem 2, consider any competitive equilibrium and any trail. In order for sellers to want to propose the contracts in the trail, the prices of all trades in the trail must be greater than their equilibrium prices. But the last buyer will only accept an offer if the price in the last contract is lower than the equilibrium price of the corresponding trade. Hence, there cannot be any locally blocking trails. Theorem 2 does not require any assumptions beyond the monotonicity of utility in transfers. As we will show in Section 5.3, competitive equilibria do not satisfy stronger cooperative solution concepts in the presence of frictions.

In light of Theorem 2, the conclusions of Examples 1 and 2 hold for trail-stable outcomes as well. Thus, trail-stable outcomes can suffer from constrained suboptimality due to pecuniary externalities despite being defined cooperatively.[25]

Theorems 1 and 2 yield sufficient conditions for the existence of trail-stable outcomes.[26]

Corollary 1. *Under FS and BCV, trail-stable outcomes exist.*

5.3 Stability

Groups of firms might still be able to commit to recontracting at a trail-stable outcome. Stability rules out such recontracting opportunities, which are called blocks, and may be a more natural solution concept in settings in which firms can coordinate easily.[27] Hatfield et al. [2013] extend the definition of stability to settings with indifferences.

Definition 5 (Hatfield et al., 2013)**.** A non-empty set of contracts $Z \subseteq X \setminus A$ *blocks* A if, for all $f \in F$ and $Y \in C^f(A_f \cup Z_f)$, we have $Z_f \subseteq Y$. An outcome is *stable* if it is individually rational and unblocked.

In a stable outcome, no group of firms can commit to recontracting among themselves while being free to drop any contracts. Unfortunately, competitive equilibria may be unstable in the presence of frictions; moreover, stable outcomes need not even exist.[28] Hence, as Fleiner et al. [2018b] argue, stability may be too stringent of a solution concept in general networks.

Example 2 *continued* (Stable outcomes need not exist in the presence of frictions). There are no stable outcomes in Example 2. Indeed, note that the no-trade outcome is unstable, since it is blocked by trade between f_1 and f_2. Note also that f_1 and f_3 cannot trade in any individually rational outcome due to the technological constraints faced by f_1 and f_2.

On the other hand, any individually rational outcome that involves trade between f_1 and f_2 is blocked by trade between f_1 and f_3. Indeed, note that ζ cannot be traded at any price greater than \$200 in an individually rational outcome, since the social surplus of trade between f_1 and f_2 is only \$20 and making a transfer of at least \$200 requires paying a transaction tax of at least \$20. But the contract $(\zeta', 250)$ blocks any outcome in which ζ is traded at price below \$250.[29]

As noted by Hatfield and Kominers [2012], requiring that the trading network is acyclic—i.e., that it forms a vertical supply chain—helps restore the existence of stable outcomes in settings with discrete, bounded prices. Appendix A

[25] As shown by Blair [1988] and Klaus and Walzl [2009], (pairwise) stable outcomes can suffer from constrained suboptimality even in two-sided many-to-many matching markets.

[26] Corollary 1 is a version of Theorem 1 in Fleiner et al. [2018b]—which generalizes Theorem 1 in Ostrovsky [2008] from supply chains to general networks—for settings with prices that are continuous and potentially unbounded.

[27] See Roth [1984, 1985], Hatfield and Milgrom [2005], Echenique and Oviedo [2006], and Hatfield and Kominers [2012, 2017].

[28] Determining whether a stable outcome exists and determining whether a particular outcome is stable are both computationally intractable problems in trading networks with cycles and discrete contracts [Fleiner, Jankó, Schlotter, and Teytelboym, 2018a]. Trail stability is more natural from a computational perspective—trail-stable outcomes can be found in polynomial time using the generalized Deferred Acceptance algorithm under full substitutability [Fleiner et al., 2018b].

[29] An alternative proof can be given using one of our lifting results (Theorem 3). Indeed, note that the no-trade outcome is not stable. However, any stable outcome must lift to a competitive equilibrium by Theorem 3, and trade does not occur in any competitive equilibrium.

shows that similar logic carries over to our setting, which features unbounded, continuous prices. The underlying reason is that stability and trail stability coincide in acyclic networks, at least under FS, as we show in Appendix A.

Even in trading networks with cycles, under FS, stability actually refines trail stability.[30]

Proposition 1. *Under FS, every stable outcome is trail-stable.*

If FS is not satisfied, then stable outcomes may not be trail-stable (see Supplementary Appendix F).

5.4 Competitive interpretations of trail stability and stability

We now develop competitive interpretations of trail stability and stability. Formally, we say that an outcome A *lifts to a competitive equilibrium* if A is a competitive equilibrium outcome—that is, if A can be supported by competitive equilibrium prices. As an outcome specifies prices for realized trades, the non-trivial part of lifting an outcome to a competitive equilibrium is finding equilibrium prices for unrealized trades.

Hatfield et al. [2013] show by example that stable outcomes do not generally lift to competitive equilibria when FS is not satisfied. Therefore, we maintain FS throughout this section. We first prove a positive result, namely that stable outcomes lift to competitive equilibria under the conditions for the existence of competitive equilibria.[31]

Theorem 3. *Under FS and BCV, stable outcomes lift to competitive equilibria.*

Frictions can cause stable outcomes to fail to exist in general networks as Example 2 shows. Therefore, for many trading networks with frictions, Theorem 3 has no bite. On the other hand, trail-stable outcomes need not lift to competitive equilibria even under FS and BCV, as the following example shows.

Example 3 (Trail-stable outcomes need not lift to competitive equilibria under FS and BCV). As depicted in Figure 2(a), there are two firms, f_1 and f_2, which interact via two trades. The firms share the same utility function

$$u^{f_i}(\Xi, t) = v(\Xi) + \sum_{\omega \in \Omega} t_\omega,$$

where v is as in Example 1. The no-trade outcome is trail-stable but inefficient. However, as utility is transferable, all competitive equilibrium outcomes are efficient. In particular, the no-trade outcome cannot lift to a competitive equilibrium.

In Example 3, both firms face hard technological constraints: they are unwilling to execute any trade individually at any finite price, but would like to complete both trades together. The no-trade outcome is trail-stable because neither the buyer nor the seller is willing to offer to buy or sell a single trade at any finite price.

To ensure that trail-stable outcomes lift to a competitive equilibrium, we impose a different regularity condition than BCV. Intuitively, we require that firms have bounded willingness to pay for any trade.

Assumption 3 (Bounded willingness to pay—BWP)**.** There exists M such that for all firms $f \in F$ and all finite sets of contracts $Y, Z \subseteq X_f$ with $Z \in C^f(Y)$:

- If $(\omega, p_\omega) \in Z_{\to f}$, then $p_\omega < M$.

[30] Proposition 1 is a version of Lemma 5 in Fleiner et al. [2018b] for settings with prices that are continuous and potentially unbounded.

[31] Theorem 3 generalizes Theorem 6 in Hatfield et al. [2013] to trading networks with distortionary frictions and income effects. Stable outcomes exist in acyclic networks even in the presence of frictions, as we show in Appendix A.

- If $(\omega, p_\omega) \in Z_{f\to}$, then $p_\omega > -M$.

BWP requires that no firm is willing to pay more than M for any trade—i.e., no firm is willing to buy any trade at a price more than M or sell any trade at a price less than $-M$. BWP rules out certain technological constraints, including those that are permitted under BCV and by Hatfield et al. [2013]. In particular, BWP does not allow a firm to require a particular input in order to produce a particular output, as such constraints make a firm willing to pay arbitrarily high prices for the input if the firm is able to procure arbitrarily high prices for the output. However, BWP allows for capacity constraints, since they never make trades desirable at extremely unfavorable prices.

BWP helps ensure that trail-stable outcome lift to competitive equilibria.[32]

Theorem 4. *Under FS and BWP, trail-stable outcomes lift to competitive equilibria.*

Theorem 4 provides a competitive interpretation of trail stability: any trail-stable outcome is consistent with price-taking equilibrium behavior by all firms (at least under FS and BWP). In light of Theorem 4, any prediction of our model that holds in every competitive equilibrium must hold in every trail-stable outcome.

Theorems 2 and 4 imply that competitive equilibria are essentially equivalent to trail-stable outcomes in our model.[33]

Corollary 2. *Under FS and BWP, competitive equilibrium outcomes and trail-stable outcomes exist and coincide.*

Corollary 2 provides competitive foundations for trail stability and cooperative foundations for competitive equilibrium: the assumption that firms coordinate on a trail-stable outcome (as in a thin market) produces the same predictions as the assumption that firms take prices in equilibrium (as in a thick market). Therefore, equilibrium analysis can be performed using scale-independent solution concepts, even in markets with frictions.

6 COMPLETE MARKETS

Trail-stable and competitive equilibrium outcomes might be constrained Pareto-inefficient in the presence of proportional transaction taxes or other distortionary frictions (see Examples 1 and 2). In the presence of transaction taxes, for example, all firms find reductions in outgoing payments more desirable than equal increases in incoming payments. As a result, firms have different marginal rates of substitution between forms of transfer, unlike in settings with complete financial markets.

Since we do not explicitly model financial markets, we formalize "equalization of marginal rates of substitution between forms of transfer" as "indifference between all forms of transfer" in defining our market completeness condition. Intuitively, if the firms share the same marginal rates of substitution between forms of transfer, then transfers can be redenominated so that the marginal rates of substitution become 1. The possibility of redenomination is precisely why, for example, the presence of multiple currencies does not cause market incompleteness *per se*.

Assumption 4 (Complete markets—CM)**.** For all $f \in F$ and $t, t' \in \mathbb{R}^{\Omega_f}$ with $\sum_{\omega \in \Omega_f} t_\omega = \sum_{\omega \in \Omega_f} t'_\omega$, we have $u^f(\Xi, t) = u^f(\Xi, t')$ for all $\Xi \subseteq \Omega_f$.

Recall that, in Examples 1 and 2, paying one unit is more costly for firms than receiving one unit (due to transaction taxes). Assumption CM rules out these differences in the costs of transfers and requires that firms only care about the

[32] Despite the fact that BWP is not satisfied in Examples 1 and 2, trail-stable outcomes lift to competitive equilibria in both examples. Thus, BWP is sufficient but not necessary for trail-stable outcomes to lift to competitive equilibria.

[33] To derive Corollary 2 formally, we need to establish that competitive equilibria exist under FS and BWP, as Theorem D.1 in the Supplementary Appendix shows.

total transfers that they receive or pay. Therefore, CM requires that a unit of transfer for one trade be equivalent to a unit of transfer for any other trade.[34] Under CM, we can write $u^f(\Xi, t) = \overline{u}^f(\Xi, q)$, where $q = \sum_{\omega \in \Omega} t_\omega$ is the total or net transfer. Note that while CM rules out distortionary frictions—such as variable sales taxes, bargaining costs, and incompleteness in financial markets—fixed transaction costs and income effects are still permitted under CM.[35]

We begin our analysis of trading networks with complete markets by recalling the definition of strong group stability, which is the most stringent stability property from the literature on matching with contracts. A strongly group stable outcome is immune to blocks by coalitions of firms that can commit to better, new contracts and maintain any existing contracts with each other and with firms outside the blocking coalition.

Definition 6 (Hatfield et al., 2013)**.** An outcome A is *strongly unblocked* if there do not exist a non-empty set $Z \subseteq X \smallsetminus A$ and sets of contracts $Y^f \subseteq A_f \cup Z_f$ for $f \in F$ such that $Y^f \supseteq Z_f$ and $U^f\left(Y^f\right) > U^f\left(A_f\right)$ for all $f \in F$ with $Z_f \neq \varnothing$. An outcome is *strongly group stable* if it is individually rational and strongly unblocked.

