Additive functions on trees

BY

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Abstract. The motivation of considering positive additive functions on trees was the characterization of extended Dynkin graphs (see I. Reiten [R]) and the application of additive functions in the representation theory (see H. Lenzing and I. Reiten [LR] and T. Hübner [H]).

We consider graphs equipped with functions of integer values, i.e.valued graphs (see also [DR]). Methods are given for the construction of additive functions on valued trees (in particular on Euclidean graphs) and for the characterization of their structure. We introduce the concept of almost additive functions, which are additive on each vertex of a graph except for one (called exceptional vertex). On (valued) trees (with fixed exceptional vertex) the almost additive functions are unique up to rational multiples. For valued trees a necessary and sufficient condition is given for the existence of positive almost additive functions.

Introduction. The Dynkin diagrams and the associated extended Dynkin diagrams occur in the representation theory of finite dimensional algebras. These diagrams can be characterised using additive and subadditive functions (see [R]). The concept of an additive function attached to a finite dimensional algebra is homological in nature (see [LR]). It is well known that among the connected quivers exactly the extended Dynkin quivers admit a positive additive function. The main motivation of this paper was the characterization of extended Dynkin graphs given by Reiten (see [R]) and some additive functions in the representation theory given by Lenzing-Reiten ([LR])and Hübner ([H]). The main result (Theorem 1.6) shows that for a valued tree there is a positive almost additive function with an exceptional vertex if and only if the tree is an enlarged Dynkin. This result answers for valued trees the Reiten's question: which graphs admot nontrivial additive functions. There are also given some inductive constructions of almost additive functions.

1. Additive and almost additive functions on valued trees.

Throughout, Δ will always be a finite graph without multiple edges and without loops (which is a finite set $I = \{1, 2, \ldots, n\}$ of the vertices, together with a set of (unordered) pairs $(i, j) \in I \times I,\ i \neq j \in I$, called the edges of Δ).

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Let $V = \mathbb{Z}^n$ be a free abelian group of rank n and let V be equipped with an - usually non - symmetric bilinear form:

$$\langle -,- \rangle \;:\; V \times V \to \mathbb{Z}.$$

We also assume that $\langle x,- \rangle = 0 \iff x = 0$; i.e. $\langle x,y \rangle = x^t C y$, where C is a non-singular integer matrix.
An automorphism $\mathcal{C}$ of V is called *Coxeter transformation* of V if

$$\langle x,y \rangle = -\langle y, \mathcal{C}x \rangle \text{ for all } x,y \in V.$$

The matrix Φ of the Coxeter transformation is uniquely determined by the matrix C, and also by $\Phi = -C^{-1}C^t$, since

$$\langle x,y \rangle = x^t C y = y^t C^t x = -y^t C C^{-1} C^t x = -\langle y, -C^{-1}C^t x \rangle.$$

The characteristic polynomial of the matrix of a Coxeter transformation $\mathcal{C}$ is called the *Coxeter polynomial* of $\mathcal{C}$. The subgroup of $\mathbb{Z}^n$ is called a *radical* if it is the set of the fixed points of the Coxeter transformation $\mathcal{C}$, i.e.

$$\mathrm{rad}(\mathcal{C}) = \big\{ x \in \mathbb{Z} \mid \mathcal{C}x = x \big\}.$$

The *spectrum* $\mathrm{Spec}(\mathcal{C})$ of $\mathcal{C}$ is the set of all eigenvalues of the matrix Φ, the *spectral radius* of $\mathcal{C}$ is given by

$$\rho(\mathcal{C}) = max\{ \mid \lambda \mid : \lambda \in Spec(\mathcal{C}) \}.$$

