# ON A THEOREM OF V. DLAB

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Abstract. A new proof of Dlab's theorem asserting that the left regular representation of an algebra is filtered by the standard modules if and only if the right regular representation of it is filtered by the proper standard modules, is given.

## 1. Introduction

Let $A$ be a finite-dimensional (associative) algebra over a field $K$; without loss of generality, assume that $A$ is connected and basic. Thus

$$_AA = \overset{n}{\underset{i=1}{\oplus}} {}_\bullet P(i), \text{ and } A_A = \overset{n}{\underset{i=1}{\oplus}} P_\bullet(i),$$

with mutually nonisomorphic indecomposable left, and right, projective modules ${}_\bullet P(i) = Ae_i$, and $P_\bullet(i) = e_iA$, respectively. In what follows, we keep the order of the indecomposable projective modules fixed and record this writing by writing $(A, \boldsymbol{e})$. Here, $\boldsymbol{e} = (e_1, e_2 \ldots, e_n)$ is the respective complete sequence of the primitive orthogonal idempotents of $A$.

Given $(A, \boldsymbol{e})$, define the sequences of the left, and right, standard $A$-modules

$${}_\bullet\Delta = \big({}_\bullet\Delta(1), {}_\bullet\Delta(2), \ldots, {}_\bullet\Delta(n)\big) \text{ and } \Delta_\bullet = \big(\Delta_\bullet(1), \Delta_\bullet(2), \ldots, \Delta_\bullet(n)\big)$$

by

$${}_\bullet\Delta(i) = Ae_i/A\varepsilon_{i+1} \operatorname{rad} Ae_i \text{ and } \Delta_\bullet(i) = e_iA/e_i \operatorname{rad} A\varepsilon_{i+1}A,$$

where $\varepsilon_{i+1} = e_{i+1} + e_{i+2} + \cdots + e_n$ for all $0 \le i \le n$ with $\varepsilon_{n+1} = 0$. Similarly, following [D], define the sequences of the left, and right, proper standard $A$-modules

$${}_\bullet\bar{\Delta} = \big({}_\bullet\bar{\Delta}(1), {}_\bullet\bar{\Delta}(2), \ldots, {}_\bullet\bar{\Delta}(n)\big) \text{ and } \bar{\Delta}_\bullet = \big(\bar{\Delta}_\bullet(1), \bar{\Delta}_\bullet(2), \ldots, \bar{\Delta}_\bullet(n)\big)$$

by

$${}_\bullet\bar{\Delta}(i) = Ae_i/A\varepsilon_i \operatorname{rad} Ae_i \text{ and } \bar{\Delta}_\bullet(i) = e_iA/e_i \operatorname{rad} A\varepsilon_iA \text{ for all } 1 \le i \le n.$$

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**Theorem 1.** [D]**.** *The left regular representation $_AA$ of an algebra $(A, \boldsymbol{e})$ is filtered by standard modules if and only if the right regular representation $A_A$ is filtered by the proper standard modules.*

Thus, denoting by $\mathcal{F}(\Omega)$, where $\Omega$ is given set of modules, the full subcategory of the category of the finite dimensional modules, consisting of all modules filtered by the elements of $\Omega$, the theorem states that

$$_AA \in \mathcal{F}(_\bullet\Delta) \text{ if and only if } A_A \in \mathcal{F}(\bar{\Delta}_\bullet).$$

The main objective of this note is to present a new proof of Dlab's theorem. This seems to be merited by the fact that the original proof in [D] is not easily accessible.

The algebras $(A, \boldsymbol{e})$ satisfying $A_A \in \mathcal{F}(\bar{\Delta}_\bullet)$ are called in [ADL] standardly stratified and characterized there as follows.

**Theorem 2** [ADL]**.** *Let $(A, \boldsymbol{e})$ be a $K$-algebra. Then the following statements are equivalent:*

(i) $(A, \boldsymbol{e}) \in \mathcal{F}(\bar{\Delta}_\bullet)$.

(ii) $\mathrm{Ext}^2_A\left(\bar{\Delta}_\bullet,\ D(_\bullet\Delta\right) = 0$.

