ON A FRAME ENERGY PROBLEM

GERGELY AMBRUS

Abstract. We consider the extremum problem

$$\max \min_{k} \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

where the maximum is taken among vector systems $(v_k)_1^N \subset \mathbb{R}^d$ satisfying $c_1 \leqslant |v_k|^2 \leqslant c_2$ for every k. We show that in the case $\sigma = 0$, uniform tight frames are the only optimal configurations. We also give quantitative bounds on how efficient these tight frames are for small values of σ^2.

1. Introduction

The following problem has been communicated to me by Prof. Jianfeng Hou [7].

Problem 1. *Assume that $d \geqslant 2$, $N \geqslant d$, $0 < c_1 < c_2$ are positive bounds, and $\sigma \geqslant 0$ is a small positive constant. Determine the quantity*

$$\max_{\substack{v_1,\ldots,v_N \in \mathbb{R}^d \\ c_1 \leqslant |v_i|^2 \leqslant c_2 \ \forall i}} \ \min_{1 \leqslant k \leqslant N} N \log \left(1 + \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} \right). \tag{1}$$

The above question is closely related to the notion of *frames*, introduced originally by Duffin and Schaeffer [4]. A vector system $(v_i)_1^N \subset \mathbb{R}^d$ is called a frame if there exist $0 < A \leqslant B < \infty$ such that

$$A|w|^2 \leqslant \sum_{l=1}^{N} \langle w, v_l \rangle^2 \leqslant B|w|^2$$

holds for every vector $w \in \mathbb{R}^d$. The quantity $\sum_{k=1}^n \sum_{l=1}^n \langle v_k, v_l \rangle^2$ is called the *frame potential* of the vector system $(v_i)_1^n$, introduced by Benedetto and Fickus [1] (see [6] and [3] for further generalizations). Frame theory has become a well-studied topic in recent years, with plenty of real-world applications. As we are going to see, Problem 1 is another example of this phenomenon.

In this short note, we solve Problem 1 in the special case $\sigma = 0$, and also discuss results in the general setting. We will concentrate on the case when σ is small, since this is needed for practical applications in signal processing.

We start by making some general comments. We would like to determine the vector systems which are extremal with respect to (1). Naturally, the

G. Ambrus was supported by the NKFIH grant no. PD125502.

factor N may be omitted from the target function. Also, for any strictly monotone increasing function f, the maxima of

$$\min_{1\leqslant k\leqslant N} f\left(\frac{|v_k|^2}{\sigma^2+\sum_{l\neq k}\langle v_k,v_l\rangle^2}\right)$$

and

$$\min_{1\leqslant k\leqslant N} \frac{|v_k|^2}{\sigma^2+\sum_{l\neq k}\langle v_k,v_l\rangle^2}$$

are attained at the same vector configurations (subject to arbitrary boundary conditions). Since $\log(1+x)$ is strictly monotone increasing on $[0,\infty)$, we may consider the latter target function.

Our result will invoke the definition of frames. To this end, we introduce the following notion. The *tensor product* of the vectors $u,v\in\mathbb{R}^d$ is the $\mathbb{R}^d\to\mathbb{R}^d$ linear map $u\otimes v$ satisfying

$$(u\otimes v)z = u\langle z,v\rangle$$

for every $z\in\mathbb{R}^n$. In matrix form, $u\otimes v$ is the $d\times d$ matrix with entries

$$(u\otimes v)_{ab} = (u)_a(v)_b\,,$$

with $1\leqslant a,b\leqslant d$. As an immediate consequence of the definition, we derive that

$$\mathrm{tr}(u\otimes v) = \langle u,v\rangle\,.$$

Given a vector system $(v_i)_1^N\subset\mathbb{R}^d$, we define its *frame operator* [1] A as

$$A(v_1,\dots,v_N) = \sum_{i=1}^N v_i\otimes v_i\,. \tag{2}$$

A set of vectors $v_1,\dots,v_N$ in $\mathbb{R}^d$ is called a *tight frame* if its frame operator is a constant multiple of the identity operator, that is,

$$\sum_{i=1}^N v_i\otimes v_i = \lambda\, I_d \tag{3}$$

with a real constant $\lambda\in\mathbb{R}$. This is equivalent to requiring that

$$\sum_{i=1}^N \langle w,v_i\rangle^2 = \lambda|u|^2$$

holds for every vector $w\in\mathbb{R}^d$.