A *valued graph* (see [DR]) $(\Delta, \mathbf{v})$ is a graph Δ together with a valuation $\mathbf{v}$ defined as follows:
For each edge $\overset{i}{\bullet} \!\!\frac{\quad\quad}{}\!\! \overset{j}{\bullet}$, there exist two non-negative integers $v_{i,j}$ and $v_{j,i}$: $\overset{i}{\bullet} \xrightarrow{(v_{i,j},v_{j,i})} \overset{j}{\bullet}$ such that

$$v_{i,j} f_j = v_{j,i} f_i$$

holds with some positive integers f_i, f_j $(i,j \in I)$.
Furthermore, set $v_{i,j} = v_{j,i} = 0$ if there is no edge between i and j.
Any graph T can be considered as a valued graph $(T, \mathbf{v})$ with a trivial valuation ($v_{i,j} = v_{j,i} = 0$ if there is no edge between i and j $v_{i,j} = 1$ otherwise).
In case $v_{i,j} = v_{j,i} = 1$ we write simply $\overset{i}{\bullet} \!\!\frac{\quad\quad}{}\!\! \overset{j}{\bullet}$ instead of $\overset{i}{\bullet} \xrightarrow{(1,1)} \overset{j}{\bullet}$, (i.e. we omit the label of valuation).

The matrix $A_\Delta = \big(a_{i,j}\big)$, where $a_{i,j} = v_{i,j}$ is called the *adjacency matrix* of the valued graph $(\Delta, \mathbf{v})$. By the definition of the valued graph the matrix $A = A_\Delta$ is symmetrizable, which means that DA is a symmetric matrix where $D = \big(d_{i,j}\big)$ is a diagonal matrix defined by $d_{i,i} = f_i$ and $d_{i,j} = 0$ otherwise.

Let Ω be an orientation of the graph $(\Delta, \mathbf{v})$. Denote by $Q = Q_{(\Delta,\Omega)}$ this oriented graph. Suppose there are no oriented cycles in Q. The Coxeter transformation is defined only for a quiver, i.e. for a finite oriented graph. Since, for a tree, our considerations will not depend on a particular orientation, (see [BLM]), we may speak about the Coxeter polynomial and spectral radius of the Coxeter transformation of a valued tree and we always choose the orientation such that for all $i, j \in I$ we have $\overset{i}{\bullet} \longrightarrow \overset{j}{\bullet}$ if $i < j$. Consequently, we may speak about the Coxeter polynomial and spectral radius of the Coxeter transformation of a valued tree.

Let us remark that the Coxeter transformation $\mathcal{C}$ for $Q = Q(\Delta, \mathbf{v})$ is defined by the matrix $C = D - DA_+$, where DA_+ is the upper triangular part of the symmetric matrix DA.

The following statement was proved for bipartite finite oriented graph without oriented cycle and it determines the relationship between the spectra of a valued tree and the spectra of its Coxeter transformation.

Lemma 1.1. [BLM] *Let $T = (T, \mathbf{v})$ be a valued tree.*

a) If any $\lambda \neq 0$ then $\lambda + \lambda^{-1} \in \mathrm{Spec}(T)$ if and only if $\lambda^2 \in \mathrm{Spec}(\mathcal{C}_T)$.

b) If T is not Dynkin, then there exists a real number $\lambda \geq 1$ such that $\rho(T) = \lambda + \lambda^{-1}$ and $\rho(\mathcal{C}_T) = \lambda^2$. Moreover, Δ is Euclidean if and only if $\lambda = 1$.

A function $\varphi : I \to \mathbb{Z}$ with integer values is said to be a *subadditive function* on an (arbitrary) graph Δ with adjacency matrix A and set of vertices I if

$$\sum_{j\in I} a_{i,j}\varphi(j) \leq 2\varphi(i) \text{ for all } i \in I,$$

and it is said to be an *additive function on* Δ if

$$\sum_{j\in I} a_{i,j}\varphi(j) = 2\varphi(i) \text{ for all } i \in I. \tag{1}$$

It is known that the existence of a positive subadditive non-additive function on a finite connected graph implies the existence of a positive definite associated quadratic form and the existence of a positive additive function implies the existence of a positive semidefinite associated quadratic form (see [R]). In the first case the graph is Dynkin and in the second it is Euclidean.

