

CHARACTERIZATIONS OF SOME CLASSES
OF QUASI-HEREDITARY ALGEBRAS

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ABSTRACT. Inspired by the work of Mirolla and Vilonen [MV] describing the categories of perverse sheaves as module categories over certain finite dimensional algebras, Dlab and Ringel introduced [DR2] an explicit recursive construction of these algebras in terms of the algebras $A(\gamma)$. In particular, they characterized the quasi-hereditary algebras of Cline-Parshall-Scott [PS] and constructed them in this way. The present paper provides a characterization of lean, shallow and replete algebras in terms of this recursive process. A detailed presentation of the results will appear elsewhere.

Let D be a division K -algebra, C a basic K -algebra, ${}_D S_C$ and ${}_C T_D$ finite-dimensional bimodules with K acting centrally. Let $\gamma : {}_C T_D \otimes_D S_C \rightarrow {}_C C_C$, be a bimodule homomorphism whose image lies in $\text{rad } C$. Let $B = D \ltimes ({}_D S_C \otimes_C T_D)$ the "split" K -algebra with the coordinate-wise addition and multiplication given by

$$(d_1, s_1 \otimes t_1)(d_2, s_2 \otimes t_2) = (d_1 d_2, d_1 s_2 \otimes t_2 + s_1 \otimes t_1 d_2 + s_1 \gamma(t_1 \otimes s_2) \otimes t_2).$$

Clearly, B is a local K -algebra with $\text{rad } B = S_C \otimes_C T$. It follows that S has the structure of a B - C -bimodule by $(d, s \otimes t) \cdot s' = ds' + s\gamma(t \otimes s')$ and T the structure of a C - B -bimodule by $t' \cdot (d, s \otimes t) = t'd + \gamma(t' \otimes s)t$. In [DR2], the 2×2 matrix $A = \begin{pmatrix} B & S \\ T & C \end{pmatrix}$ with multiplication given by

$$\begin{pmatrix} b & s \\ t & c \end{pmatrix} \begin{pmatrix} b' & s' \\ t' & c' \end{pmatrix} = \begin{pmatrix} bb' + (0, s \otimes t') & b \cdot s' + sc' \\ t \cdot b' + ct' & \gamma(t \otimes s') + cc' \end{pmatrix},$$

is shown to be a $A(\gamma)$ ring, viz. the quotient of the tensor algebra over the $(C \times D) - (C \times D)$ -bimodule $T \otimes S$ by the ideal generated by the elements $t \otimes s - \gamma(t \otimes s)$.

Let $e_1 = \begin{pmatrix} (1, 0) & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{e}_C = (e_2, e_3, \dots, e_n)$ be a complete sequence of primitive orthogonal idempotents of C so that $\sum_{i=1}^n e_i = 1 : A_A = \bigoplus_{i=1}^n e_i A$. Write $\mathbf{e} = \mathbf{e}_A = (e_1, e_2, \dots, e_n)$, $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$ for $1 \leq i \leq n$ and $\varepsilon_{n+1} = 0$.

Let $\Delta_C(i)$ and $\Delta_C^o(i)$ be the right and left standard modules of (C, \mathbf{e}_C) , respectively (see [D] for the basic definitions and notation). Dlab and Ringel

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have shown in [DR2] that (A, \mathbf{e}) is quasi-hereditary if and only if (C, \mathbf{e}_C) is quasi-hereditary and S_C and ${}_C T$ have Δ_C -filtration and Δ_C^o -filtration, respectively; in fact, they have shown that all basic quasi-hereditary algebras over a perfect field K can be obtained by iterating this construction, starting with a division K -algebra C . In the present note, we are going to characterize lean algebras, as well as shallow and replete quasi-hereditary algebras A in terms of properties of C , ${}_D S_C$, ${}_C T_D$ and the homomorphism γ .