In the special case when all the vectors v_i are of norm 1, we are talking about a *unit norm tight frame* (UNTF). By comparing traces in (3), it immediately follows that in this case, $\lambda = N/d$. It was proven by Benedetto and Fickus [1] that UNTF's exist for every $N\geqslant d$.

We associate to a vector system $(v_i)_1^N\subset\mathbb{R}^d$ its frame potential (or 2-frame potential [5]) defined by

$$FP\,(v_1,\dots,v_N) = \sum_{i,j=1}^N \langle v_i,v_j\rangle^2. \tag{4}$$

Let $G(v_1,\dots,v_N)$ denote the Gram matrix corresponding to the vector system $(v_i)_1^N$, that is, the $N\times N$ matrix G satisfying

$$G(v_1,\dots,v_N)_{ij} = \langle v_i, v_j\rangle\,. \tag{5}$$

If L denotes the $N\times d$ matrix with rows $v_1,\dots,v_N$, then

$$G(v_1,\dots,v_N) = LL^\top, \tag{6}$$

and on the other hand,

$$A(v_1,\dots,v_N) = L^\top L. \tag{7}$$

The frame potential of the vector system may be expressed as

$$FP(v_1,\dots,v_N) = \operatorname{tr} G^2 = \sum_{i,j=1}^N G_{ij}^2 = \|G\|_{HS}^2,$$

the square of the Hilbert-Schmidt norm of G. Thus, using (6), (7), and the property that for arbitrary $N\times N$ matrices R,S, $\operatorname{tr}(RS^\top)=\operatorname{tr}(R^\top S)$ holds,

$$FP\left((v_i)_1^N\right) = \|G\|_{HS}^2 = \operatorname{tr}(LL^\top LL^\top) = \operatorname{tr}(L^\top LL^\top L) = \|A\|_{HS}^2. \tag{8}$$

The above formula is called the *frame potential duality,* which lies at the core of the proof of the existence of UNTF's [1].

We are going to call a vector system $(v_i)_1^N\subset\mathbb{R}^d$ to be *uniform* if $|v_i|=c$ holds for every i with some constant $c\geqslant 0$. Equivalently, $(v_i)_1^N\subset cS^{d-1}$, where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$.

2. The case $\sigma = 0$

According to the above remarks, our task is to find the vector systems maximizing

$$M(v_1,\dots,v_N) = \min_{1\leqslant k\leqslant N}\frac{|v_k|^2}{\sum_{l\neq k}\langle v_k,v_l\rangle^2} \tag{9}$$

subject to $c_1\leqslant |v_i|^2\leqslant c_2$ for every $1\leqslant i\leqslant N$. We are going to call vector systems for which the maximum is attained to be *extremal.*

When $N\leqslant d$, (9) is maximized when $(v_i)_1^N$ is an orthogonal system – in this case, the denominator is 0 for every k, thus, $M(v_1,\dots,v_N)=\infty$. Clearly, only orthogonal systems correspond to this value.

In light of this remark, from now on we are going to assume that the number of the vectors exceeds d, therefore, $M(v_1,\dots,v_N)<\infty$.

Theorem 1. *Assume that $2\leqslant d<N$, and $0<c_1<c_2$. The vector system $v_1,\dots,v_N\subset\mathbb{R}^d$ is a maximizer of $M(v_1,\dots,v_N)$ defined in (9) subject to the condition $c_1\leqslant|v_i|^2\leqslant c_2$ for every $1\leqslant i\leqslant N$ if and only if it is a scaled copy of a unit norm tight frame (UNTF) with scaling factor $\sqrt{c_1}$.*

Proof. For a given vector system $(v_i)_1^N$, let

$$m_k = \frac{|v_k|^2}{\sum_{l\neq k}\langle v_k,v_l\rangle^2}. \tag{10}$$

Then $M(v_1,\dots,v_N)=\min_k m_k$. Call a configuration vector v_i *minimal,* if $m_i = M(v_1,\dots,v_N)$.

First, note that scaling down a vector v_i does not decrease $\min_k m_k$. Indeed, replace v_k by $\widetilde{v}_k = \lambda v_k$, where $\lambda \in (0,1]$ is a real constant. This does not change the value of m_k. On the other hand, for any $l \neq k$, we have $\langle \widetilde{v}_k, v_l\rangle^2 = \lambda^2 \langle v_k, v_l\rangle^2 \leqslant \langle v_k, v_l\rangle^2$, therefore, the value of m_l is not decreased either, and the same holds for M. This argument also shows that in an extremal configuration, every non-minimal v_i must be orthogonal to at least one minimal v_j, otherwise shrinking v_i would cause M to increase.