A function $\varphi : I \to \mathbb{Z}$ with integer values is said to be an *almost additive function with the exceptional vertex k* on an (arbitrary) graph Δ

with set of vertices I and with adjacency matrix A if

$$\sum_{j\in I} a_{i,j}\varphi(j) = 2\varphi(i) \text{ for every } i \neq k.$$

An additive function φ is called *positive*, if $\varphi(i) > 0$ for each $i \in I$. Also we call φ *nonnegative* if φ is nonzero and $\varphi(i) \geq 0$ for each i, and call *negative* if $-\varphi$ is positive.

Let (Δ, φ) be a graph Δ together with an (almost) additive function φ. Removing all vertices $x \in I$ with $\varphi(x) = 0$ and all edges containing such vertices x, we get a subgraph of Δ with an (almost) *additive function without zero values.* Since this removing process does not change the additive property we may suppose that all of our (almost) additive functions are without zero values.

The additive functions are uniquely determined up to integer multiples. To avoid misunderstandings, we always consider so called *normalized* (almost) additive functions with minimal integer values, i.e. the least common divisor of their values is 1. For the characterization of (almost) additive functions φ sometimes we need these functions with rational values, i.e. the rational multiples of the (almost) additive functions. To make our calculation easier we shall, sometimes, fix the value of the function φ at the exceptional vertex k to be 1. Such a function will be called a *reduced (almost) additive function.*

For a quiver Q without oriented cycle the elements of the radical of the corresponding Coxeter transformation determine an additive function on the underlying valued graph.

LEMMA 1.2. *Let Q be a finite oriented graph without oriented cycles and with underlying graph $(\Delta, \mathbf{v})$ and I the set of its vertices. The following statements are equivalent for the additive function $\varphi : I \to \mathbb{Z}$:*
a) $\hat{\varphi} = \big(\varphi(1), \ldots, \varphi(n)\big) \in rad(\mathcal{C})$
b) $\hat{\varphi}$ *is the eigenvector of the Coxeter matrix* Φ *of* Q *with eigenvalue* 1.
c) $\hat{\varphi}$ *is the eigenvector of the adjacency matrix* $(\Delta, \mathbf{v})$ *with eigenvalue* 2.
d) φ *is additive on the graph* $(\Delta, \mathbf{v})$.

Proof. The statements $a) \iff b)$ and $d) \iff c)$ follow from the definitions of the additive function and the radical of the Coxeter transformation. The equivalence of $b)$ and $c)$ follows from Lemma 1.1. ∎

The Dynkin graphs have no additive functions since for their adjacency matrix A the matrix $2I - A$ is non-singular. It is known (see [R]) that a connected graph has a positive additive function φ if and only if the graph is extended Dynkin (Euclidean). In this case φ is uniquely determined. We remark that normalized subadditive functions on a graph are, in general, not uniquely determined.

Example 1.1. Let us write the values of the functions at the corresponding vertices. On the Dynkin graph A_3 the functions $\overset{1}{\bullet}\!-\!\!-\!\!-\overset{2}{\bullet}\!-\!\!-\!\!-\overset{1}{\bullet}$ and $\overset{1}{\bullet}\!-\!\!-\!\!-\overset{2}{\bullet}\!-\!\!-\!\!-\overset{3}{\bullet}$ are different almost additive and subadditive functions. The exceptional vertex of the almost additive function is in the first case the middle vertex and in the second case it is the right one.

Let $(\Delta, \mathbf{v})$ be a valued tree, $k \in I$ and φ be an almost additive function on I with an exceptional vertex $k \in I$ with $\varphi(k) \neq 0$. Define the *deviation* $d_k \in \mathbb{Q}$ of φ at the vertex k by the equation

$$\varphi(k)(2 - d_k) = \sum_{j \in I} a_{k,j}\ \varphi(j). \tag{2}$$

Clearly, d_k is uniquely determined and $d_k = 0$ if and only if φ is *additive at the vertex* k, i.e. $\sum_{j\in I} a_{k,j}\varphi(j) = 2\varphi(k)$.