Recall that (A, \mathbf{e}) is *lean* (see [ADL]) if

$$e_i(\text{rad } A)^2 e_j = e_i(\text{rad } A) \varepsilon_m(\text{rad } A) e_j \quad \text{for all } 1 \leq i, j \leq n \text{ and } m = \min\{i, j\}.$$

Equivalently, (A, \mathbf{e}) is lean if and only if (C, \mathbf{e}_C) is lean and for $i, j \geq 2$, the products $e_i A e_1 A e_j$ ($\subseteq e_i(\text{rad } A)^2 e_j$) belong to $(\text{rad } C)^2$. Since these products generate the image of γ , we have the following statement.

PROPOSITION 1. *The algebra (A, \mathbf{e}) is lean if and only if (C, \mathbf{e}_C) is lean and $\text{Im } \gamma \subseteq (\text{rad } C)^2$.*

Denoting the *standard* right and left modules of A by $\Delta(i) = \Delta_A(i)$ and $\Delta^o(i) = \Delta_A^o(i)$, respectively, (A, \mathbf{e}) said to be *quasi-hereditary* if

$$\dim_K A = \sum_{i=1}^n (1/d_i) \dim_K \Delta(i) \dim_K \Delta^o(i), \quad (*)$$

where $d_i = \dim_K \text{End } S(i)$; here, and in what follows, $S(i)$, $P(i)$ and $V(i)$ denote the simple right A -module, $P(i) \simeq e_i A$ its projective cover and $V(i)$ the kernel of the canonical epimorphism $P(i) \rightarrow \Delta(i)$. The equality $(*)$ is equivalent to the fact that $\text{End } \Delta(i) \simeq \text{End } S(i)$ for all $1 \leq i \leq n$ and that the regular representation A_A has a Δ -filtration (which has been the original definition of Cline-Parshall-Scott; see also [DR1]). Indeed, this follows from the following series of statements (A) – (C) (cf.[D2]):

(A) For every A -module X , $[X : S(i)] = (1/d_i) \dim_K X e_i$; thus

$$[A_A : S(n)] = (1/d_n) \dim_K A e_n = (1/d_n) \dim_K \Delta^o(n).$$

(B) Always, $\dim_K A e_n A \leq (1/d_n) \dim_K \Delta(n) \dim_K \Delta^o(n)$.

(C) The equality $\dim_K A e_n A = (1/d_n) \dim_K \Delta(n) \dim_K \Delta^o(n)$ holds

if and only if $\text{End}_A \Delta(n) = \text{End}_A S(n)$ and $A e_n A \simeq \bigoplus_{\text{finite}} \Delta(n)$.

The quasi-hereditary algebra (A, \mathbf{e}) is called *shallow* if all $\text{rad } \Delta(i)$ and $\text{rad } \Delta^o(i)$ ($1 \leq i \leq n$) are semi-simple. This is equivalent to the fact (see [ADL]) that

$$e_i(\text{rad } A)^2 A e_j = e_i(\text{rad } A) \varepsilon_M(\text{rad } A) e_j \quad \text{for all } 1 \leq i, j \leq n \text{ and } M = \text{Max}\{i, j\}.$$

As a consequence, both Δ_C -filtration of S_C and Δ_C^o -filtration of ${}_C T$ are in this case *top* filtrations (see [ADL]), and (C, \mathbf{e}_C) is a shallow quasi-hereditary algebra. The following example shows that these properties are not sufficient for (A, \mathbf{e}) to be shallow.

EXAMPLE. Let (A, \mathbf{e}) be the path algebra of the quiver $2 \longrightarrow 1 \longrightarrow 3$; then (A, \mathbf{e}) is quasi-hereditary (in fact, hereditary). Since $e_2(\text{rad } A)^2 e_3 \neq 0$ and $e_2(\text{rad } A)e_3(\text{rad } A)e_3 = 0$, (A, \mathbf{e}) is not shallow. However, $C = \varepsilon_2 A \varepsilon_2$ is shallow quasi-hereditary algebra and both S_C and ${}_C T$ are simple C -modules. The key missing property is leanness.