Therefore, there exists a uniform vector system of norm $\sqrt{c_1}$ which is extremal. By scaling, we may assume that $c_1 = 1$. First, we characterize these uniform extremal vector systems, and we will treat the non-uniform case afterwards.

Clearly,

$$\min_{1 \leqslant k \leqslant N} \frac{1}{\sum_{l \neq k} \langle v_k, v_l\rangle^2}$$

is maximized iff its reciprocal is minimized. Thus, we may study the extremum problem

$$\min_{v_1,\ldots,v_N \in S^{d-1}} \max_{1 \leqslant k \leqslant N} \sum_{l \neq k} \langle v_k, v_l\rangle^2. \tag{11}$$

Since $|v_i| = 1$, this is attained at the same configurations as the minmax of

$$E_k := \sum_{l=1}^{N} \langle v_k, v_l\rangle^2.$$

By frame potential duality (8),

$$\begin{aligned} N \max_{1 \leqslant k \leqslant N} E_k &\geqslant \sum_{k,l=1}^{N} \langle v_k, v_l\rangle^2 \\ &= \|G(v_1,\ldots,v_N)\|_{HS}^2 \\ &= \|A(v_1,\ldots,v_N)\|_{HS}^2 . \end{aligned}$$

Since (2) shows that $\mathrm{tr}A = N$, the Cauchy-Schwarz inequality applied to the diagonal entries of A implies that

$$\|A(v_1,\ldots,v_N)\|_{HS}^2 \geqslant \frac{N^2}{d} ,$$

therefore,

$$\min_{v_1,\ldots,v_N \in S^{d-1}} \max_{k} \sum_{l=1}^{N} \langle v_k, v_l\rangle^2 \geqslant \frac{N}{d} .$$

Equality may only hold when $A = \frac{N}{d} I_d$, that is, the vectors v_i form a UNTF.

Let us now consider the general case. Let $(v_i)_1^N$ be an extremal vector system. First, if the system is uniform, then the above argument shows that it must be a scaled copy of a UNTF with scaling factor $\sqrt{c_1}$. In this case,

$$\frac{|v_k|^2}{\sum_{l \neq k} \langle v_k, v_l\rangle^2} = \frac{d}{c_1(N-d)} \tag{12}$$

holds for every $1 \leqslant k \leqslant N$. Thus, the answer to Problem 1 is

$$\max_{\substack{v_1,\dots,v_N \in \mathbb{R}^d \\ c_1 \leqslant |v_i|^2 \leqslant c_2 \ \forall i}} \min_{1 \leqslant k \leqslant N} N \log\left(1 + \frac{|v_k|^2}{\sum_{l \neq k} \langle v_k, v_l \rangle^2}\right) = N \log\left(1 + \frac{d}{c_1(N-d)}\right).$$

Assume now that there is an extremal system containing a vector with norm exceeding $\sqrt{c_1}$. As shown at the beginning of the proof, scaling down any v_k does not decrease the individual values m_k. Shrinking the vectors of norm exceeding $\sqrt{c_1}$ one–by–one results in a uniform configuration, without changing the value of M (by the maximality condition). Thus, the characterization result for uniform vector systems show that the *direction vectors* of the system $(v_i)_1^N$ must form a UNTF. Moreover, again by the maximality condition and (12),

$$m_k = \frac{d}{c_1(N-d)}$$

holds for every k. Therefore, none of the values m_k may change during the scaling process, and every vector v_i of the original system is minimal. This is only possible if any vector of norm exceeding $\sqrt{c_1}$ is orthogonal to all the other vectors. Thus, the system $(v_i)_1^N$ must be the union of an orthogonal base of an r-dimensional subspace L consisting of vectors of norm between $\sqrt{c_1}$ and $\sqrt{c_2}$, and a $\sqrt{c_1}$-norm tight frame of $L^\top$ consisting of $N-r$ vectors. However, in this case, the value of (9) is

$$M(v_1, \dots, v_N) = \frac{d-r}{c_1(N-d)}$$

by (12). This shows that the vector system may only be extremal when $r = 0$, that is, the vector system is a scaled copy of a UNTF. □

3. Results for $\sigma^2 > 0$

The answer to Problem 1 clearly depends on the value of σ – not only the extremal value does so, but the structure of the extremal vector systems as well. To illustrate this, assume that σ is very large compared to $c_2 N$. In this case, the dominant term of $\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2$ is the first one. Therefore, the extremum of (1) is attained when $|v_i|^2 = c_2$ for every i – that is, the vector norms are *maximal*, opposite to the case $\sigma = 0$.