(iii) $\mathcal{F}(\bar{\Delta}_\bullet) = \left\{X \in \mathrm{mod}{-}A \mid \mathrm{Ext}^1_A\left(X,\ D(_\bullet\Delta)\right) = 0\right\}$.

(ii') $\mathrm{Ext}^k_A(\bar{\Delta}_\bullet,\ D(_\bullet\Delta)) = 0$ *for all* $k \geq 1$.

(iii') $\mathcal{F}(\bar{\Delta}_\bullet) = \left\{X \in \mathrm{mod}{-}A \mid \mathrm{Ext}^k_A\left(X,\ D(_\bullet\Delta)\right) = 0 \text{ for all } k \geq 1\right\}$.

Here, the notation $\mathrm{Ext}^k_A\left(\bar{\Delta}_\bullet,\ \mathrm{D}(_\bullet\Delta)\right) = 0$ means $\mathrm{Ext}^k_A\left(\bar{\Delta}_\bullet(i),\ \mathrm{D}(_\bullet\Delta(j))\right) = 0$ for all $1 \leq i, j \leq n$, $\mathrm{D}(M)$ denotes $\mathrm{Hom}_K(M, K)$, etc.

## 2. Main results

In analogy to Theorem 2, we are going to prove the following theorem.

**Theorem 3.** *Let $(A, \boldsymbol{e})$ be a $K$-algebra. Then the following statements are equivalent:*

(i) $(A, \boldsymbol{e}) \in \mathcal{F}(_\bullet\Delta)$.

(ii) $\mathrm{Ext}^2_A(_\bullet\Delta,\ D(\bar{\Delta}_\bullet)) = 0$.

(iii) $\mathcal{F}(_\bullet\Delta) = \left\{X \in \mathrm{mod}{-}A \mid \mathrm{Ext}^1_A\left(X,\ D(\bar{\Delta}_\bullet)\right) = 0\right\}$.

(ii') $\mathrm{Ext}^k_A(_\bullet\Delta,\ D(\bar{\Delta}_\bullet)) = 0$ *for all* $k \geq 1$.

(iii') $\mathcal{F}(_\bullet\Delta) = \left\{X \in \mathrm{mod}{-}A \mid \mathrm{Ext}^k_A\left(X,\ D(\bar{\Delta}_\bullet)\right) = 0 \text{ for all } k \geq 1\right\}$.

Observe that the statements $(ii)$ of Theorem 2 and $(ii)$ of Theorem 3 are equivalent. Hence, the following Corollary follows (for equivalence of $(iv)$, see [D] or [ADL]).

**Corollary.** *Let $(A, \boldsymbol{e})$ be a $K$-algebra. Then all five statements of Theorem 2 and five statements of Theorem 3, as well as*

$$\dim_K A = \sum_{i=1}^{n} \frac{1}{d_i} \dim_K\left(_\bullet\Delta(i)\right) \cdot \dim_K\left(\bar{\Delta}_\bullet(i)\right) \qquad (iv)$$

*with* $d_i = \dim_K \operatorname{End}\big(\bar{\Delta}_\bullet(i)\big)$ *are equivalent.*

In order to prepare our proof of Theorem 3, let us formulate the following lemma. Recall the notion of the trace filtration of a (left) module ${}_AX$

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \ldots \supseteq X^{(i)} \supseteq X^{(i+1)} \supseteq \cdots \supseteq X^{(n)} \supseteq X^{(n+1)} = 0,$$

where $X^{(i)}$ is the trace $\tau_{A\varepsilon_i}(X) = A\varepsilon_i X$ of the projective module $A\varepsilon_i$ in $X$.