In Definition 6, Y^f is the set of contracts that f signs in the block. Note that Y^f need not be f's best choice from the set of available contracts. In particular, strong group stability rules out blocks in which firms only improve their utility by selecting all of the blocking contracts. Hence, as Hatfield et al. [2013] show, strong group stability is stronger than stability. Moreover, Y^f can contain existing contracts that the counterparties no longer want. In particular, strong group stability rules out blocks in which different members of the blocking coalition can make selections from the set of existing contracts that are incompatible with one another or involve firms outside the coalition. Hence, strong group stability also refines properties such as (strong) setwise stability [Echenique and Oviedo, 2006, Klaus and Walzl, 2009] and the core.[36]

It appears extremely unlikely that firms would rationally deviate from a strongly group stable outcome, and competitive equilibria are strongly group stable in complete markets.[37]

Theorem 5 (First Welfare Theorem)**.** *Under CM, competitive equilibrium outcomes are strongly group stable.*

Since strongly group stable outcomes are stable and in the core, Theorem 5 implies that competitive equilibrium outcomes are stable and in the core in complete markets. As core outcomes are Pareto-efficient, Theorem 5 is a version of the First Welfare Theorem [Debreu, 1951].

Combining Theorem 5 with our results on markets with frictions, we obtain that all of the solution concepts described in this paper are essentially equivalent in complete markets (under FS and BWP).

Corollary 3. *Under FS, BWP, and CM, competitive equilibrium outcomes, strongly group stable outcomes, stable outcomes, and trail-stable outcomes exist and coincide.*

When markets are complete, we can also restate BCV more simply using only total transfers, since firms are indifferent regarding the sources of transfers.

[34] In particular, any transferable utility economy satisfies CM.
[35] When assumed jointly, FS and CM restrict income effects for certain agents. In particular, intermediaries that buy or sell more than one trade cannot experience income effects. However, firms that act only as buyers or only as sellers can experience limited income effects. Moreover, firms that buy or sell only one trade at a time can experience arbitrary income effects. In incomplete markets, on the other hand, all firms can experience income effects even under FS.
[36] As pointed out by Hatfield et al. [2013], strong group stability also refines strong stability [Hatfield and Kominers, 2015], and group stability [Konishi and Ünver, 2006].
[37] Theorem 5 extends Theorem 5 in Hatfield et al. [2013], which shows that competitive equilibrium outcomes are strongly group stable, to settings with income effects or risk aversion.

Assumption 2′ (Bounded CVs under CM—BCV-CM)**.** For all $f \in F$, we have

$$\inf_{\overline{u}^f(\Xi,q)\geq\overline{u}^f(\varnothing,0)} q > -\infty.$$

In complete markets, under FS and BCV-CM, we obtain an equivalence between competitive equilibrium and (strong group) stability.[38]

Corollary 4. *Under FS, BCV-CM, and CM, competitive equilibrium outcomes, strongly group stable outcomes, and stable outcomes exist and coincide.*[39]

7 CONCLUSION

This paper develops a model of differentiated markets with frictions based on matching in trading networks. Competitive equilibria exist in our model when trades are fully substitutable (and mild regularity conditions are satisfied) but may be inefficient. In the presence of frictions, competitive equilibria may be unstable but still essentially coincide with trail-stable outcomes. In complete markets, on the other hand, competitive equilibria are essentially equivalent to stable outcomes and trail-stable outcomes, even in the presence of income effects. Our results provide new cooperative foundations for competitive equilibrium and competitive foundations for trail stability that apply in thin markets and in markets with frictions.

We leave three theoretical open questions. First, can the complete markets condition be relaxed while still guaranteeing that competitive equilibrium outcomes are stable? Second, to what extent can the condition that firms have bounded willingness to pay for trades be relaxed while still ensuring that trail-stable outcomes lift to competitive equilibria? Third, can externalities or peer effects (as analyzed by Pycia [2012], Pycia and Yenmez [2017], and Rostek and Yoder [2017]) be incorporated into our analysis?

REFERENCES

Atila Abdulkadiroğlu and Tayfun Sönmez. 1999. House allocation with existing tenants. *Journal of Economic Theory* 88, 2 (1999), 233–260.

Hiroyuki Adachi. 2017. Stable matchings and fixed points in trading networks: A note. *Economics Letters* 156 (2017), 65–67.

Tommy Andersson and Lars-Gunnar Svensson. 2014. Non-Manipulable House Allocation With Rent Control. *Econometrica* 82, 2 (2014), 507–539.

Kenneth J. Arrow. 1953. Le rôle des valeurs boursières pour la répartition la meilleure des risques. *Colloques Internationaux du Centre National de la Recherche Scientifique* 11 (1953), 41–47. Translated in "The Role of Securities in the Optimal Allocation of Risk Bearing" (1964), *Review of Economic Studies 31*(2), 91–96.

Orhan Aygün and Tayfun Sönmez. 2013. Matching with Contracts: Comment. *American Economic Review* 103, 5 (2013), 2050–2051.

Elizabeth Baldwin and Payk Klemperer. Forthcoming, 2018. Understanding preferences: "Demand types," and the existence of equilibrium with indivisibilities. *Econometrica* (Forthcoming, 2018).

Charles Blair. 1988. The Lattice Structure of the Set of Stable Matchings with Multiple Partners. *Mathematics of Operations Research* 13, 4 (1988), 619–628.

Laurens Cherchye, Thomas Demuynck, Bram De Rock, and Frederic Vermeulen. 2017. Household Consumption When the Marriage Is Stable. *American Economic Review* 107, 6 (2017), 1507–1534. https://doi.org/10.1257/aer.20151413

Pierre-André Chiappori, Sonia Oreffice, and Climent Quintana-Domeque. 2012. Fatter attraction: Anthropometric and socioeconomic matching on the marriage market. *Journal of Political Economy* 120, 4 (2012), 659–695.

Pierre-André Chiappori, Bernard Salanié, and Yoram Weiss. 2017. Partner choice, investment in children, and the marital college premium. *American Economic Review* 107, 8 (2017), 2109–2167.

Eugene Choo and Aloysius Siow. 2006. Who marries whom and why. *Journal of Political Economy* 114, 1 (2006), 175–201.

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[38] Corollary 4 generalizes Theorem 5 and the first part of Theorem 9 in Hatfield et al. [2013], which deals with transferable utility economies.

[39] A similar argument to the proof of Corollary 4 shows that chain stability (in the sense of Hatfield et al. [2018]) coincides with (strong group) stability, trail stability, and competitive equilibrium under FS, BCV-CM, and CM. Indeed, the proof of Proposition 1 shows that chain-stable outcomes are trail-stable under FS, and the proof of Theorem 3 shows that chain-stable outcomes lift to competitive equilibria under FS and BCV. Hatfield et al. [2018] and Ikebe et al. [2015] prove similar equivalence results for transferable utility economies. However, competitive equilibria are not chain-stable in the presence of distortionary frictions. For instance, there are no chain-stable outcomes in Example 2.

Vincent P. Crawford and Elsie Marie Knoer. 1981. Job Matching with Heterogeneous Firms and Workers. *Econometrica* 49, 2 (1981), 437–450.

Gerard Debreu. 1951. The coefficient of resource utilization. *Econometrica* 19, 3 (1951), 273–292.

Jacques H. Drèze. 1975. Existence of an exchange equilibrium under price rigidities. *International Economic Review* 16, 2 (1975), 301–320.

Arnaud Dupuy and Alfred Galichon. 2014. Personality traits and the marriage market. *Journal of Political Economy* 122, 6 (2014), 1271–1319.

Arnaud Dupuy, Alfred Galichon, Sonia Jaffe, and Scott Duke Kominers. 2017. Taxation in Matching Markets. (2017). Mimeo.

Federico Echenique and Jorge Oviedo. 2006. A theory of stability in many-to-many matching markets. *Theoretical Economics* 1, 1 (2006), 233–273.

Francis Ysidro Edgeworth. 1881. *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences.* C. Kegan Paul & Company.

Tamás Fleiner. 2003. A Fixed-Point Approach to Stable Matchings and Some Applications. *Mathematics of Operations Research* 28, 1 (2003), 103–126. https://doi.org/stability,matching

Tamás Fleiner, Zsuzsanna Jankó, Ildikó Schlotter, and Alexander Teytelboym. 2018a. *Complexity of Stability in Trading Networks.* Mimeo.

Tamás Fleiner, Zsuzsanna Jankó, Akihisa Tamura, and Alexander Teytelboym. 2018b. *Trading Networks with Bilateral Contracts.* Mimeo.

Jeremy T. Fox. 2010. Identification in matching games. *Quantitative Economics* 1, 2 (2010), 203–254.

Jeremy T. Fox. 2017. Specifying a Structural Matching Game of Trading Networks with Transferable Utility. *American Economic Review* 107, 5 (2017), 256–260. https://doi.org/10.1257/aer.p20171114

Jeremy T. Fox. 2018. Estimating matching games with transfers. *Quantitative Economics* 9, 1 (2018), 1–38.

Jeremy T. Fox and Patrick Bajari. 2013. Measuring the efficiency of an FCC spectrum auction. *American Economic Journal: Microeconomics* 5, 1 (2013), 100–146.

Jeremy T. Fox, David H. Hsu, and Chenyu Yang. 2018. Unobserved Heterogeneity in Matching Games with an Application to Venture Capital. *Journal of Political Economy* 126, 4 (2018), 1339–1373.

D. Gale and L. S. Shapley. 1962. College Admissions and the Stability of Marriage. *Amer. Math. Monthly* 69, 1 (1962), 9–15.

Alfred Galichon, Scott Duke Kominers, and Simon Weber. Forthcoming, 2018. Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility. *Journal of Political Economy* (Forthcoming, 2018).

Alfred Galichon and Bernard Salanié. 2014. *Cupid's Invisible Hand: Social Surplus and Identification in Matching Models.* Mimeo.

Faruk Gul and Ennio Stacchetti. 1999. Walrasian Equilibrium with Gross Substitutes. *Journal of Economic Theory* 87, 1 (1999), 95–124.

Oliver D. Hart. 1975. On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory* 11, 3 (1975), 418–443.

John William Hatfield and Scott Duke Kominers. 2012. Matching in Networks with Bilateral Contracts. *American Economic Journal: Microeconomics* 4, 1 (2012), 176–208.

John W. Hatfield and Scott Duke Kominers. 2015. Multilateral Matching. *Journal of Economic Theory* 156 (2015), 175–206.

John William Hatfield and Scott Duke Kominers. 2017. Contract design and stability in many-to-many matching. *Games and Economic Behavior* 101 (2017), 78–97.

John William Hatfield, Scott Duke Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp. 2013. Stability and Competitive Equilibrium in Trading Networks. *Journal of Political Economy* 121, 5 (2013), 966–1005.

John William Hatfield, Scott Duke Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp. 2018. *Chain Stability in Trading Networks.* Mimeo.

John William Hatfield, Scott Duke Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp. Forthcoming, 2019. Full Substitutability. *Theoretical Economics* (Forthcoming, 2019).

John W. Hatfield and Paul Milgrom. 2005. Matching with Contracts. *American Economic Review* 95, 4 (2005), 913–935.

John William Hatfield, Charles R. Plott, and Tomomi Tanaka. 2012. Understanding Price Controls and Nonprice Competition with Matching Theory. *American Economic Review* 102, 3 (2012), 371–375.

John William Hatfield, Charles R. Plott, and Tomomi Tanaka. 2016. Price Controls, Non-Price Quality Competition, and the Nonexistence of Competitive Equilibrium. *Games and Economic Behavior* 99 (2016), 134–163.

P. Jean-Jacques Herings. 2015. Equilibrium and matching under price controls. (2015). Mimeo.

Christian A. L. Hilber and Teemu Lyytikäinen. 2017. Transfer Taxes and Household Mobility: Distortion on the Housing or Labor Market? *Journal of Urban Economics* 101 (2017), 57–73.

Yoshiko T. Ikebe, Yosuke Sekiguchi, Akiyoshi Shioura, and Akihisa Tamura. 2015. Stability and competitive equilibria in multi-unit trading networks with discrete concave utility functions. *Japan Journal of Industrial and Applied Mathematics* 32, 2 (2015), 373–410.