However, in practical applications, the value of the error term σ is almost negligible. Therefore, we will concentrate on the case when σ is small. First, we restrict the study to uniform vector systems.

Theorem 2. *Assume that $\sigma \leqslant c_1 \sqrt{(N-d)/d}$. Then there is a UNTF with scaling factor $\sqrt{c_1}$ which maximizes*

$$\min_{1 \leqslant k \leqslant N} N \log \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} \tag{13}$$

among vector systems $(v_i)_1^N \subset \sqrt{c} S^{d-1}$ satisfying $c_1 \leqslant c \leqslant c_2$.

Proof. Let $|v_i|^2 = c$ for every i with $c \in [c_1, c_2]$. The previous arguments show that maximizing (13) on $\sqrt{c}S^{d-1}$ is equivalent to solving

$$\min_{v_1,\ldots,v_N \in \sqrt{c}\,S^{d-1}} \max_{1\leqslant k\leqslant N} \frac{\sigma^2 + \sum_{l\neq k}\langle v_k, v_l\rangle^2}{c}. \tag{14}$$

For a fixed value of c, the contribution of the term σ^2/c is constant, therefore it may be omitted from the target function, and the results of the previous section apply. Therefore, the extremum value is attained when the vector system is a UNTF with scaling factor $\sqrt{c}$, and the extremal value of (14) is

$$\frac{\sigma^2}{c} + c\,\frac{N-d}{d}. \tag{15}$$

Thus, we need to minimize the above quantity as a function of c over the interval $[c_1, c_2]$. Since $N > d$, the (15) is decreasing on the interval $[0, \sigma\sqrt{d/(N-d)}]$ and is increasing for $c > \sigma\sqrt{d/(N-d)}$. Thus, when $c_1 > \sigma\sqrt{d/(N-d)}$, the minimum over the interval $[c_1, c_2]$ is taken at $c = c_1$, and the answer to Problem 1, restricted to uniform vector systems, is

$$\begin{aligned}\max_{\substack{v_1,\ldots,v_N \in \sqrt{c}S^{d-1}\\ c_1\leqslant c\leqslant c_2}} \min_{1\leqslant k\leqslant N} N\log\left(1+\frac{|v_k|^2}{\sigma^2+\sum_{l\neq k}\langle v_k,v_l\rangle^2}\right)\\ = N\log\left(1+\frac{c_1}{\sigma^2+c_1^2(N-d)/d}\right).\end{aligned} \tag{16}$$

□

Let us now consider the general case. Although extremal vector systems are not necessarily uniform, we show that for small σ, there exists an extremal vector system containing relatively few vectors of non-minimal norm.

Theorem 3. *There exists a vector system which is extremal with respect to Problem 1 containing at most*

$$d\frac{c_1^2-\sigma^2}{c_1^2-d\sigma^2}$$

vectors of norm strictly larger than $\sqrt{c_1}$.

Proof. Let $(v_i)_1^N$ be a vector system satisfying the boundary conditions $c_1 \leqslant |v_i|^2 \leqslant c_2$ for every i. By re-indexing, we may assume that the norms of the vectors form an increasing sequence, and let $m \leqslant N$ be an index – not necessarily the smallest – so that $|v_i|^2 > c_1$ holds for every $i > m$. Introduce the *simultaneous scaling* by a factor $\lambda < 1$ of $(v_i)_1^N$ by setting

$$\widetilde{v_i} = v_i$$

for $1 \leqslant i \leqslant m$, and

$$\widetilde{v_i} = \lambda v_i$$

for $m+1 \leqslant i \leqslant N$. If $\lambda < 1$ is close enough to 1, all vectors of the simultaneously scaled configuration have norm between $\sqrt{c_1}$ and $\sqrt{c_2}$.