PROPOSITION 2. *The algebra (A, \mathbf{e}) is a shallow if and only if (C, \mathbf{e}_C) is a shallow quasi-hereditary algebra, S_C has a top Δ_C -filtration, ${}_C T$ has a top Δ_C^o -filtration and $\text{Im } \gamma \subseteq (\text{rad } C)^2$.*

Proof (of sufficiency). We need only to show that, under the conditions for $C, S_C, {}_C T$ and γ ,

$$e_i(\text{rad } A^2)e_j \subset e_i(\text{rad } A)\varepsilon_M(\text{rad } A)e_j \text{ for all } 1 \leq i, j \leq n \text{ and } M = \text{Max}\{i, j\}.$$

First, let $i, j \geq 2$. Then, in view of the fact that (C, \mathbf{e}_C) is shallow,

$$e_i(\text{rad } A)\varepsilon_M(\text{rad } A)e_j = e_i(\text{rad } C)\varepsilon_M(\text{rad } C)e_j = e_i(\text{rad } C)^2 e_j,$$

which, in turn equals to $e_i(\text{rad } A)\varepsilon_2(\text{rad } A)e_j$. Since, by Proposition 1, (A, \mathbf{e}) is lean, we get

$$e_i(\text{rad } A)^2 e_j = e_i(\text{rad } A)\varepsilon_m(\text{rad } A)e_j \subset e_i(\text{rad } A)\varepsilon_2(\text{rad } A)e_j,$$

as required.

If $i = 1, j \geq 2$, the inclusion $e_1(\text{rad } A)^2 e_j \subset e_1(\text{rad } A)\varepsilon_j(\text{rad } A)e_j$ follows from the fact that the Δ -filtration of $\text{rad } P(1)$, induced by the top Δ_C -filtration of S_C (which exists by [DR2]), is a top filtration. A similar argument can be applied to $i \geq 2, j = 1$.

Recall that the quasi-hereditary algebra (A, \mathbf{e}) is said to be *replete* if all $V(i) = e_i A \varepsilon_{i+1} A$ are projective, top submodules of $\text{rad } P(i) = e_i(\text{rad } A)$, and all $V^o(i) = A \varepsilon_{i+1} A e_i$ are projective, top submodules of $\text{rad } P^o(i) = (\text{rad } A)e_i$ (see [ADL]). If (A, \mathbf{e}) is a replete quasi-hereditary algebra, then (C, \mathbf{e}_C) is a replete quasi-hereditary algebra and both S_C and ${}_C T$ are projective C -modules. Again, these conditions alone do not imply that (A, \mathbf{e}) is replete. In the above Example, (C, \mathbf{e}_C) is replete and both S_C and ${}_C T$ are simple (projective) C -modules, but (A, \mathbf{e}) is not replete. The missing property is leanness again.

PROPOSITION 3. *The algebra (A, \mathbf{e}) is a replete quasi-hereditary algebra if and only if (C, \mathbf{e}_C) is a replete quasi-hereditary algebra S_C and ${}_C T$ are projective C -modules and $\text{Im } \gamma \subseteq (\text{rad } C)^2$.*

Proof (of sufficiency) We want to show that $V(i)$ is a projective, top submodule of $\text{rad } P(i)$. Consider first $i \geq 2$. Since $V_C(i) = e_i(\text{rad } A)\varepsilon_{i+1}A\varepsilon_2$ is projective, top C -submodule of the C -module $\text{rad } P_C(i) = e_i(\text{rad } A)\varepsilon_2$, $V(i) = V_C(i) \otimes_C \varepsilon_2 A$ is a projective A -module.

Moreover, since A is lean by Proposition 1, we have the equality

$$e_i(\text{rad } A)^2 e_{i+1} = e_i(\text{rad } A)\varepsilon_2(\text{rad } A)\varepsilon_{i+1},$$

and thus can identify the top of $V(i)$ in the top of $\text{rad } P(i)$ with the top of $V_C(i)$. This yields a top embedding of $V(i)$ in $\text{rad } P(i)$. Furthermore, since S_C is projective C -module, $V(1) = \text{rad } P(1) = S_C \otimes_C \varepsilon_2 A$ is projective A -module (trivially embedded in $\text{rad } P(1)$ as a top submodule).

One can use similar arguments to deal with the left A -modules $V^o(i)$.

Let us conclude our brief note with the remark that similar characterizations of the left and right medial algebras (for a definition, see [ADL]) can be made combining the conditions of Propositions 2 and 3.

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