Ravi Jagadeesan. 2017. Complementary inputs and the existence of stable outcomes in large trading networks. (2017). Mimeo.

Alexander S. Kelso and Vincent P. Crawford. 1982. Job Matching, Coalition Formation, and Gross Substitutes. *Econometrica* 50, 6 (1982), 1483–1504.

Bettina Klaus and Markus Walzl. 2009. Stable many-to-many matchings with contracts. *Journal of Mathematical Economics* 45, 7–8 (2009), 422–434.

Hideo Konishi and M. Utku Ünver. 2006. Credible group stability in many-to-many matching problems. *Journal of Economic Theory* 129, 1 (2006), 57–80.

Michael Ostrovsky. 2008. Stability in Supply Chain Networks. *American Economic Review* 98, 3 (2008), 897–923.

Marek Pycia. 2012. Stability and Preference Alignment in Matching and Coalition Formation. *Econometrica* 80, 1 (2012), 323–362.

Marek Pycia and M. Bumin Yenmez. 2017. *Matching with Externalities.* Mimeo.

Marzena Rostek and Nathan Yoder. 2017. Matching with multilateral contracts. (2017). Mimeo.

Alvin E. Roth. 1984. Stability and Polarization of Interests in Job Matching. *Econometrica* 52, 1 (1984), 47–58.

Alvin E Roth. 1985. Conflict and coincidence of interest in job matching: Some new results and open questions. *Mathematics of Operations Research* 10, 3 (1985), 379–389.

Lloyd Shapley and Herbert Scarf. 1974. On cores and indivisibility. *Journal of Mathematical Economics* 1, 1 (1974), 23–37.

Lloyd S. Shapley and Martin Shubik. 1972. The Assignment Game I: The Core. *International Journal of Game Theory* 1, 1 (1972), 111–130.

Alfred Tarski. 1955. A lattice-theoretical fixpoint theorem and its applications. *Pacific J. Math.* 5, 2 (1955), 285–309.

Alexander Westkamp. 2010. Market structure and matching with contracts. *Journal of Economic Theory* 145, 5 (2010), 1724–1738.

Appendix

A ACYCLIC NETWORKS

In acyclic trading networks, or supply chains, no firm can be simultaneously upstream and downstream from another firm even via intermediaries [Hatfield and Kominers, 2012, Ostrovsky, 2008, Westkamp, 2010].

Assumption A.1 (Acyclicity—AC)**.** There do not exist $n \geq 1$ and trades $\omega_1, \dots, \omega_n$ such that $\mathrm{s}(\omega_{i+1}) = \mathrm{b}(\omega_i)$ for all $1 \leq i \leq n$, where $\omega_{n+1} = \omega_1$.

As shown by Ostrovsky [2008] and Hatfield and Kominers [2012], imposing acyclicity can help ensure the existence of stable outcomes in trading networks with frictions. In acyclic networks, trail stability is tautologically equivalent to chain stability (in the sense of Ostrovsky [2008]). The following lemma relates stability and trail stability in acyclic networks.

Lemma A.1. *Under FS and AC, every trail-stable outcome is stable.*

Proposition 1 and Lemma A.1 imply that trail-stable and stable outcomes coincide in supply chains under FS, yielding a continuous-price version of Theorem 7 in Hatfield and Kominers [2012].

We now derive several results concerning acyclic networks as corollaries of our results on general trading networks with frictions. First, competitive equilibria are stable under FS and AC (by Theorem 2 and Lemma A.1).

Corollary A.1. *Under FS and AC, every competitive equilibrium outcome is stable.*

Theorem 1 and Corollary A.1 imply that FS, BCV, and AC are together sufficient for the existence of stable outcomes.[40]

Corollary A.2. *Under FS, BCV, and AC, stable outcomes exist.*

In light of Lemma A.1, trail-stable outcomes must lift to competitive equilibria in supply chains under FS and BCV by Theorem 3. Hence, imposing acyclicity allows us to replace BWP with BCV in Theorem 4.

Corollary A.3. *Under FS, BCV, and AC, trail-stable outcomes lift to competitive equilibria.*

B AN EQUIVALENT DEFINITION OF FULL SUBSTITUTABILITY

This appendix states a version of Theorem A.1 in Hatfield et al. [2019]. More precisely, we show that full substitutability implies *strong full substitutability*, a condition that deals with indifferences more directly.

Strong full substitutability combines four conditions, which are each similar to conditions defined in Appendix A in Hatfield et al. [2019]. The first condition, *increasing-price full substitutability for sales*, requires that sales are substitutable to each other and complementary to purchases as prices rise (i.e., as the set of available purchases shrinks and the set of available sales expands). The analogous condition for purchases is *decreasing-price full substitutability for purchases*. We also consider similar two other conditions, *decreasing-price full substitutability for sales* and *increasing-price full substitutability for purchases*, which are not exactly analogous to the first two conditions due to income effects.

Assumption 1′ (Strong FS—SFS)**.** For all $f \in F$, finite $Y, Y' \subseteq X_f$, and $Z \in C^f(Y)$:

[40] Corollary A.2 is a version of Theorem 1 in Ostrovsky [2008] and Theorem 3 in Hatfield and Kominers [2012] for settings in which prices are continuous. However, Corollary A.2 holds even when willingness to pay is unbounded (i.e., BWP is not satisfied), unlike the existence results proved by Ostrovsky [2008] and Hatfield and Kominers [2012].

- (Increasing-price full substitutability for sales—IFSS) If $Y_{\to f} \supseteq Y'_{\to f}$ and $Y_{f\to} \subseteq Y'_{f\to}$, then there exists $Z' \in C^f(Y')$ with $Z' \cap Y_{f\to} \subseteq Z$.
- (Decreasing-price full substitutability for purchases—DFSP) If $Y_{f\to} \supseteq Y'_{f\to}$ and $Y_{\to f} \subseteq Y'_{\to f}$, then there exists $Z' \in C^f(Y')$ with $Z' \cap Y_{\to f} \subseteq Z$.

For all $f \in F$, finite $Y, Y' \subseteq X_f$, and $y \in Y$ such that there exists $Z \in C^f(Y)$ with $y \in Z$:

- (Decreasing-price full substitutability for sales—DFSS) If $Y_{\to f} \subseteq Y'_{\to f}$ and $Y_{f\to} \supseteq Y'_{f\to} \ni y$, then there exists $Z' \in C^f(Y')$ with $y \in Z'$.
- (Increasing-price full substitutability for purchases—IFSP) If $Y_{f\to} \subseteq Y'_{f\to}$ and $Y_{\to f} \supseteq Y'_{\to f} \ni y$, then there exists $Z' \in C^f(Y')$ with $y \in Z'$.

The main theorem of this section asserts that FS and SFS are equivalent.

Theorem B.1. *FS is equivalent to SFS.*

We use Theorem B.1 to deal with indifferences in the proofs of several of our results. Although Hatfield et al. [2019] rule out income effects, Theorem B.1 is logically independent of Theorem A.1 in Hatfield et al. [2019] as we derive a weaker conclusion.

Supplementary appendix

C PROOF OF THEOREM ??

Fix a firm $f \in F$. We first translate the statement of Theorem B.1 to demand-language, taking care to account for the possibility that a trade is not available at any finite price. We then apply a perturbation argument similar to the proof of Theorem B.1 in Hatfield et al. [2019] to prove a demand language version of Theorem B.1, which is equivalent to Theorem B.1. We note that the notation and the lemma (Lemma C.1) discussed in this section are also used in the proof of Theorem 1.

C.1 Passing to demand language

We use infinite prices to denote unavailable trades for the sake of notational convenience. Formally, define a set of prices by

$$\mathsf{P} = (\mathbb{R} \cup \{-\infty\})^{\Omega_{f\to}} \times (\mathbb{R} \cup \{\infty\})^{\Omega_{\to f}},$$

where $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R} \cup \{\infty\}$ are topologized with the disjoint union topologies. Given $p \in \mathsf{P}$ and $\Xi \subseteq \Omega_f$, let

$$U^f(\Xi|p) = u^f\left(\Xi, \left(p_{\Xi_{f\to}}, (-p)_{\Xi_{\to f}}, 0_{\Omega_f \smallsetminus \Xi}\right)\right)$$

denote f's utility of trading set Ξ of contracts at price vector p, where we write $u^f(\Xi, t) = -\infty$ if $t_\omega = -\infty$ for some $\omega \in \Omega_f$. Define the *extended demand correspondence* $\mathbf{D}^f : \mathsf{P} \rightrightarrows \mathcal{P}(\Omega_f)$ by

$$\mathbf{D}^f(p) = \underset{\Xi \subseteq \Omega_f}{\arg\max}\, U^f(\Xi|p).$$

Note that the restriction of the extended demand correspondence to $\mathbb{R}^{\Omega_f}$ is precisely the demand correspondence D^f. We write full substitutability in demand language similarly to Hatfield et al. [2019].

Definition C.1 (Hatfield et al., 2019)**.** $\mathbf{D}^f$ is *(demand-language) fully substitutable* if for all $p \leq p' \in \mathsf{P}$ with $|\mathbf{D}^f(p)| = |\mathbf{D}^f(p')| = 1$, we have

$$\Xi' \cap \{\omega \in \Omega_{f\to} \mid p_\omega = p'_\omega\} \subseteq \Xi$$
$$\Xi \cap \{\omega \in \Omega_{\to f} \mid p_\omega = p'_\omega\} \subseteq \Xi',$$

where $\mathbf{D}^f(p) = \{\Xi\}$ and $\mathbf{D}^f(p') = \{\Xi'\}$.

We now write the constitutent conditions of strong full substitutability in demand language similarly to Hatfield et al. [2019].

Definition C.2. $\mathbf{D}^f$ is *(demand-language) increasing-price fully substitutable for sales* if for all $p \leq p' \in \mathsf{P}$ and $\Xi \in \mathbf{D}^f(p)$, there exists $\Xi' \in \mathbf{D}^f(p')$ with

$$\Xi' \cap \{\omega \in \Omega_{f\to} \mid p_\omega = p'_\omega\} \subseteq \Xi.$$

Definition C.3. $\mathbf{D}^f$ is *(demand-language) decreasing-price fully substitutable for sales* if for all $p \geq p' \in \mathsf{P}$ and $\psi \in \Xi \in \mathbf{D}^f(p)$ with $\psi \in \Omega_{f\to}$ and $p_\psi = p'_\psi$, there exists $\Xi' \in \mathbf{D}^f(p')$ with $\psi \in \Xi'$.

The original substitutability conditions are equivalent to their demand-language analogues, as the following lemma shows formally.

Lemma C.1. *C^f is fully substitutable (resp. increasing-price fully substitutable for sales, decreasing-price fully substitutable for sales) if and only if $\mathbf{D}^f$ is.*

Proof. Given a finite set of contracts $Y \subseteq X$, define a price vector $p_f(Y) \in \mathbb{R}^{\Omega_f}$ by

$$p_f(Y)_\omega = \begin{cases} \sup_{(\omega,q)\in Y} q & \text{for } \omega \in \Omega_{f\to} \\ \inf_{(\omega,q)\in Y} q & \text{for } \omega \in \Omega_{\to f} \end{cases},$$

so that $p_f(Y)_\omega$ is the most favorable price at which ω is available in Y. Due to the definitions of C^f and $\mathbf{D}^f$, we have

$$C^f(Y) = \left\{\left\{\left(\omega, p_f(Y)_\omega\right) \mid \omega \in \Psi\right\} \mid \Psi \in \mathbf{D}^f\left(p_f(Y)\right)\right\}$$

for all finite sets $Y \subseteq X$. It follows that C^f is fully substitutable (resp. increasing-price fully substitutable for sales, decreasing-price fully substitutable for sales) whenever $\mathbf{D}^f$ is. Note also that

$$\mathbf{D}^f(p) = \left\{\tau(Y) \mid Y \in C^f\left(\{(\omega, p_\omega) \mid p_\omega \in \mathbb{R}\}\right)\right\}$$

for all $p \in \mathsf{P}$. It follows that $\mathbf{D}^f$ is fully substitutable (resp. increasing-price fully substitutable for sales, decreasing-price fully substitutable for sales) whenever C^f is. □

Note that $\mathbf{D}^f$ is upper hemi-continuous by Berge's Maximum Theorem. Considering perturbations shows that extended demand is generically single-valued on P.