**Lemma ( cf.** [DR] **or** [DK]**).** *Let* $X = X^{(1)} \supseteq \ldots \supseteq X^{(i)} \supseteq \cdots \supseteq X^{(n+1)} = 0$ *and* $Y = Y^{(1)} \supseteq \ldots \supseteq Y^{(i)} \supseteq \cdots \supseteq Y^{(n+1)} = 0$ *be the trace filtrations of the (left) $A$-modules $X$ and $Y$. Let $f : X \to Y$ be an epimorphism. Then the induced morphisms*

$$f_i : X^{(i)} \to Y^{(i)} \text{ and } \bar{f}_i : X^{(i)}/X^{(i+1)} \to Y^{(i)}/Y^{(i+1)}$$

*are epimorphisms,* $\operatorname{Ker} f_i = \operatorname{Ker} f \cap X^{(i)}$, $\operatorname{Ker} \bar{f}_i \cong \operatorname{Ker} f_i / \operatorname{Ker} f_{i+1}$ *and*

$$\operatorname{Ker} f = (\operatorname{Ker} f)^{(1)} \supseteq \cdots \supseteq (\operatorname{Ker} f)^{(i)} = \operatorname{Ker} f \cap X^{(i)} \supseteq \cdots \supseteq (\operatorname{Ker} f)^{(n+1)} = 0$$

*is the trace filtration of* $\operatorname{Ker} f$. *Thus, if $X$ and $Y$ belong to $\mathcal{F}({}_\bullet\Delta)$, so does* $\operatorname{Ker} f$.

*Proof.* The statements follow immediately from the following fact: If $f : X \to Y$ is an epimorphism and $P$ a projective module, then the induced map

$$f_P : \tau_P(X) \to \tau_P(Y)$$

is an epimorphism, $\operatorname{Ker} f_P = \operatorname{Ker} f \cap \tau_P(Y)$ and

$$0 \to \operatorname{Ker} f/\big(\operatorname{Ker} f \cap \tau_P(Y)\big) \to X/\tau_P(X) \to Y/\tau_P(Y) \to 0$$

is an exact sequence. Thus, if both $A/\tau_P(A)$-modules $X/\tau_P(X)$ and $Y/\tau_P(Y)$ are projective, then also $\operatorname{Ker} f/\big(\operatorname{Ker} \cap \tau_P(Y)\big)$ is projective.

*Proof of Theorem 3.* First, we can see immediately that the implications $(iii) \Rightarrow (i)$, $(iii') \Rightarrow (ii')$ and $(ii') \Rightarrow (ii)$ are trivial.

Also, the implication $(i) \Rightarrow (ii')$ follows easily by induction. Recall that always

$$\operatorname{Ext}^1_A({}_\bullet\Delta,\ \mathrm{D}(\bar{\Delta}_\bullet)) = 0.$$

Thus, for every $X \in \mathcal{F}({}_\bullet\Delta)$, $\operatorname{Ext}^k_A\big(X,\ \mathrm{D}(\bar{\Delta}_\bullet)\big) = 0$. Now, given $X \in \mathcal{F}({}_\bullet\Delta)$, consider an exact sequence $0 \to Y \to P \to X \to 0$ with a free module $P$. By $(i)$ $P$ belongs to $\mathcal{F}({}_\bullet\Delta)$, and thus, by Lemma, $Y \in \mathcal{F}({}_\bullet\Delta)$. Moreover,

$$\operatorname{Ext}^{k+1}_A\big(X,\ \mathrm{D}(\bar{\Delta}_\bullet(i))\big) \cong \operatorname{Ext}^k_A\big(Y,\ \mathrm{D}(\bar{\Delta}_\bullet(i))\big), \text{ for all } k \geq 1 \text{ and } 1 \leq i \leq n.$$

In particular, we get $(ii')$.

Similarly, $(iii)$ implies $(iii')$. Indeed, in this case $A_A \in \mathcal{F}({}_\bullet\Delta)$ and, hence following the preceding argument, we get

$$\begin{aligned}\mathcal{F}({}_\bullet\Delta) &= \left\{X \mid \mathrm{Ext}^1_A\left(X,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0\right\} \subseteq \left\{X \mid \mathrm{Ext}^k_A\left(X,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0 \text{ for all } k \geq 1\right\} \\ &\subseteq \left\{X \mid \mathrm{Ext}^1_A\left(X,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0\right\} = \mathcal{F}({}_\bullet\Delta),\end{aligned}$$

and thus all inclusions are equalities.