In Example 1.1 the deviation of $\hat{\varphi}_1 = (1, 2, 1)$ at the middle vertex is 1 and the deviation of $\hat{\varphi}_2 = (1, 2, 3)$ at the right vertex is $\frac{4}{3}$, i.e. on a tree we can define different almost additive functions by choosing different exceptional vertices. The following theorem gives the answer to the question about the uniqueness of an almost additive function with fixed exceptional vertex.

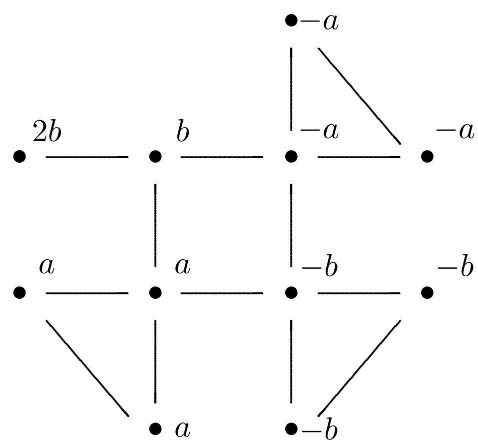
We denote by $T \setminus \{k\}$ the tree obtained from T by deleting the vertex k and all adjacent edges.

Theorem 1.3. *Let $(T, \mathbf{v})$ be a valued tree. Let $k \in I$ be an exceptional vertex of an almost additive function φ on T without zero values. Then φ is uniquely determined up to a rational multiple.*

Proof. We prove by induction on the number n of vertices of T. For $n = 1$ the statement is obvious. If $n > 1$ then remove the exceptional vertex k and all adjacent edges from T. By the induction hypothesis we have unique almost additive functions on the connected components of $T \setminus \{k\}$. The exceptional vertices of these almost additive functions are the vertices which were connected to k. This implies the uniqueness of our additive functions on T with exceptional vertex k and our statement follows. ∎

Remark that if the underlying graph is not a tree then, as the next counterexample shows, the uniqueness does not hold.

Example 1.2. Consider the following graph where the labels of the vertices are the values of a function φ on the graph. If a and b are arbitrary relative prime numbers then, as one can easily check, φ is an almost additive function with exceptional vertex labeled by $2b$.



It is easy to see that if a graph with a strictly positive almost additive function has positive deviation then it can be extended to a positive almost additive function. This is possible by connecting a new vertex to the exceptional vertex.

Theorem 1.4. *Let φ be an almost additive function on the valued tree $(T, \mathbf{v})$ with exceptional vertex k. Suppose $\varphi(k) \neq 0$. Denote by $\chi_T(x)$ the Coxeter polynomial of T. Then the deviation of the almost additive function φ at k is*

$$d_k = \frac{\chi_T(1)}{\chi_{T\setminus\{k\}}(1)}, \tag{3}$$

thus d_k is uniquely determined.

Proof. Denote by A the adjacency matrix of T and let e_k be the kth row vector of the $n \times n$ identity matrix. We may suppose that φ is reduced i.e. $\varphi(k) = 1$. For the almost additive function φ and the deviation d_k of the exceptional vertex k we have

$$(2I - A - d_k e_k^t e_k)\hat{\varphi}^t = 0 \tag{4}$$

where $\hat{\varphi}$ is the vector introduced in Lemma 1.2.
If $\mid 2I - A \mid= 0$ then φ is additive on T which implies by Lemma 1.2 the equalities $\chi_T(1) = 0$ and $d_k = 0$. By $\varphi(k) \neq 0$ the restriction of φ is not additive on $T \setminus \{k\}$ and $\chi_{T\setminus\{k\}}(1) \neq 0$. Thus, the statement in case $\mid 2I - A \mid= 0$ follows.