Claim C.1. *The set $\{p \in \mathsf{P} \mid |\mathbf{D}^f(p)| = 1\}$ is open and dense in P.*

Proof. Let $\mathfrak{S} = \{p \in \mathsf{P} \mid |\mathbf{D}^f(p)| = 1\}$. The set $\mathfrak{S}$ is open because $\mathbf{D}^f$ is upper hemi-continuous and $\mathcal{P}(\Omega_f)$ is discrete. To see that $\mathfrak{S}$ is dense, note that for all $\Xi \neq \Xi' \subseteq \Omega$, the set

$$\{p \in \mathsf{P} \mid U^f(\Xi|p) = U^f\left(\Xi'|p\right) \neq -\infty\}$$

is nowhere dense. Indeed, if $U^f(\Xi|p) = U^f(\Xi'|p) \neq -\infty$, we have $U^f(\Xi|p') \neq U^f(\Xi'|p')$ for any $p' = \left(p_{\Omega\smallsetminus\{\omega\}}, p_\omega + \epsilon\right)$ and $\omega \in (\Xi \smallsetminus \Xi') \cup (\Xi' \smallsetminus \Xi)$. □

C.2 Theorem B.1 in demand-language

The following technical result exploits the upper hemi-continuity of extended demand and uses perturbations to perform certain selections from the extended demand correspondence.

Claim C.2. *Let $p \in \mathbb{R}^\Omega$ and let $\mathfrak{B} \subseteq \mathbb{R}^{\Omega_f}$ be open and dense in some neighborhood of 0.*

(a) For all $\Psi \in \mathbf{D}^f(p)$, there exists $\epsilon \in \mathfrak{B}$ such that $\mathbf{D}^f(p+\epsilon) = \{\Psi'\} \subseteq \mathbf{D}^f(p)$ with $\Psi' \subseteq \Psi$.

(b) If $\psi \in \Psi \in \mathbf{D}^f(p)$, then there exists $\epsilon \in \mathfrak{B}$ such that $\mathbf{D}^f(p+\epsilon) = \{\Psi'\} \subseteq \mathbf{D}^f(p)$ with $\psi \in \Psi'$.

Proof. By shrinking $\mathfrak{B}$ if necessary, we can assume that $\mathbf{D}^f(p+\epsilon) \subseteq \mathbf{D}^f(p)$ for all $\epsilon \in \mathfrak{B}$ (by upper hemi-continuity).

We begin by proving Part (a). First, we show that there exists $\epsilon \in \mathfrak{B}$ such that $\Psi' \subseteq \Psi$ for all $\Psi' \in \mathbf{D}^f(p+\epsilon)$. Take $\epsilon = \left(0_\Psi, \delta_{\Omega_{\to f}\smallsetminus\Psi}, -\delta_{\Omega_{f\to}\smallsetminus\Psi}\right)$, where $\delta > 0$ is such that $\epsilon \in \mathfrak{B}$. Note that $U^f(\Xi|p+\epsilon) \leq U^f(\Xi|p)$ for all $\Xi \subseteq \Omega$ with equality if and only if $\Xi \subseteq \Psi$. It follows that $\Psi' \subseteq \Psi$ for all $\Psi' \in \mathbf{D}^f(p+\epsilon)$.

To complete the proof of Part (a), we perturb ϵ. More precisely, let $\mathfrak{B}'$ be an open neighborhood of $0 \in \mathbb{R}^{\Omega_f}$ such that $\mathbf{D}^f(p+\epsilon+\epsilon') \subseteq \mathbf{D}^f(p+\epsilon)$ for all $\epsilon' \in \mathfrak{B}'$—such a $\mathfrak{B}'$ exists by upper hemi-continuity. By Claim C.1, there exists $\epsilon' \in \mathfrak{B}'$ such that $\epsilon + \epsilon' \in \mathfrak{B}$ and $|\mathbf{D}^f(p+\epsilon+\epsilon')| = 1$.

The proof of Part (b) is similar. Note that ψ is never demanded if $|p_\psi| = \infty$. Hence, we must have $p_\psi \in \mathbb{R}$. First, we show that there exists $\epsilon \in \mathfrak{B}$ such that $\psi \in \Psi'$ for all $\Psi' \in \mathbf{D}^f(p+\epsilon)$. Without loss of generality, assume that $\psi \in \Omega_{f\to}$. Take $\epsilon = \left(0_{\Omega_f \smallsetminus \{\psi\}}, \delta_\psi\right)$, where $\delta > 0$ is such that $\epsilon \in \mathfrak{B}$. Note that $U^f(\Xi|p+\epsilon) \geq U^f(\Xi|p)$ for all $\Xi \subseteq \Omega$ with equality if and only if $\psi \notin \Xi$. It follows that $\psi \in \Psi'$ for all $\Psi' \in \mathbf{D}^f(p+\epsilon)$.

To complete the proof of Part (b), we perturb ϵ as in the proof of Part (a). □

Using suitable selections, Claim C.2 implies a demand-language version of Theorem B.1.

Claim C.3. *If $\mathbf{D}^f$ is fully substitutable, then $\mathbf{D}^f$ is increasing-price fully substitutable for sales.*

Proof. Let $p \leq p' \in \mathsf{P}$, and let $\Xi \in \mathbf{D}^f(p)$. Let

$$\mathfrak{B} = \left\{\epsilon \in \mathbb{R}^{\Omega_f} \mid \mathbf{D}^f(p'+\epsilon) \subseteq \mathbf{D}^f(p') \text{ and } |\mathbf{D}^f(p'+\epsilon)| = 1\right\},$$

which is non-empty and dense in a neighborhood of 0 by Claim C.1 and upper hemi-continuity. By Claim C.2(a), there exists $\epsilon \in \mathfrak{B}$ such that $\mathbf{D}^f(p+\epsilon) = \{\Psi\}$ with $\Psi \subseteq \Xi$. Note that $\mathbf{D}^f(p'+\epsilon) = \{\Xi'\}$ for some $\Xi' \in \mathbf{D}^f(p')$ by construction.

Because $\mathbf{D}^f$ is fully substitutable, we have

$$\Xi' \cap \{\omega \in \Omega_{f\to} \mid p_\omega = p'_\omega\} \subseteq \Psi \subseteq \Xi.$$

It follows that $\mathbf{D}^f$ is increasing-price fully substitutable for sales. □

Claim C.4. *If $\mathbf{D}^f$ is fully substitutable, then $\mathbf{D}^f$ is decreasing-price fully substitutable for sales.*

Proof. Let $p \geq p' \in \mathsf{P}$, let $\Xi \in \mathbf{D}^f(p)$, and suppose that $\psi \in \Xi$ satisfies $p_\psi = p'_\psi$. Let

$$\mathfrak{B} = \left\{\epsilon \in \mathbb{R}^{\Omega_f} \mid \mathbf{D}^f(p'+\epsilon) \subseteq \mathbf{D}^f(p') \text{ and } |\mathbf{D}^f(p'+\epsilon)| = 1\right\},$$

which is non-empty and dense in a neighborhood of 0 by Claim C.1 and upper hemi-continuity. By Claim C.2(b), there exists $\epsilon \in \mathfrak{B}$ such that $\mathbf{D}^f(p+\epsilon) = \{\Psi\}$ with $\psi \in \Psi$. Note that $\mathbf{D}^f(p'+\epsilon) = \{\Xi'\}$ for some $\Xi' \in \mathbf{D}^f(p')$ by construction.

Since $\mathbf{D}^f$ is fully substitutable, we must have $\psi \in \Xi' \in \mathbf{D}^f(p)$. Thus, $\mathbf{D}^f$ is decreasing-price fully substitutable for sales. □

C.3 Proof of Theorem B.1

Clearly SFS implies FS. It remains to prove the converse. Suppose that C^f is fully substitutable. Lemma C.1 and Claim C.3 imply that C^f is increasing-price fully substitutable for sales. Lemma C.1 and Claim C.4 imply that C^f is decreasing-price fully substitutable for sales. Similarly, C^f must be decreasing-price and increasing-price fully substitutable for purchases. Thus, C^f is strongly fully substitutable.

D PROOF OF THEOREM ??

The strategy of the proof is to reduce Theorem 1 to a different existence result, Theorem D.1.

Theorem D.1. *Under FS and BWP, competitive equilibria exist.*

We first modify utility functions so that BWP is satisfied (Lemma D.1), ensuring that our modification preserves FS (Lemma D.2). We then show that any competitive equilibrium in the modified economy yields a competitive equilibrium in the original economy (Lemma D.4). We conclude the proof of Theorem 1 by applying Theorem D.1, which guarantees that competitive equilibria exist in the modified economy. We then prove Theorem D.1.

We note that the modification and the lemmata discussed in this section are also used in the proof of Theorem 3.

D.1 The modified economy

For $f \in F$, let

$$\mathcal{K}^f = - \inf_{u^f(\Xi,t) \geq u^f(\varnothing,0)} \sum_{\omega \in \Omega_f} t_\omega,$$

which is finite by BCV. Let $\Pi \geq 1 + \sum_{f \in F} \mathcal{K}^f$ be arbitrary.

We modify the economy by giving agents to option to make any trade for a cost of Π.[41] Formally, for $f \in F$, define $\widehat{u}^f : \mathcal{P}(\Omega_f) \times \mathbb{R}^{\Omega_f} \to \mathbb{R}$ by

$$\widehat{u}^f(\Xi, t) = \max_{\Xi \subseteq \Psi \subseteq \Omega_f} u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi}\right)\right).$$

The function $\widehat{u}^f$ is clearly continuous and strictly increasing in the $\mathbb{R}^{\Omega_f}$ factor. Consider a modified economy in which utility functions are given by $\widehat{u}^f$ for $f \in F$. The remainder of this subsection verifies that the modified economy satisfies BWP and FS.

We first show that the modified economy satisfies BWP. Intuitively, note that this property is precisely what giving firms the option to make any trade for a cost of Π achieves.

Lemma D.1. *Under BCV, the modified economy satisfies BWP.*

Proof. We claim that BWP is satisfied with $M = \Pi + 1$. Let $f \in F$, let $\omega \in \Omega_f \smallsetminus \Xi$, let $\Xi \subseteq \Omega_f$, and let $t \in \mathbb{R}^{\Omega_f}$ be such that $t_\omega = 0$. Note that, for all $\omega \in \Psi \subseteq \Omega_f$, we have

$$u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, -M_\omega, (t - \Pi)_{\Psi \smallsetminus \Xi \smallsetminus \{\omega\}}\right)\right) < u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi}\right)\right)$$

whenever $u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi}\right)\right) \in \mathbb{R}$, because $M > \Pi = \Pi - t_\omega$. Hence, we have

$$\begin{aligned}
\widehat{u}^f\left(\Xi \cup \{\omega\}, \left(t_{\Omega_f \smallsetminus \{\omega\}}, -M_\omega\right)\right) &= \max_{\Xi \cup \{\omega\} \subseteq \Psi \subseteq \Omega_f} u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, -M_\omega, (t - \Pi)_{\Psi \smallsetminus \Xi \smallsetminus \{\omega\}}\right)\right) \\
&< \max_{\Xi \cup \{\omega\} \subseteq \Psi \subseteq \Omega_f} u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi \smallsetminus \{\omega\}}\right)\right) \\
&\leq \max_{\Xi \subseteq \Psi \subseteq \Omega_f} u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi \smallsetminus \{\omega\}}\right)\right) \\
&= \widehat{u}^f(\Xi, t).
\end{aligned}$$

Therefore, firm f will never choose a contract (ω, p_ω) with $p_\omega > M$ (resp. $p_\omega < -M$) if $\omega \in \Omega_{\to f}$ (resp. $\omega \in \Omega_{\to f}$). Since f, ω, Ξ, and t were arbitrary, the claim follows. □

The following claim, which asserts that giving a firm the option to make one trade for a cost of Π preserves full substitutability, will be used to prove that FS holds in the modified economy.