Consequently, in order to complete the proof, it remains to show that $(ii)$ implies $(iii)$, i.e. to show, assuming $(ii)$, that

$$\mathcal{F}({}_\bullet\Delta) \supseteq \left\{X \in \text{mod}-A \mid \mathrm{Ext}^k_A\left(X,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0\ \right\} = \mathcal{F}.$$

Thus, let $X \in \mathcal{F}$ and let

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \ldots \supseteq X^{(i)} \supseteq X^{(i+1)} \supseteq \cdots \supseteq X^{(n)} \supseteq X^{(n+1)} = 0$$

be the trace filtration of $X$. We shall proceed by induction on the length of the trace filtration. Assume that

$$\left\{Y \in \text{mod}-A \mid \mathrm{Ext}^1_A\left(Y,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0 \text{ and } Y^{(i)} = 0\right\} \subseteq \mathcal{F}({}_\bullet\Delta)$$

and consider $X$ satisfying

$$\mathrm{Ext}^1_A\left(X,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0 \text{ and } X^{(i+1)} = 0.$$

We have two exact sequences, namely

$$0 \to X^{(i)} \to X \to Y \to 0 \text{ with } Y^{(i)} = 0, \tag{$*$}$$

and

$$0 \to Z \to \Phi(i) \to X^{(i)} \to 0, \tag{$**$}$$

where $\Phi(i)$ is a direct sum of a finite number of ${}_\bullet\Delta(i)$'s, and $\Phi(i) \to X^{(i)}$ is a $A/A\varepsilon_{i+1}A$ − projective cover of $X^{(i)}$. Therefore, the induced map

$$\mathrm{Hom}\left(X^{(i)},\ \mathrm{D}(\bar{\Delta}_\bullet(i))\right) \to \mathrm{Hom}\left(\Phi(i),\ \mathrm{D}(\bar{\Delta}_\bullet(i)\right)$$

is surjective.
Now, since for $t \geq i$, $\mathrm{Hom}\left(P(t), Y\right) = 0$,

$$\mathrm{Ext}^1_A\left(Y,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0 \text{ for } j \geq i.$$

Furthermore,

$$\mathrm{Hom}\left(X^{(i)},\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0 \text{ for } j < i,$$

and thus the exact sequence

$$0 = \mathrm{Hom}\left(X^{(i)},\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) \to \mathrm{Ext}^1\left(Y,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) \to \mathrm{Ext}^1\left(X,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0$$

derived from $(*)$ shows that

$$\mathrm{Ext}^1\left(Y,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0 \text{ for } j < i.$$

We conclude, by induction, that $Y$ belongs to $\mathcal{F}({}_\bullet\Delta)$. Therefore, in view of $(ii)$, $\mathrm{Ext}^2_A\left(Y,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0$, and the exact sequence

$$0 = \mathrm{Ext}^1\left(X,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) \to \mathrm{Ext}^1\left(X^{(i)},\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) \to \mathrm{Ext}^2_A\left(Y,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0$$

derived from $(*)$ yields $\mathrm{Ext}^1\left(X^{(i)},\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0$ for all $1 \le j \le n$.

Recall that

$$\mathrm{Hom}({}_\bullet\Delta(i),\ \mathrm{D}(\bar{\Delta}_\bullet(j))) = 0 \text{ for all } i \neq j$$

and consider the exact sequence

$$\begin{aligned}\mathrm{Hom}\left(X^{(i)},\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) \to \mathrm{Hom}\left(\Phi(i),\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) \to \mathrm{Hom}\left(Z,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) \to \\ \to \mathrm{Ext}^1\left(X^{(i)},\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0 \qquad (***)\end{aligned}$$

derived from $(**)$. Thus, for $i \neq j$,

$$\mathrm{Hom}\left(Z,\ \mathrm{D}(\bar{\Delta}_\bullet(j))\right) = 0.$$

For $i = j$, the first map in $(***)$ is surjective and, thus, $\mathrm{Hom}\left(Z,\ \mathrm{D}(\bar{\Delta}_\bullet(i))\right) = 0$ as well. The fact that $\mathrm{Hom}\left(Z,\ \mathrm{D}(\bar{\Delta}_\bullet)\right) = 0$ means, however, that $Z = 0$ and thus $X^{(i)} \cong \Phi(i)$. Consequently, $X \in \mathcal{F}({}_\bullet\Delta)$, as required.

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