Assume that $\mid 2I - A \mid \neq 0$. Suppose that the almost additive function φ is reduced i.e. $\varphi(k) = 1$. This implies that $(2I - A)\hat{\varphi}^t = d_k e_k^t e_k \hat{\varphi}^t$ hence $\hat{\varphi}^t = d_k(2I - A)^{-1} e_k^t e_k \hat{\varphi}^t$ and $\frac{1}{d_k}\hat{\varphi}^t = (2I - A)^{-1} e_k^t$. Moreover, since $\varphi(k) = 1$ we have $\hat{\varphi}^{-1} = e_k$ and $\frac{1}{d_k} = e_k(2I - A)^{-1} e_k^t$ which is the entry of the (k, k)'s position of the inverse of the non-singular matrix $2I - A$. Thus, we have $\frac{1}{d_k} = f_{T\setminus\{k\}}(2)/f_T(2)$, where $f_\Gamma(x)$ is the characteristic polynomial of the graph Γ and $f_T(2) \neq 0$ since φ is not additive.

From the uniqueness of $(2I - A)^{-1}$ the uniqueness of d_k follows. In view of the correspondence between the spectrum of the graph and the spectrum of the corresponding Coxeter transformation by Lemma 1.1. we have

$$f_{T\setminus\{k\}}(2)/f_T(2) = \chi_{T/\{k\}}(1)/\chi_T(1).$$

■

Corollary 1.5. *The deviation of each almost additive function on a Dynkin graph is strictly positive and on an Euclidean graph it is zero.*

Proof. Let T be a Dynkin graph. With the notation of Theorem 1.4 we have

$$d_k = \frac{\chi_T(1)}{\chi_{T\setminus\{k\}}(1)}.$$

It is known that the Coxeter polynomial of Dynkin graphs has only cyclotomic polynomials as its irreducible factors. The sum of the coefficients of products of cyclotomic polynomials is positive and 1 is a root of the Coxeter polynomial of an Euclidean graph. The Coxeter polynomials of a Dynkin graph can be decomposed into irreducible cyclotomic factors (see [BLM]), at 1 they have positive value. T is Dynkin therefore $T \setminus \{k\}$ is also Dynkin. It follows that d_k is positive. ■

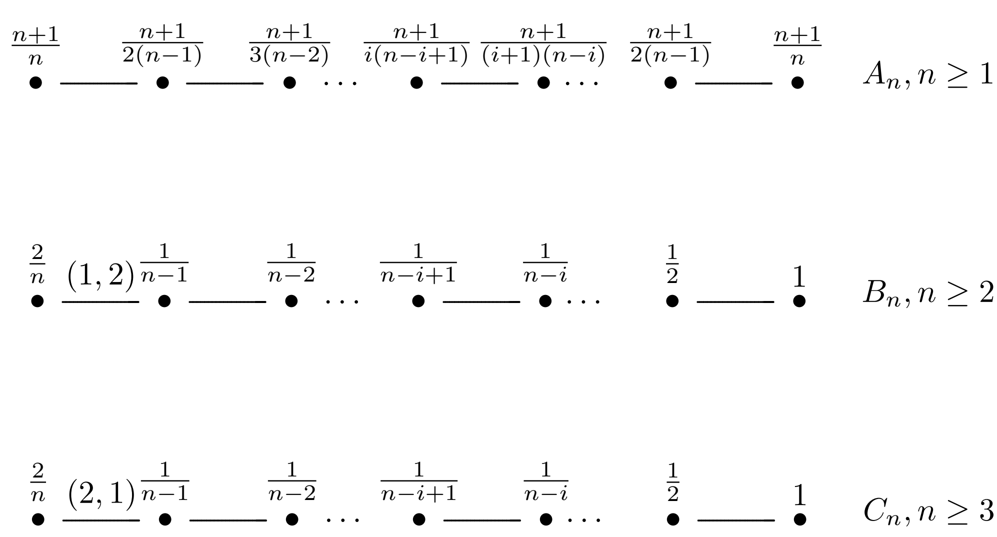
The question about the sign of the deviation in case of wild graphs is much more complicated. One graph T is said to be *enlarged Dynkin* if it can be decomposed into Dynkin graphs by removing exactly one vertex and all edges adjacent to it in T. Clearly, the Dynkin graphs with $n > 2$ vertices are also enlarged Dynkin's.