Claim D.1. *Let Π be a positive real number. Given a utility function u^f and $\varphi \in \Omega_f$, define $\widehat{u}^f_\varphi : \mathcal{P}(\Omega_f) \to \mathbb{R}^{\Omega_f} \to \mathbb{R}$ by*

$$\widehat{u}^f_\varphi(\Xi, t) = \max\left\{u^f(\Xi, t), u^f\left(\Xi \cup \{\varphi\}, \left(t_{\Omega_f \smallsetminus \{\varphi\}}, (t - \Pi)_\varphi\right)\right)\right\}.$$

[41] Hatfield et al. [2019] show that such *trade endowments* preserve full substitutability when preferences are quasilinear (see Theorem 2 in Hatfield et al., 2019).

If u^f is fully substitutable, then so is $\widehat{u}^f_\varphi$.

Proof. The proof of this claim is similar to the proof of Lemma A.2 in Hatfield et al. [2013] and uses the notation of Appendix C.1. Lemma C.1 guarantees that $\mathbf{D}^f$ is fully substitutable. Let $\widehat{\mathbf{D}}^f$ denote the extended demand correspondence for the utility function $\widehat{u}^f_\varphi$. Without loss of generality, assume that $\varphi \in \Omega_{\to f}$.

We first show that $\widehat{\mathbf{D}}^f$ is fully substitutable. Let $p \leq p' \in \mathsf{P}$ be such that $\left|\widehat{\mathbf{D}}^f(p)\right| = \left|\widehat{\mathbf{D}}^f(p')\right| = 1$. Let $\widehat{\mathbf{D}}^f(p) = \{\Xi\}$ and let $\widehat{\mathbf{D}}^f(p') = \{\Xi'\}$. Define $q \in \mathsf{P}$ by

$$q = \left(p_{\Omega_f \smallsetminus \{\omega\}}, \min\{\Pi, p_\omega\}_\omega\right).$$

and define $q' \in \mathsf{P}$ similarly. Note that $q \leq q'$ always holds. We divide into cases based on the order between p_φ, p'_φ, and Π to show that

$$\begin{aligned} \Xi' \cap \{\omega \in \Omega_{f\to} \mid p_\omega = p'_\omega\} &\subseteq \Xi \\ \Xi \cap \{\omega \in \Omega_{\to f} \mid p_\omega = p'_\omega\} &\subseteq \Xi'. \end{aligned} \tag{D.1}$$

Case 1: $p_\varphi \leq p'_\varphi \leq \Pi$. In this case, we have $p = q$, $p' = q'$, $\widehat{\mathbf{D}}^f(p) = \mathbf{D}^f(q)$ and $\widehat{\mathbf{D}}^f(p') = \mathbf{D}^f(q')$, and so (D.1) follows from the full substitutability of $\mathbf{D}^f$.

Case 2: $p_\varphi \leq \Pi < p'_\varphi$. In this case, we have $p = q$ and $\widehat{\mathbf{D}}^f(p) = \mathbf{D}^f(q)$. Let $\omega \in \Omega_{f\to} \smallsetminus \Xi$ satisfy $p_\omega = p'_\omega$—note that $\omega \neq \varphi$ by construction. By IFSS, there exists $\Psi' \in \mathbf{D}^f(q')$ with $\omega \notin \Psi'$. Since $\Xi' = \Psi' \smallsetminus \{\varphi\}$, we have $\omega \notin \Xi'$. Similarly, if $\omega \in \Xi_{\to f}$ satisfies $p_\omega = p'_\omega$, IFSP implies that there exists $\Psi' \in \mathbf{D}^f(q')$ with $\omega \in \Psi'$. Since $\Xi' = \Psi' \smallsetminus \{\varphi\}$, we have $\omega \in \Xi'$. (D.1) follows.

Case 3: $\Pi < p_\varphi \leq p'_\varphi$. Let $\Psi \in \mathbf{D}^f(q)$ be arbitrary, and note that $\Xi = \Psi \smallsetminus \{\varphi\}$. Let $\omega \in \Omega_{f\to} \smallsetminus \Psi$ satisfy $p_\omega = p'_\omega$. By IFSS, there exists $\Psi' \in \mathbf{D}^f(q')$ with $\omega \notin \Psi'$. Since $\Xi' = \Psi' \smallsetminus \{\varphi\}$, we have $\omega \notin \Xi'$. Similarly, if $\omega \in \Psi_{\to f}$ satisfies $p_\omega = p'_\omega$, IFSP implies that there exists $\Psi' \in \mathbf{D}^f(q')$ with $\omega \in \Psi'$. Since $\Xi' = \Psi' \smallsetminus \{\varphi\}$, we have $\omega \in \Xi'$. (D.1) follows.

The cases exhaust all possibilities, completing the proof that $\widehat{\mathbf{D}}^f$ is fully substitutable. By Lemma C.1, $\widehat{u}^f_\varphi$ must be fully substitutable as well. □

Claim D.1 and a straightforward inductive argument imply that FS holds in the modified economy.

Lemma D.2. *Under FS, the modified economy satisfies FS.*

D.2 Outcomes in the modified economy

This subsection shows that competitive equilibria in the modified economy give rise to competitive equilibria in the original economy (Lemma D.4). The following lemma, which is also used in the proof of Theorem 3, shows that agent f can only produce $\mathcal{K}^f$ units of surplus in the modified economy and that trade endowments can only be used at social cost Π. As will be seen in the proof of Lemma D.4, it follows that trade endowments cannot be used in any competitive equilibrium.

Lemma D.3. *Let $\Xi \subseteq \Omega_f$ and let $t \in \mathbb{R}^{\Omega_f}$. Suppose $\widehat{u}^f(\Xi, t) \geq \widehat{u}^f(\varnothing, 0)$. Under BCV:*

(a) We have $\sum_{\omega \in \Omega_f} t_\omega \geq -\mathcal{K}^f$.
(b) If we have $u^f(\Xi, t) < \widehat{u}^f(\Xi, t)$, then we have $\sum_{\omega \in \Omega_f} t_\omega \geq \Pi - \mathcal{K}^f$.

Proof. Note that $\widehat{u}^f(\varnothing, 0) \geq u^f(\varnothing, 0)$ and thus we have $\widehat{u}^f(\Xi, t) \geq u^f(\varnothing, 0)$. Let $\Xi \subseteq \Psi \subseteq \Omega_f$ be such that

$$\widehat{u}^f(\Xi, t) = u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi}\right)\right).$$

The definition of $\mathcal{K}^f$ implies that

$$-\mathcal{K}^f \leq \sum_{\omega \in \Omega_f \smallsetminus \Psi \cup \Xi} t_\omega + \sum_{\omega \in \Psi \smallsetminus \Xi} (t_\omega - \Pi) = -\Pi \cdot |\Psi \smallsetminus \Xi| + \sum_{\omega \in \Omega_f} t_\omega$$

so that

$$|\Pi| \cdot |\Psi \smallsetminus \Xi| - \mathcal{K}^f \leq \sum_{\omega \in \Omega_f} t_\omega. \tag{D.2}$$

As $|\Psi \smallsetminus \Xi| \geq 0$ always holds, Part (a) follows from (D.2). If $u^f(\Xi, t) < \widehat{u}^f(\Xi, t)$, then we must have $\Psi \neq \Xi$. As $|\Psi \smallsetminus \Xi| \geq 1$ in this case, Part (b) follows from (D.2) as well. □

We now show that competitive equilibria in the modified economy give rise to competitive equilibria in the original economy.

Lemma D.4. *Under BCV, any competitive equilibrium in the modified economy is a competitive equilibrium in the original economy.*

Proof. Let $[\Xi; p]$ be a competitive equilibrium in the modified economy. For $f \in F$, let $t^f = \left(p_{\Xi_{f\to}}, (-p)_{\Xi_{\to f}}, 0_{\Omega_f \smallsetminus \Xi}\right)$. Since $[\Xi; p]$ is a competitive equilibrium in the modified economy, we have $\widehat{u}^f(\Xi_f, t^f) \geq \widehat{u}^f(\varnothing, 0)$ for all $f \in F$. Note that

$$\sum_{f \in F} \sum_{\omega \in \Omega_f} t^f_\omega = \sum_{f \in F} \sum_{\omega \in \Xi_f} t^f_\omega = \sum_{\omega \in \Xi} (p_\omega - p_\omega) = 0.$$

In light of the fact that $\Pi > \sum_{f \in F} \mathcal{K}^f$, it follows from Lemma D.3 that $u^f(\Xi_f, t^f) \geq \widehat{u}^f(\Xi_f, t^f)$ for all $f \in F$.

Let $f \in F$ be arbitrary. For any $\Psi \subseteq \Omega_f$, we have

$$\begin{aligned} u^f(\Xi_f, t^f) \geq \widehat{u}^f(\Xi_f, t^f) &\geq \widehat{u}^f\left(\Psi, \left(p_{\Psi_{f\to}}, (-p)_{\Psi_{\to f}}, 0_{\Omega_f \smallsetminus \Psi}\right)\right) \\ &\geq u^f\left(\Psi, \left(p_{\Psi_{f\to}}, (-p)_{\Psi_{\to f}}, 0_{\Omega_f \smallsetminus \Psi}\right)\right), \end{aligned}$$

where the second inequality is because $[\Xi; p]$ is a competitive equilibrium in the modified economy and the third inequality follows from the definition of $\widehat{u}^f$. It follows that $\Xi_f \in D^f(p)$. Since f was arbitrary, $[\Xi; p]$ is a competitive equilibrium in the original economy. □

D.3 Completion of the proof of Theorem 1

Theorem D.1 and Lemmata D.1 and D.2 imply the modified economy has a competitive equilibrium $[\Xi; p]$, which is a competitive equilibrium in the original economy by Lemma D.4.

D.4 Proof of Theorem D.1

Let M be as in BWP. Intuitively, we consider a grid of size ϵ in $[-2M, 2M]^\Omega$, chosen so that there are no indifferences. We then use the Gale-Shapley operator of Hatfield and Kominers [2012] and Fleiner et al. [2018b] to produce an ϵ-equilibrium. Sending $\epsilon \longrightarrow 0$, we obtain a competitive equilibrium.

Formally, a vector $\delta \in (-\epsilon, \epsilon)^{\Omega}$ is *ϵ-regular* if D^f is single-valued on $[-2M, 2M]^{\Omega_f} \cap \left(\epsilon\mathbb{Z}^{\Omega_f} + \delta_{\Omega_f}\right)$ for all $f \in F$. The following claim asserts that there are many regular vectors.

Claim D.2. *For any $\epsilon > 0$, the set of ϵ-regular vectors is dense in $(-\epsilon, \epsilon)^{\Omega}$.*

Proof. For a firm $f \in F$, let

$$\mathfrak{S}_f = \left\{ p \in \mathbb{R}^{\Omega} \mid \left| D^f \left(p_{\Omega_f} \right) \right| = 1 \right\},$$

which is open and dense in $\mathbb{R}^{\Omega_f}$ by Claim C.1. Let $n = \left\lfloor \frac{2M}{\epsilon} \right\rfloor + 1$ and let $T = ([-n, n] \cap \mathbb{Z})^{\Omega}$.