As in (10), let

$$\mu_k = \frac{|v_k|^2}{\sigma^2+\sum_{l\neq k}\langle v_k,v_l\rangle^2}$$

and

$$\widetilde{\mu}_k = \frac{|\widetilde{v}_k|^2}{\sigma^2 + \sum_{l \neq k} \langle \widetilde{v}_k, \widetilde{v}_l \rangle^2}$$

We study the effect of the simultaneous scaling on the values μ_k. When $1 \leqslant k \leqslant m$, then $|v_k|$ is unchanged, while the denominator does not increase (it decreases if and only if there is a v_i with $i \geqslant m+1$ which is non-orthogonal to v_k). Thus,

$$\widetilde{\mu}_k \geqslant \mu_k$$

for every $1 \leqslant k \leqslant m$.

Assume that $m+1 \leqslant k \leqslant N$. Then,

$$\widetilde{\mu}_k = \frac{\lambda^2 |v_k|^2}{\sigma^2 + \lambda^2 \sum_{l=1}^{m} \langle v_k, v_l \rangle^2 + \lambda^4 \sum_{l=m+1,\ l \neq k}^{N} \langle v_k, v_l \rangle^2}\,.$$

Calculating the derivative of $\widetilde{\mu}_k$ with respect to λ at $\lambda = 1$, one obtains that its sign agrees to that of

$$\sigma^2 - \sum_{l=m+1,\ l \neq k}^{N} \langle v_k, v_l \rangle^2\,. \tag{17}$$

In particular, if

$$\sigma^2 < \sum_{l \geqslant m+1,\ l \neq k} \langle v_k, v_l \rangle^2$$

holds for every $k \geqslant m+1$, then (17) is negative for every $m+1 \leqslant k \leqslant N$. Thus, for sufficiently small values of $\varepsilon > 0$, the simultaneous scaling with factor $\lambda = 1 - \varepsilon$ does not decrease any of the terms μ_k, thus, it does not decrease their minimum either.

Assume now that $(v_i)_1^N$ is an extremal vector system which, among the extremal configurations, minimizes $\sum_{i=1}^N |v_i|^2$. We will show that not too many vectors can have norm exceeding $\sqrt{c_1}$. We may suppose that among these vectors, v_{N-1} and v_N have the largest absolute value inner product. By applying the simultaneous scaling defined above with $m = N-2$, we obtain a new vector system. By the minimality condition, this may not be extremal, which, in light of the above observations, shows that

$$\sigma^2 > \langle v_{N-1}, v_N \rangle^2 \tag{18}$$

must hold. The following Lemma implies an upper bound on the number of vectors with non-minimal norm.

Lemma 1 (Welch [8]). *Assume that M vectors $w_1, \dots, w_M \subset \mathbb{R}^d$ are given so that $|w_i|^2 \geqslant c_1$ for every i. Then*

$$\max_{i \neq j} \langle w_i, w_j \rangle^2 \geqslant \frac{c_1^2 (M-d)}{d(M-1)}\,.$$

We note that a stronger bound has recently been proven by Bukh and Cox [2], but for our needs, the above estimate is sufficient. Below, we provide a quick proof to it.

Proof of Lemma 1. Clearly, we may assume that all vectors w_i have norm $\sqrt{c_1}$, as shrinking the vectors decreases the inner products studied. Denote by ν the above maximum. Let G denote the Gram matrix of the vector system $(w_i)_1^M$ and let A be the associated frame operator. Then

$$\|A\|_{HS}^2 = \|G\|_{HS}^2 = Mc_1^2 + \sum_{i \neq j} \langle w_i, w_j \rangle^2 \leqslant Mc_1^2 + M(M-1)\nu. \tag{19}$$

On the other hand, $\mathrm{tr} A = Mc_1$, hence the Cauchy-Schwarz inequality implies that

$$\|A\|_{HS}^2 \geqslant \frac{M^2 c_1^2}{d}$$

Combined with (19) this shows the desired estimate. □

Returning back to the original argument, by (18) and Lemma 1 we conclude that in any extremal vector system $(v_i)_1^N$ with minimal $\sum_{i=1}^N |v_i|^2$, the number M of vectors of norm strictly larger than $\sqrt{c_1}$ must satisfy

$$\frac{c_1^2(M-d)}{d(M-1)} < \sigma^2 .$$

Solving the inequality for M leads to

$$M < d\frac{c_1^2 - \sigma^2}{c_1^2 - \sigma^2 d} .$$ □

We conclude the article by showing that for small values of σ, the answer to Problem 1 does not differ drastically from (16). Indeed, assume that $(v_i)_1^N$ is an extremal vector system provided by Theorem 3. Let $A = \sum_{i=1}^N v_i \otimes v_i$ be the associated frame operator. As before,

$$\|A\|_{HS}^2 \geqslant \frac{\left(\sum_{i=1}^N |v_i|^2\right)^2}{d} . \tag{20}$$