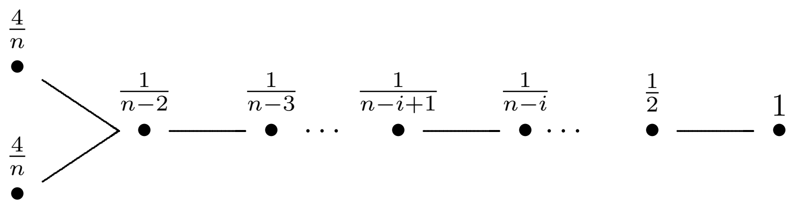
Theorem 1.6. *Let T be a valued tree. Then there exists a positive almost additive function φ with any exceptional vertex $k \in I$ if and only if T is an enlarged Dynkin graph. Fixing a vertex $k \in I$, the almost additive function φ with exceptional vertex k is unique.*

Proof. The existence of almost additive function is clear. Let φ be a positive almost additive function on T with exceptional vertex k. Suppose

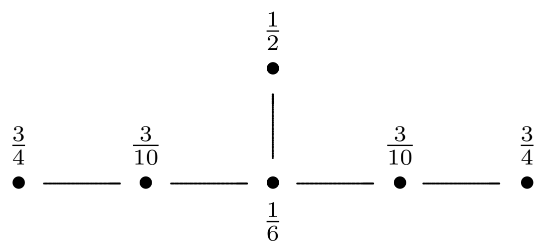
φ is reduced i.e. $\varphi(k) = 1$. By (4) φ is the eigenvector of the matrix $A - d_k e_k^t e_k$ corresponding to the eigenvalue 2. We may assume that $\mid 2I - A \mid \neq 0$, otherwise by Lemma 1.2 φ is additive on T and by Corollary 1.5 T is Euclidean which is enlarged Dynkin and the statement for such a graph holds.
By the Perron Frobenius Theorem the positive eigenvector $\hat{\varphi}$ corresponds to the maximal eigenvalue 2 of the matrix $A - d_k e_k^t e_k$. By the interlacing property the maximal eigenvalue of the adjacency matrix of $T \setminus \{k\}$ is less than 2. Thus, by Lemma 1.1 the maximum of the absolut value of the eigenvalue of the Coxeter transformation is less than 1 and the tree $T \setminus \{k\}$ is Dynkin, i.e. T is an enlarged Dynkin graph.
Conversely, suppose T is an enlarged Dynkin graph i.e. it can be decomposed into Dynkin graphs by removing the vertex k and the corresponding edges. It is easy to check that to every Dynkin graph and for each of its vertex $k \in I$ there exists at least one positive almost additive function φ with the exceptional vertex k. Let d_k be the deviation of the almost additive function with the exceptional vertex k.
The uniqueness follows from the Perron-Frobenius Theory since φ is (strictly) positive, thus $\hat{\varphi}$ is the only eigenvector of the matrix $A - d_k e_k^t e_k$ corresponding to the unique maximal eigenvalue 2. ∎

2. Inductive constructions of almost additive functions. Since the vertices of Dynkin graphs are well characterized by the deviation of uniquely determined positive almost additive functions corresponding to the vertices, on the next picture we show the list of these graphs labeling the vertices with the deviation values.