Note that δ is ϵ-regular if $\delta + \epsilon T \subseteq \mathfrak{S}_f$. For any $\mathfrak{t} \in T$, the set of vectors δ such that $\delta + \epsilon\mathfrak{t} \in \mathfrak{S}_f$ is open and dense in $(-\epsilon, \epsilon)^{\Omega}$ since $\mathfrak{S}_f$ is open and dense in $\mathbb{R}^{\Omega_f}$. As T is finite, it follows that the set of ϵ-regular vectors contains an open and dense subset of $(-\epsilon, \epsilon)^{\Omega}$. □

An arrangement $[\Xi; p]$ is an ϵ-equilibrium if every agent f demands Ξ_f when given access to all sales, as well as purchases in Ξ, at prices p, and other purchases at prices $p + \epsilon$.

Definition D.1. An arrangement $[\Xi; p]$ is an *ϵ-equilibrium* if $p \in [-2M, 2M]^{\Omega_f}$ and $\Xi \in D^f \left(\hat{p}^{f,\Xi,\epsilon}\right)$ for all f, where

$$\hat{p}^{f,\Xi,\epsilon}_{\omega} = \begin{cases} p_{\omega} & \text{if } \omega \in \Xi \text{ or } f = \mathrm{s}(\omega) \\ p_{\omega} + \epsilon & \text{if } \omega \notin \Xi \text{ and } f = \mathrm{b}(\omega) \end{cases}.$$

The following claim shows that ϵ-equilibria exist.

Claim D.3. *For all $0 < \epsilon < M$, under FS and BWP, there exists an ϵ-equilibrium.*

Proof. Let δ be an ϵ-regular vector, which exists by Claim D.2. Let $\mathsf{P}_{\omega} = [-2M, 2M] \cap (\epsilon\mathbb{Z} + \delta_{\omega})$, and let

$$\widehat{X} = \bigcup_{\omega \in \Omega} (\{\omega\} \times \mathsf{P}_{\omega}) \subseteq X.$$

Note that C^f is single-valued on $\mathcal{P}\left(\widehat{X}_f\right)$ by ϵ-regularity, and so write $C^f(Y) = \{\mathsf{C}^f(Y)\}$ for $Y \subseteq \widehat{X}_f$.

Following Hatfield and Kominers [2012], define $\Phi : \mathcal{P}\left(\widehat{X}\right)^2 \to \mathcal{P}\left(\widehat{X}\right)^2$ by

$$\Phi(X^B, X^S) = \left(\Phi^B(X^B, X^S), \Phi^S(X^B, X^S)\right)$$
$$\Phi^B(X^B, X^S) = (\widehat{X} \smallsetminus X^S) \cup \bigcup_{f \in F} \mathsf{C}^f \left(X^B_{\to f} \cup X^S_{f\to}\right)_{f\to}$$
$$\Phi^S(X^B, X^S) = (\widehat{X} \smallsetminus X^B) \cup \bigcup_{f \in F} \mathsf{C}^f \left(X^B_{\to f} \cup X^S_{f\to}\right)_{\to f}.$$

As in Fleiner [2003], Hatfield and Milgrom [2005], Hatfield and Kominers [2012], and Fleiner et al. [2018b], order $\mathcal{P}\left(\widehat{X}\right)^2$ by letting $(X^B, X^S) \sqsubseteq (\bar{X}^B, \bar{X}^S)$ if $X^B \supseteq \bar{X}^B$ and $X^S \subseteq \bar{X}^S$. As Hatfield and Kominers [2012] and Fleiner et al. [2018b] have shown, Φ is isotone (with respect to $\sqsubseteq$) under FS. The Tarski [1955] fixed point theorem guarantees that Φ has a fixed point (X^B, X^S).

Given $f \in F$, since (X^B, X^S) is a fixed-point of Φ, we have

$$X^B_{f\to} = (\widehat{X}_{f\to} \smallsetminus X^S_{f\to}) \cup \mathsf{C}^f \left(X^B_{\to f} \cup X^S_{f\to}\right)_{f\to}. \tag{D.3}$$

Since $\mathsf{C}^f\left(X^B_{\to f} \cup X^S_{f\to}\right)_{f\to} \subseteq X^S_{f\to}$, it follows that $X^B_{f\to} \cup X^S_{f\to} = \widehat{X}_{f\to}$. Taking unions over f, we have

$$\widehat{X} = \bigcup_{f\in F} X_{f\to} = \bigcup_{f\in F}\left(X^B_{f\to} \cup X^S_{f\to}\right) = X^B \cup X^S. \tag{D.4}$$

(D.3) also implies that

$$X^B_{f\to} \cap X^S_{f\to} = \mathsf{C}^f\left(X^B_{\to f} \cup X^S_{f\to}\right)_{f\to}.$$

Similarly, we have

$$X^B_{\to f} \cap X^S_{\to f} = \mathsf{C}^f\left(X^B_{\to f} \cup X^S_{f\to}\right)_{\to f},$$

and it follows that

$$(X^B \cap X^S)_f = \mathsf{C}^f\left(X^B_{\to f} \cup X^S_{f\to}\right). \tag{D.5}$$

Let $\omega \in \Omega$ be arbitrary. Since $\epsilon < M$, we have $\max \mathsf{P}_\omega > M$ and $\min \mathsf{P}_\omega < -M$. Thus, we have $(\omega, \max \mathsf{P}_\omega), (\omega, \min \mathsf{P}_\omega) \notin X^B \cap X^S$ by BWP and (D.5). If $(\omega, \max \mathsf{P}_\omega) \notin X^B$, then adding $(\omega, \max \mathsf{P}_\omega)$ to X^B and removing it from X^S preserves (D.4) and (D.5) by BWP for $f = \mathsf{s}(\omega)$. Thus, we can assume that $(\omega, \max \mathsf{P}_\omega) \in X^B \smallsetminus X^S$. Similarly, we can assume that $(\omega, \min \mathsf{P}_\omega) \in X^S \smallsetminus X^B$. Define

$$p_\omega = \max\left\{p'_\omega \mid (\omega, p'_\omega) \in X^S\right\},$$

which exists as P_ω is finite and $(\omega, \min \mathsf{P}_\omega) \in X^S$.

We claim that $[\Xi; p]$ is an ϵ-equilibrium, where $\Xi = \tau(X^B \cap X^S)$. Note that since $(\omega, \max \mathsf{P}_\omega) \notin X^S$ for all $\omega \in \Omega$, we have $\hat{p}^{f,\Xi,\epsilon}_\omega \in \mathsf{P}_\omega$ for all $\omega \in \Omega$ and $f \in F$. The definition of p_ω also ensures that $\left(\omega, \hat{p}^{\mathsf{b}(\omega),\Xi,\epsilon}_\omega\right) \in X^B$ and $\left(\omega, \hat{p}^{\mathsf{s}(\omega),\Xi,\epsilon}_\omega\right) = (\omega, p_\omega) \in X^S$ for all $\omega \in \Omega$. It follows that $\left(X^B \cap X^S\right)_f \subseteq \kappa\left(\left[\Omega_f; \hat{p}^{f,\Xi,\epsilon}\right]\right) \subseteq X^B_{\to f} \cup X^S_{f\to}$ for all $f \in F$. Hence, (D.5) implies that $\Xi_f \in D^f\left(\hat{p}^{f,\Xi,\epsilon}\right)$ for all $f \in F$, so that $[\Xi; p]$ is an ϵ-equilibrium. □

As $[-2M, 2M]$ is sequentially compact, Claim D.3 implies that there exists an arrangement $[\Xi; p]$, a sequence $n_1 < n_2 < \cdots$ of positive integers, and a sequence $p_1, p_2, \ldots \in [-2M, 2M]^\Omega$ such that $[\Xi; p_k]$ is a $\frac{1}{n_k}$-equilibrium for all k and $p_k \to p$. Note that $\hat{p}_k^{f,\Xi,\frac{1}{n_k}} \to p_{\Omega_f}$ for all $f \in F$ because $\frac{1}{n_k} \to 0$. Because $\Xi_f \in D^f\left(\hat{p}_k^{f,\Xi,\frac{1}{n_k}}\right)$ for all k and D^f is upper hemi-continuous, it follows that $\Xi_f \in D^f\left(p_{\Omega_f}\right)$ for all $f \in F$. Thus, $[\Xi; p]$ is a competitive equilibrium.

E OTHER PROOFS OMITTED FROM THE TEXT

E.1 Proof of Theorem 2

Competitive equilibrium outcomes are clearly individually rational. It remains to show that no trail locally blocks a competitive equilibrium outcome. Let $[\Xi; p]$ be a competitive equilibrium and let $A = \kappa([\Xi; p])$. Suppose for the sake of deriving a contradiction that there is a locally blocking trail $(z_1, \ldots, z_n)$.

Let $z_i = (\omega_i, p'_i)$. Let $f_i = \mathsf{s}(x_i)$ and let $f_{n+1} = \mathsf{b}(x_n)$. As $A_{f_1} \notin C^{f_1}(A_{f_1} \cup \{x_1\})$ and $[\Xi; p]$ is a competitive equilibrium, we must have $p'_1 > p_{\omega_1}$. Similarly, we must have $p'_2 > p_{\omega_2}$. A simple inductive argument shows that $p'_n > p_{\omega_n}$. But we must have $p'_n < p_{\omega_n}$ since $A_{f_{n+1}} \notin C^{f_{n+1}}(A_{f_{n+1}} \cup \{x_n\})$. Thus, there are no locally blocking trails.

E.2 Proof of Proposition 1

We adapt the proof of Lemma 5(ii) in Fleiner et al. [2018b] to our setting. Inspired by Fleiner et al. [2018b], we say that a circuit $(z_1, \ldots, z_n)$ is *locally blocking* if every pair of adjacent contracts is demanded by their common agent in every choice set.

Definition E.1. Let Y be an outcome. A sequence of contracts $(z_1, \ldots, z_n)$ is a *locally blocking circuit* if:

- for all $1 \leq i \leq n$, we have $\{z_{i-1}, z_i\} \subseteq W$ for all $W \in C^{f_i}\left(Y_{f_i} \cup \{z_{i-1}, z_i\}\right)$, where $f_i = \mathrm{s}(z_i) = \mathrm{b}(z_{i-1})$.

Here, we write $z_0 = z_n$.

To prove Proposition 1, we show (as in Fleiner et al., 2018b) that every shortest locally blocking circuit or locally blocking trail gives rise to a blocking set.

Claim E.1. *Let Y be an individually rational outcome. Under FS, if $(z_1, \ldots, z_n)$ is shortest among all locally blocking circuits and locally blocking trails for Y, then the set $\{z_1, \ldots, z_n\}$ blocks Y.*

Proof. We prove the contrapositive of the claim. Suppose that $(z_1, \ldots, z_n)$ is a locally blocking circuit or locally blocking trail. If $(z_1, \ldots, z_n)$ is a locally blocking trail and there exists $W \in C^{f_{i+1}}(\{z_i, z_{i+1}\})$ with $z_i \notin W$, then $(z_{i+1}, \ldots, z_n)$ is a locally blocking trail. Similarly, if $(z_1, \ldots, z_n)$ is a locally blocking trail and there exists $W \in C^{f_{i+1}}(\{z_i, z_{i+1}\})$ with $z_{i+1} \notin W$, then $(z_1, \ldots, z_i)$ is a locally blocking trail.

Now, suppose that $Z = \{z_1, \ldots, z_n\}$ does not block Y. Then, there is a firm f, a contract $z_j \in Z_f$, and a set $W \in C^f\left(Y_f \cup Z_f\right)$ with $z_j \notin W$. Without loss of generality, we can assume that $f = \mathrm{s}(z_j)$, so that $f = f_j$. We show that there is a locally blocking circuit or locally blocking trail that is shorter than $(z_1, \ldots, z_n)$. By the logic of the previous paragraph, we can assume that $\{z_i, z_{i+1}\} \subseteq W$ for all $W \in C^{f_{i+1}}(\{z_i, z_{i+1}\})$ if $(z_1, \ldots, z_n)$ is a locally blocking trail, as otherwise there is a shorter locally blocking trail.