Let

$$\mu = \min_k \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

be the quantity for which we have to provide an upper bound. Then

$$|v_k|^2 \geqslant \mu\sigma^2 + \mu \sum_{l \neq k} \langle v_k, v_l \rangle^2 = \mu\sigma^2 + \mu \sum_{l=1}^N \langle v_k, v_l \rangle^2 - \mu |v_k|^4$$

holds for every k. By summing over k,

$$\sum_{k=1}^N |v_k|^2 \geqslant N\mu\sigma^2 + \mu\|A\|_{HS}^2 - \mu \sum_{k=1}^N |v_k|^4. \tag{21}$$

Introduce $R = \sum_{k=1}^N |v_k|^2$. By Theorem 3,

$$Nc_1 \leqslant R \leqslant Nc_1 + d\frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2}(c_2 - c_1), \tag{22}$$

and also

$$\sum_{k=1}^N |v_k|^4 \leqslant Nc_1^2 + d(c_2^2 - c_1^2)\frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2} .$$

Therefore, (20) and (21) leads to

$$R \geqslant \mu \left(N\sigma^2 + \frac{R^2}{d} - Nc_1^2 - d(c_2^2 - c_1^2)\frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2} \right)$$

showing that

$$\mu \leqslant \frac{R}{N\sigma^2 + \frac{R^2}{d} - Nc_1^2 - d(c_2^2 - c_1^2)\frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2}}. \tag{23}$$

In order to obtain an upper bound for μ, we have to maximize this quantity as a function of R over the interval given by (22). By a simple calculation one obtains that if

$$N^2 c_1^2 > Nd(c_1^2 - \sigma^2) + d^2(c_2^2 - c_1^2)\frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2}$$

holds (which is guaranteed when $N \gg d$ and $\sigma \ll c_1$), then the function is decreasing over the whole interval. Therefore, its maximum value is attained at $R = Nc_1$, leading to the bound

$$\mu \leqslant \frac{c_1}{\sigma^2 + c_1^2(N-d)/d - d^2(c_2^2 - c_1^2)(c_1^2 - \sigma^2)/(c_1^2 - d\sigma^2)}$$

by (23). This provides a quantitative estimate on the difference between (1) and (16), showing that for practical applications, a UNTF of norm $\sqrt{c_1}$ is a well-justified choice.

4. Acknowledgement

I am grateful to Prof. Bai Bo, Prof. Jianfeng Hou, and other members of the Theory Lab at Huawei Technologies for communicating the problem to me and for the successful collaboration, and to Prof. Ed Saff for his valuable advices.

References

[1] J.J. Benedetto, M. Fickus, *Finite normalized tight frames.* Adv. Comp. Math. **18** (2003), 357–385.

[2] B. Bukh and C. Cox, *Nearly orthogonal vectors and small antipodal spherical codes.* `arXiv` manuscript no. 1803.02949, 1–22. 2019.

[3] X. Chen, V. Gonzalez, E. Goodman, S. Kang, and K.A. Okoudjou, *Universal optimal configurations for the p-frame potentials.* Adv. Comput. Math. **46** (2020), no. 4.

[4] R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series.* Trans. Amer. Math. Soc. **72** (1952), 341-366.

[5] M. Ehler and K.A. Okoudjou, *Minimization of the probabilistic p-frame potential.* Journal of Statistical Planning and Inference **142** (2012), no. 3, 645–659.

[6] M. Fickus, B.D. Johnson, K. Kornelson, and K.A. Okoudjou, *Convolutional frames and the frame potential.* Appl. Comput. Harmon. Anal. **19** (2005), no. 1., 77–91.

[7] J. Hou, *personal communication.* 2020.

[8] L. Welch, *Lower bounds on the maximum cross correlation of signals (Corresp.)* IEEE Transactions on Information Theory, **20** (1974), no. 3., 397–399.

Gergely Ambrus, Alfréd Rényi Institute of Mathematics, POB 127 H-1364 Budapest, Hungary.

E-mail address: `ambrus@renyi.hu`