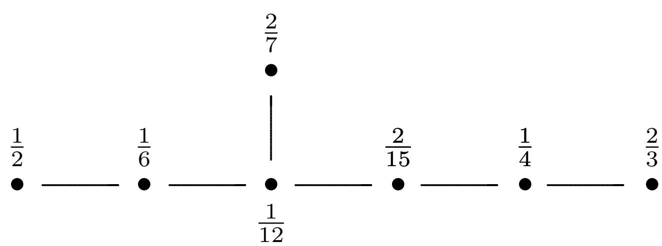




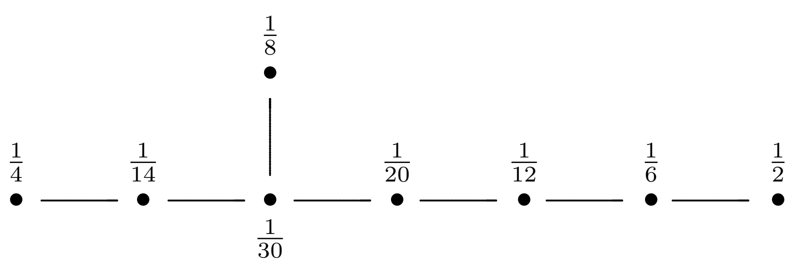
$D_n, n \geq 4$



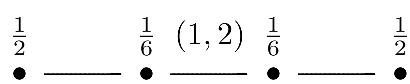
E_6



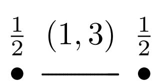
E_7



E_8



F_4



G_2

The connection between the existence of an additive function on a graph and existence of positive almost additive functions on its subgraphs seems to be an interesting problem. The following statement characterizes the almost additive functions on trees which consist of trees (with almost additive functions at their exceptional vertices) hanging on a new vertex at their exceptional vertices. We shall call such trees *one-point extension* of the original trees.

THEOREM 2.1. *Let* $T_1 = (T_1, \mathbf{v_1})$, $T_2 = (T_2, \mathbf{v_2}), \ldots, T_s = (T_s, \mathbf{v_s})$ *be valued trees with Let* $\varphi_1, \varphi_2, \ldots, \varphi_s$ *be almost additive functions on* T_1,$T_2, \ldots, T_s$ *with exceptional vertices* $k_1 \in (T_1)_0$, $k_2 \in (T_2)_0, \ldots, k_s \in (T_s)_0$ *and with deviations* $d_{k_1}, d_{k_2}, \ldots, d_{k_s}$. *Let* T *be a graph obtained from* T_1, $T_2, \ldots T_s$ *with one-point* (k) *extension at the vertices* $k_1, k_2, \ldots, k_s$. *Then the extended almost additive function on* T *with the exceptional vertex* k *has the deviation* $2 - (\frac{1}{d_{k_1}} + \frac{1}{d_{k_2}} + \ldots + \frac{1}{d_{k_s}})$.

Proof. We may suppose without restricting the generalization that $\varphi_1, \ldots, \varphi_s$ are reduced almost additive functions, i.e. $\varphi(k_l) = 1$ for all $1 \leq l \leq s$. Let $S_{k_l} = \sum_{j \sim k_l} \varphi_{k_l}(j)$, where $k_l \sim j$ means that the vertices $j \in (T_l)_0$ are connected to the vertex k_l. By definition of the deviation $2 - S_{k_l} = d_{k_l}$ holds. At the graph T the extension of the almost additive functions $\varphi_2, \ldots, \varphi_s$ will be almost additive at the vertex k_l if (preserving the values except for the values on the corresponding exceptional vertices) there exists $r_l \in \mathbb{Z}$ such that $\varphi(k) + r_l S_{k_l} = 2r_l\varphi(k_l) = 2r_l$ for $1 \leq k_l \leq s$. Thus

$$\frac{\varphi(k)}{r_l} = 2 - S_{k_l} = d_{k_l}.$$

For the deviation d_k at the vertex k we have

$$d_k = 2 - \frac{r_1 + r_2 + \ldots + r_s}{\varphi(k)} = 2 - (\frac{1}{d_{k_1}} + \ldots + \frac{1}{d_{k_s}}).$$

■

Theorem 2.1 explains how can we construct Dynkin and Euclidean (in other words extended Dynkin) graphs from Dynkin‘s by using almost additive functions. Extending the Dynkin graph by one vertex at any vertex with deviation $\frac{1}{2}$ we get an Euclidean graph since the deviation of A_1 (a simple graph with one vertex) is equal to 2. Taking a Dynkin graph and any of its vertices k with $d_k > 0.5$ we may enlarge our graph with a new vertex connected to k such that the enlarged graph remains Dynkin. For example E_6 and E_7 can be enlarged (to E_7 and E_8 respectively) by connecting a new vertex to the vertices with deviations $\frac{3}{4}$ and $\frac{2}{3}$. Also B_n, C_n, D_n can be enlarged by a new vertex at the vertices with deviation 1.