By Theorem B.1, SFS must be satisfied. We divide into cases based on whether $j = 1$ and whether we have a trail or a circuit to complete the proof of the claim.

Case 1: $j = 1$ and $(z_1, \ldots, z_n)$ is a locally blocking trail. By IFSS, there exists $W' \in C^f(Y_f \cup Z_{f\rightarrow})$ with $z_1 \notin W'$. Among all such W', take W to minimize $|W' \smallsetminus Y_f|$. As $Y_f \notin C^f(Y_f \cup \{z_1\})$, we have $Y_f \notin C^f(Y_f \cup Z_{f\rightarrow})$, and hence $W' \nsubseteq Y_f$.

Let $z_k \in W \smallsetminus Y_f$ be arbitrary. By IFSS, we must have $Y_f \notin C^f(Y_f \cup \{z_k\})$, so that $(z_k, \ldots, z_n)$ is a shorter locally blocking trail.

Case 2: $j \neq 1$ or $(z_1, \ldots, z_n)$ is a locally blocking circuit. In either case, z_{j-1} is well-defined. By IFSS, there exists $W' \in C^f(Y_f \cup \{z_{j-1}\} \cup Z_{f\rightarrow})$ with $z_j \notin W'$. Among all such W', take W to minimize $|W' \smallsetminus Y_f|$. As $\{z_{j-1}, z_j\} \subseteq B$ for all $B \in C^f(Y_f \cup \{z_{j-1}, z_j\})$, we have $z_{j-1} \in W$ by DFSP.

Let $z_k \in W \smallsetminus Y_f$ be arbitrary. By IFSS, we must have $z_k \in B$ for all $B \in C^f(Y_f \cup \{z_{j-1}, z_k\})$. If $k < j$, then $(z_k, \ldots, z_{j-1})$ is a shorter locally blocking circuit. If $k > j$, then $(z_1, \ldots, z_{j-1}, z_k, \ldots, z_n)$ is a shorter locally blocking circuit or locally blocking trail.

The cases exhaust all possibilities, completing the proof of the claim. □

Claim E.1 implies Proposition 1.

E.3 Proof of Theorem 3

Let A be any stable outcome, and let $\Xi = \tau(A)$. For $\omega \in \tau(A)$, let p_ω be the unique price such that $(\omega, p_\omega) \in A$.

For $f \in F$, let

$$\mathcal{K}^f = - \inf_{u^f(\Xi,t) \geq u^f(\varnothing,0)} \sum_{\omega \in \Omega_f} t_\omega,$$

which is finite by BCV. Let

$$\Pi = 1 + \sum_{f \in F} \mathcal{K}^f + 2 \sum_{\omega \in \Xi} |p_\omega|.$$

Recall the definition of $\widehat{u}^f : \mathcal{P}(\Omega_f) \times \mathbb{R}^{\Omega_f} \to \mathbb{R}$ from the proof of Theorem 1, which is

$$\widehat{u}^f(\Xi, t) = \max_{\Xi \subseteq \Psi \subseteq \Omega_f} u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi}\right)\right).$$

Consider a modified economy in which utility functions are given by $\widehat{u}^f$ for $f \in F$.

Claim E.2. *The outcome A is stable in the modified economy.*

Proof. The outcome A is clearly individually rational in the modified economy. It remains to prove that A is not blocked in the modified economy. Suppose for the sake of deriving a contradiction that there is a blocking set Z in the modified economy.

Let $\widehat{C}^f$ denote the choice function of f in the modified economy. For $f \in F$ and $Y^f \in \widehat{C}^f(A_f \cup Z_f)$, note that $U^f\left(Y^f\right) \geq U^f(\varnothing)$ and thus

$$-\mathcal{K}^f \leq \sum_{(\omega,p'_\omega) \in Y^f_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega) \in Y^f_{\to f}} p'_\omega \leq \sum_{(\omega,p'_\omega) \in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega) \in Z_{\to f}} p'_\omega + \sum_{\omega \in \Xi_f} |p_\omega|,$$

where the first inequality is due to Lemma D.3(a), so that

$$\sum_{(\omega,p'_\omega) \in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega) \in Z_{\to f}} p'_\omega + \sum_{\omega \in \Xi_f} |p_\omega| + \mathcal{K}^f \geq 0.$$

But note that

$$\sum_{f \in F} \left(\sum_{(\omega,p'_\omega) \in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega) \in Z_{\to f}} p'_\omega + \sum_{\omega \in \Xi_f} |p_\omega| + \mathcal{K}^f \right) = 2 \sum_{\omega \in \Xi} |p_\omega| + \sum_{f \in F} \mathcal{K}^f = \Pi - 1.$$

It follows that

$$\sum_{(\omega,p'_\omega) \in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega) \in Z_{\to f}} p'_\omega + \sum_{\omega \in \Xi_f} |p_\omega| + \mathcal{K}^f \leq \Pi - 1 < \Pi$$

for all $f \in F$, so that

$$\sum_{(\omega,p'_\omega) \in Y^f_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega) \in Y^f_{\to f}} p'_\omega \leq -\mathcal{K}^f + \Pi - 1 < -\mathcal{K}^f + \Pi.$$

Hence, Lemma D.3(b) implies that $\widehat{U}^f\left(Y^f\right) = U^f\left(Y^f\right)$ for all $f \in F$.

Let $W \in C^f(A_f \cup Z_f)$ be arbitrary. In light of the previous paragraph and the fact that $U^f(W) \leq \widehat{U}^f(W)$, we must have $W \in \widehat{C}^f(A_f \cup Z_f)$. Since Z blocks A in the modified economy, we must have $Z_f \subseteq W$. Hence, Z blocks A in the original economy, which contradicts the hypothesis that A is stable in the original economy. □

Claim E.2 guarantees that A is stable in the modified economy. By Proposition 1, A is trail-stable in the modified economy. Lemmata D.1 and D.2 ensure that FS and BCV are satisfied in the modified economy. Hence, there exists a competitive equilibrium $[\Xi; p]$ in the modified economy with $\kappa([\Xi; p]) = A$ by Theorem 4 (which is proved independently). Lemma D.4 guarantees that $[\Xi; p]$ is a competitive equilibrium in the modified economy.

E.4 Proof of Theorem 4

We set prices for unrealized trades that are as high as possible while remaining (weakly) undesirable to sellers. Call a trail $(z_1, \ldots, z_n)$ *locally semi-blocking* if the seller of z_i wants to propose z_i when given access to z_{i-1} for all $i > 1$, and the seller of z_1 wants to propose z_1. We consider a contract desirable to a seller if it appears in a locally semi-blocking trail.

Definition E.2. A trail $(z_1, \ldots, z_n)$ *locally semi-blocks* an outcome A if:

- $A_{f_1} \notin C^{f_1}(A_{f_1} \cup \{z_1\})$, where $f_1 = \mathrm{s}(z_1)$;
- $\{z_i, z_{i+1}\} \subseteq Y$ for all $Y \in C^{f_{i+1}}(A_{f_{i+1}} \cup \{z_i, z_{i+1}\})$ for $1 \le i \le n-1$, where $f_{i+1} = \mathrm{b}(z_i) = \mathrm{s}(z_{i+1})$.

Formally, let A be an outcome and let $\Xi = \tau(A)$. Let M be as in BWP. Let X^B be the set of contracts that appear in some locally semi-blocking trail. Thus, X^B consists of all contracts that are strictly desirable to their sellers.[42] For $\omega \in \Omega$, define

$$p_\omega = \min \left\{ M, \inf_{(\omega, p'_\omega) \in X^B} p'_\omega \right\}, \tag{E.1}$$

so that p_ω is the minimum of M and the highest price at which ω is weakly undesirable to its seller. We prove that $\kappa([\Xi; p]) = A$ and that $[\Xi; p]$ is a competitive equilibrium.

Claim E.3. *Under BWP, if A is individually rational, then we have $\kappa([\Xi; p]) = A$.*

Proof. Suppose that $(\omega, p'_\omega) \in A$. BWP implies that $p'_\omega < M$. As $u^{\mathrm{s}(\omega)}$ is strictly increasing in transfers and A is individually rational, we have $(\omega, p''_\omega) \in X^B$ if and only if $p''_\omega > p'_\omega$. It follows that $p_\omega = p'_\omega$. Since $\omega \in \Xi$ was arbitrary, the claim follows. □

Claim E.4. *Under FS and BWP, if A is trail-stable, then $[\Xi; p]$ is a competitive equilibrium.*

Proof. Suppose for the sake of deriving a contradiction that $\Xi_f \notin D^f\left(p_{\Omega_f}\right)$. As A is individually rational, it follows from Claim E.3 that $\Xi' \notin D^f\left(p_{\Omega_f}\right)$ for all $\Xi' \subseteq \Xi_f$.

We perturb prices slightly to ensure that sellers have strict incentives to propose contracts. Due to the upper hemi-continuity of demand, we can ensure that a sufficiently small perturbation does not affect the property that f demands no subset of Ξ_f. Formally, define

$$\mathfrak{O} = \{p' \in \mathbb{R}^{\Omega_f} \mid D^f(p') \cap \mathcal{P}(\Xi_f) = \varnothing\}.$$

Since D^f is upper hemi-continuous, the set $\mathfrak{O}$ contains an open neighborhood of p_{Ω_f}. By (E.1), there exists $q \in \mathfrak{O}$ such that $q_{\Xi_f \cup \Omega_{f\to}} = p_{\Xi_f \cup \Omega_{f\to}}$, we have $q_\omega = p_\omega$ whenever $p_\omega = M$, and $(\omega, p_\omega) \in X^B$ whenever $\omega \in \Omega_{\to f} \smallsetminus \Xi$ and $p_\omega < M$. We consider the prices q instead of the prices p.

By Theorem B.1, SFS must be satisfied. To produce a contradiction, we consider the set of trades that f could demand at price vector q that contains fewest trades outside Ξ_f. Formally, let $\Psi \in D^f(q)$ minimize $|\Psi' \smallsetminus \Xi|$ over all $\Psi' \in D^f(q)$.

[42] In the fixed-point interpretation of trail-stable outcomes [Adachi, 2017, Fleiner et al., 2018b], X^B is the set of contracts that are available to their buyers.

Consider the corresponding set of contracts $W = \kappa([\Psi; q])$. Note that $W \nsubseteq A$ and $W_{\to f} \smallsetminus A \subseteq X^B$ by construction and BWP. We divide into cases based on whether $W \smallsetminus A$ contains any contracts that are sold by f to produce a contradiction.

Case 1: $W \smallsetminus A \nsubseteq X_{\to f}$. In this case, we either produce a locally blocking trail or show that any sale in $W \smallsetminus A$ must appear in some locally semi-blocking trail. Formally, let $z \in W_{f\to} \smallsetminus A_{f\to}$ be arbitrary and let $\omega = \tau(z)$. By IFSS, we have $z \in W'$ for all $W' \in C^f(A \cup \{z\} \cup W_{\to f})$. Let $W^0 \in C^f(A \cup \{z\} \cup W_{\to f})$ minimize $|W' \smallsetminus A|$ over all $W' \in C^f(A \cup \{z\} \cup W_{\to f})$.

As $q_\omega = p_\omega$, the trail $((\omega, p'_\omega))$ cannot be locally semi-blocking for any $p'_\omega < q_\omega$ by (E.1). Hence, we have that $A_f \in C^f(A_f \cup \{z\})$ by the upper hemi-continuity of demand. It follows that $W^0_{\to f} \smallsetminus A_{\to f} \neq \varnothing$. Since $W_{\to f} \smallsetminus A \subseteq X^B$, there must exist a locally semi-blocking trail $(z_1, \ldots, z_n)$ with $z_n \in W^0_{\to f}$. By DFSP, we have $z_n \in W'$ for all $W' \in C^f(A_f \cup \{z_n, z\})$. We divide into cases based on whether there exists $W'' \in C^f(A_f \cup \{z_n, z\})$ with $z \notin W''$ to derive contradictions.

Subcase 1.1: There exists $W'' \in C^f(A_f \cup \{z_n, z\})$ with $z \notin W''$. Then, the trail $(z_1, \ldots, z_n)$ is locally blocking, contradicting the assumption that A is trail-stable.

Subcase 1.2: $z \in W''$ for all $W'' \in C^f(A_f \cup \{z_n, z\})$. Then, $(z_1, \ldots, z_n, z)$ is a locally semi-blocking trail. Since u^f is continuous, there exists $p'_\omega < p_\omega$ such that $(z_1, \ldots, z_n, (\omega, p'_\omega))$ is a locally semi-blocking trail, contradicting (E.1).

Case 2: $W \smallsetminus A \subseteq X_{\to f}$. Let $z \in W \smallsetminus A$ be arbitrary, and let $(z_1, \ldots, z_n)$ be a locally semi-blocking trail with $z_n = z$. By DFSP, we have $A_f \notin C^f(A_f \cup \{z\})$. Thus, $(z_1, \ldots, z_n)$ is a locally blocking trail, contradicting the assumption that A is trail-stable.

The cases exhaust all possibilities. We have produced contradictions in all cases, completing the proof of the claim. □

Claims E.3 and E.4 together imply the theorem.

E.5 Proof of Corollary 2

Competitive equilibria exist by Theorem D.1. Competitive equilibrium outcomes are trail-stable by Theorem 2. Trail-stable outcomes lift to competitive equilibria by Theorem 4.

E.6 Proof of Theorem 5

We prove the contrapositive. Let $[\Xi; p]$ be an arrangement and suppose that $A = \kappa([\Xi; p])$ is not strongly group stable. If A is not individually rational, then clearly $[\Xi; p]$ is not a competitive equilibrium. Thus, we can assume that A is not strongly unblocked—that is, that there exists a non-empty set of contracts $Z \subseteq X \smallsetminus A$ and, for each $f \in F$ with $Z_f \neq \varnothing$, a set of contracts $Y^f \subseteq Z_f \cup A_f$ with $Y^f \supseteq Z_f$ and $U^f\left(Y^f\right) > U^f\left(A_f\right)$ (see Definition 6).

Let $F' = \{f \in F \mid Z_f \neq \varnothing\}$. For each $f \in F'$, let

$$\mathcal{M}^f = \sup\left\{q \;\middle|\; \overline{u}^f\left(\tau(Y^f), \sum_{\omega \in \tau(Y^f)_{f\to}} p_\omega - \sum_{\omega \in \tau(Y^f)_{\to f}} p_\omega - q\right) \geq U^f\left(A_f\right)\right\}$$

denote the negative of the compensating variation for f from the change from $\tau(A_f)$ to $\tau(Y^f)$ at price vector p. For $\omega \in \tau(Z)$, let $\tilde{p}_\omega$ be the unique price such that $(\omega, \tilde{p}_\omega) \in Z$. Define $\tilde{p}_\omega = p_\omega$ for $\omega \in \Omega \smallsetminus \tau(Z)$. The definition of Y^f

ensures that

$$\overline{u}^f\left(\tau(Y^f), \sum_{\omega\in\tau(Y^f)_{f\to}} \tilde{p}_\omega - \sum_{\omega\in\tau(Y^f)_{\to f}} \tilde{p}_\omega\right) > U^f\left(A_f\right)$$

for all $f \in F'$. It follows that

$$\begin{aligned}\mathcal{M}^f &> \sum_{\omega\in\tau(Y^f)_{f\to}} p_\omega - \sum_{\omega\in\tau(Y^f)_{\to f}} p_\omega - \sum_{\omega\in\tau(Y^f)_{f\to}} \tilde{p}_\omega + \sum_{\omega\in\tau(Y^f)_{\to f}} \tilde{p}_\omega \\ &= \sum_{\omega\in\tau(Y^f)_{f\to}} (p_\omega - \tilde{p}_\omega) + \sum_{\omega\in\tau(Y^f)_{\to f}} (\tilde{p}_\omega - p_\omega).\end{aligned}$$

Because $p_\omega = \tilde{p}_\omega$ for $\omega \notin Z$ and $Z_f \subseteq Y^f$, we have

$$\mathcal{M}^f > \sum_{\omega\in\tau(Z_f)_{f\to}} (p_\omega - \tilde{p}_\omega) + \sum_{\omega\in\tau(Z_f)_{\to f}} (\tilde{p}_\omega - p_\omega).$$

Summing over $f \in F'$, we have $\sum_{f\in F'} \mathcal{M}^f > 0$. Thus, there exists $f \in F'$ with $\mathcal{M}^f > 0$. For such f, we have

$$\begin{aligned}u^f\left(\tau(Y^f), \left(p_{\tau(Y^f)_{f\to}}, (-p)_{\tau(Y^f)_{\to f}}, 0_{\Omega_f\smallsetminus\tau(Y^f)}\right)\right) &> U^f\left(A_f\right) \\ &= u^f\left(\Xi_f, \left(p_{\Xi_{f\to}}, (-p)_{\Xi_{\to f}}, 0_{\Omega_f\smallsetminus\Xi}\right)\right),\end{aligned}$$

so that $\Xi_f \notin D^f\left(p_{\Omega_f}\right)$. Therefore, $[\Xi; p]$ is not a competitive equilibrium.

E.7 Proof of Corollary 3

Competitive equilibrium outcomes exist and coincide with trail-stable outcomes by Corollary 2, and are strongly group stable by Theorem 5. Strongly group stable outcomes are always stable. Stable outcomes are trail-stable by Proposition 1.

E.8 Proof of Corollary 4

Competitive equilibria exist by Theorem 1. Competitive equilibrium outcomes are strongly group stable by Theorem 5. Strongly group stable outcomes are always stable. Stable outcomes lift to competitive equilibria by Theorem 3.

E.9 Proof of Lemma A.1

The proof is similar to the proof of Theorem 7 in Hatfield and Kominers [2012]. By Theorem B.1 in Appendix B, we can assume that SFS is satisfied.

We prove the contrapositive. Let A be outcome that is not stable. If A is not individually rational, then clearly A is not trail-stable. Thus, we can assume that A is blocked by a non-empty blocking set Z.

Since Z is non-empty and the network is assumed to be acyclic, there is a firm f_1 with $Z_{\to f_1} = \varnothing$ and $Z_{f_1\to} \neq \varnothing$. Let $z_1 \in Z_{f_1\to}$ be arbitrary. By IFSS, we have $A_{f_1} \notin C^{f_1}(A_{f_1} \cup \{z_1\})$. Let $f_2 = \mathrm{b}(z_1)$.

If $A_{f_2} \notin C^{f_2}(A_{f_2} \cup \{z_1\})$, then (z_1) is a locally blocking trail. Thus, we can assume that $A_{f_2} \in C^{f_2}\left(A_{f_2} \cup \{z_1\}\right)$. DFSP implies that $z_1 \in W'$ for all $W' \in C^{f_2}(A_{f_2} \cup \{z_1\} \cup Z_{f_2\to})$. Let $W \in C^{f_2}(A_{f_2} \cup \{z_1\} \cup Z_{f_2\to})$ minimize $|W' \smallsetminus A|$ among all $W' \in C^{f_2}(A_{f_2} \cup \{z_1\} \cup Z_{f_2\to})$. By IFSS, we must have $W = \{z_1, z_2\}$ for some $z_2 \in Z_{f_2\to}$. Note that $A_{f_2} \notin C^{f_2}(A_{f_2} \cup \{z_1, z_2\})$ by construction.

A similar argument to the previous paragraph shows that (z_1, z_2) is a locally blocking trail or there exists $z_3 \in Z$ with $\mathrm{s}(z_3) = \mathrm{b}(z_2)$ such that $A_{f_2} \notin C^{f_2}(A_{f_2} \cup \{z_2, z_3\})$. By induction and due to acyclicity, we obtain a locally blocking trail. Thus, A is not trail-stable.

F EXAMPLES OMITTED FROM THE TEXT

The following two examples remove the frictions from Examples 1 and 2, respectively, showing that competitive equilibrium cannot be Pareto-comparable and that adding an outside option that is not used cannot shut down trade. Thus, distortionary frictions are crucial to the conclusions of Examples 1 and 2.

Example 3 *continued* (Cyclic economy with transferable utility). In Example 3, the competitive equilibria are $[\{\zeta, \psi\}; p]$, where $\left|p_\zeta - p_\psi\right| \leq 10$. All competitive equilibria are Pareto-efficient, as guaranteed by the First Welfare Theorem (see, e.g., Hatfield et al. [2013]), and trade occurs in every competitive equilibrium.

Example F.1 (Cyclic economy with transferable utility and an outside trade—Hatfield and Kominers, 2012). As depicted in Figure 2(b), consider the economy of Example 3 with an additional firm f_3, which interacts with f_1 via trade ω'. Firm f_i has utility function

$$u^{f_i}(\Xi, t) = v^{f_i}(\Xi) + \sum_{\omega \in \Omega_f} t_\omega,$$

where valuations v^{f_1}, v^{f_2}, and v^{f_3} are as in Example 2.

Trade ζ' cannot be realized in equilibrium due to the technological constraints of f_1 and f_2. Hence, we must have $p_{\zeta'} \geq 300$ in any competitive equilibrium, since f_3 must weakly prefer $\varnothing$ over $\{\zeta'\}$ in equilibrium. In order for trade to occur, f_1 must prefer ζ over ζ', and so we must have $p_\zeta \geq p_{\zeta'}$. Hence, the competitive equilibria are $[\{\zeta, \psi\}; p]$, where $|p_\zeta - p_\psi| \leq 10$ and $p_\zeta \geq p_{\zeta'} \geq 300$. Essentially, adding an outside option simply forces p_ζ to be at least \$300 without shutting down trade between f_1 and f_2.

The next example shows that a regularity condition, such as BCV, is needed in addition to FS to ensure that competitive equilibria exist.

Example F.2 (Competitive equilibria need not exist under FS alone). Consider two firms, b and s, and one trade ω between them with $\mathrm{s}(\omega) = s$ and $\mathrm{b}(\omega) = b$. Suppose that s is not willing to sell ω at any (finite) price, but b would buy ω at any (finite) price. Note that the market does not clear at any price—b always demands ω and s never demands ω. The issue is that the variation needed to exactly compensate b for going from autarky to trade is $-\infty$. If b's compensating variation were $-p$, then autarky could be sustained in equilibrium at any price above p.

The last example shows that FS needed for stable outcomes to be trail-stable.

Example F.3 (Stable outcomes may not be trail-stable without FS). As depicted in Figure 2(a), there are two firms, f_1 and f_2, which interact via two trades, ζ and ψ. Firm f_i has utility function

$$u^{f_i}(\Xi, t) = v^{f_i}(\Xi) + \sum_{\omega \in \Omega_f} t_\omega,$$

where

$$\begin{aligned} v^{f_1}(\varnothing) = v^{f_2}(\varnothing) &= 0 \\ v^{f_1}(\{\zeta\}) = v^{f_1}(\{\psi\}) &= 1 \\ v^{f_1}(\{\zeta, \psi\}) &= -\infty \\ v^{f_2}(\{\zeta\}) = v^{f_2}(\{\psi\}) &= -\infty \\ v^{f_2}(\{\zeta, \psi\}) &= 1. \end{aligned}$$

Note that trades ζ and ψ are not complementary for firm f_1, which implies that f_1's preferences are not fully substitutable.

The no-trade outcome $\varnothing$ is stable, as no non-empty set of contracts is individually rational for both f_1 and f_2. However, the trail $((\zeta, 0), (\psi, 0))$ locally blocks the outcome $\varnothing$. Thus, the no-trade outcome is stable but not trail-stable.