In this way we have a *new method to find the complete list of the Euclidean graphs.*

The following statement presents the solution of the problem of determining additive functions on a tree in a special case.

THEOREM 2.2. *Let $T_1 = (T_1, \mathbf{v_1})$ and $T_2 = (T_2, \mathbf{v_2})$ be valued trees with $i \in (T_1)_0$ and $j \in (T_2)_0$. Let φ_1 and φ_2 be almost additive functions with exceptional vertices i and j with $\varphi(i), \varphi(j) \neq 0$ and with corresponding deviations d_i and d_j. Let T be a graph obtained from T_1 and T_2 by connecting them by an edge with $\overset{i}{\bullet} \;\text{——}\; \overset{j}{\bullet}$. Then there exists a uniquely determined additive function on T if and only if $d_i d_j = 1$.*

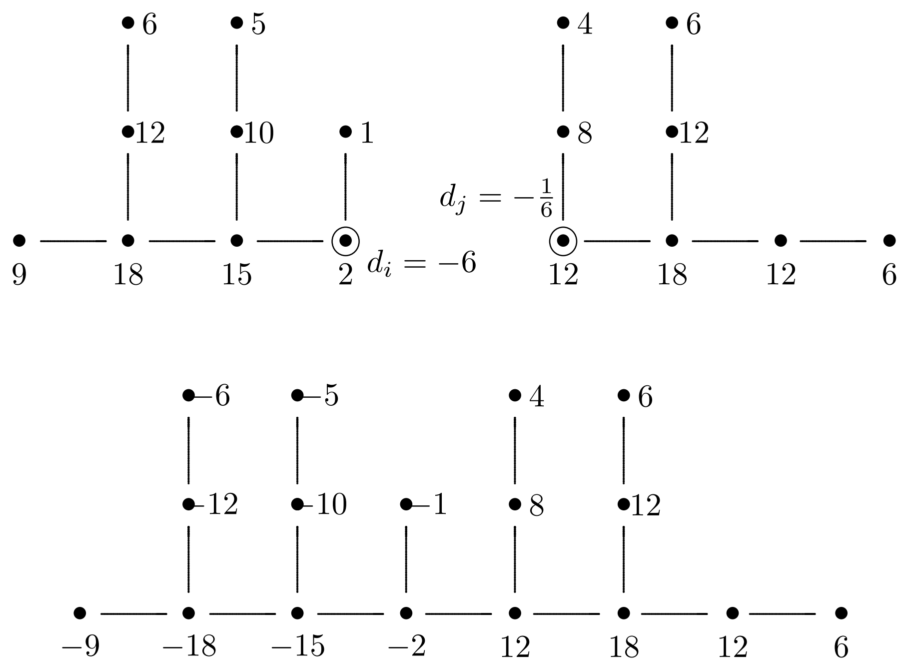
Proof. We require the almost additivity with the exceptional vertices i and j. Therefore we should find the integers l_1 and l_2 such that

$$l_1 d_i = l_2 \varphi_2(i) \text{ and } l_2 d_j = l_1 \varphi_1(j). \tag{5}$$

Since $\varphi_2(i) = 1$ and $\varphi_2(j) = 1$ the system of equations (5) has a solution if and only if $d_i d_j = 1$. ∎

The following example shows how to construct an additive function from two suitable almost additive functions with deviations d_i and d_j (by Theorem 2.2).

EXAMPLE 2.1.



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