ON ZEROS OF RECIPROCAL POLYNOMIALS

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ABSTRACT. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geq 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \ldots, \left[\frac{m}{2}\right]$) are on the unit circle, provided that the "coefficient condition"

$$|A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|$$

is satisfied.

Moreover, if the "coefficient condition" holds, then all zeros e^{iu_j} $(j = 1, 2, \ldots, m)$ can be arranged such that

$$\left| e^{i\frac{2\pi j}{m+1}} - e^{iu_j} \right| < \frac{\pi}{m+1} \quad (j = 1, \ldots, m).$$

If $m = 2n + 1$ is odd, then $-1 = e^{iu_{n+1}}$ is always a zero, and all zeros of P_{2n+1} are single.
If $m = 2n$ is even, if the "coefficient condition" holds with equality and if

$$\text{sgn } A_{2n} = \text{ sgn } (-1)^{k+1}(A_k - A_{2n})$$

for all $k = 1, 2, \ldots, n$ with $A_k - A_{2n} \neq 0$, then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_{2n}. Otherwise all zeros of P_{2n} are single.

1. Introduction

The Coxeter transformation was introduced in the representation theory of finite dimensional algebras (see [2]). The characteristic polynomial of the Coxeter transformation of an oriented graph whose underlying graph is a wild star is a Salem polynomial (see [3], [4]).

Allowing circles in the underlying graph, the spectral properties of the Coxeter transformations get much more complicated. These properties are related to polynomials of the form

$$l(z^m + z^{m-1} + \cdots + z + 1) + \left(z^k + z^{m-k}\right) \quad (z \in \mathbb{C})$$

where m, k are fixed non-negative integers with $m \geq 2$, $1 \leq k \leq \left[\frac{m}{2}\right]$ and l is a fixed real number.

The zeros of the first expression $l(z^m + z^{m-1} + \cdots + z + 1)$ are

$$\epsilon_j = e^{i\frac{j}{m+1}2\pi} \quad (j = 1, 2, \ldots, m)$$

the $(m + 1)$st roots of unity except 1, *they are on the unit circle*. It is surprising that adding $z^k + z^{m-k}$ to the first expression the polynomial obtained inherits this property. Moreover, *not just all zeros remain on the unit circle but they move away from ϵ_j just a little* even if we add a

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linear combination $\sum_{k=1}^{\left[\frac{m}{2}\right]} a_k(z^k+z^{m-k})$ to the expression $l(z^m+z^{m-1}+\cdots+z+1)$, *provided that* $|l|$ *is large enough.* This leads to the main result of the paper: giving a sufficient condition for reciprocal polynomials to have all of their zeros on the unit circle and also giving the location of the zeros.

Our basic tool is a transformation of semi-reciprocal polynomials called the Chebyshev transformation. Although this transformation seems to be well known we could not find a suitable reference. In Section 2, based on [1], we summarize the properties of the Chebyshev transformation. In Section 3 we formulate our results and prove them. In Section 4 we discuss the necessity of our sufficient condition.

2. The Chebyshev transformation

A polynomial p of the form $p(z) = \sum_{j=0}^{m} a_j z^j \quad (z \in \mathbb{C})$ where $a_j \in \mathbb{C}$ are given numbers with $a_m \neq 0,\ a_j = a_{m-j} \ \left(j = 0, \ldots, \left[\frac{m}{2}\right]\right)$ is called a *reciprocal polynomial of degree* m.

We need a more general class of reciprocal polynomials (of even degree).

Definition 1. *A polynomial* p *of the form*

$$p(z) = \sum_{j=0}^{2n} a_j z^j \quad (z \in \mathbb{C}) \tag{1}$$

where $n \in \mathbb{N}, a_0, \ldots, a_{2n} \in \mathbb{R}$ *and*

$$a_j = a_{2n-j} \ (j = 0, \ldots, n-1) \tag{2}$$

is called a real semi-reciprocal polynomial of degree at most $2n$*. If* $a_{2n} \neq 0$ *we call* p *a real reciprocal polynomial of degree* $2n$*.*

Denote by $\mathcal{R}_{2n}$ the set of all real semi-reciprocal polynomials of degree at most $2n$.

If $p \in \mathcal{R}_{2n},\ p \neq o$ (o=the zero polynomial), then there is an integer $k,\ 0 \leq k \leq n$, such that

$$a_{2n} = a_{2n-1} = \cdots = a_{n+k+1} = 0 = a_{n-k-1} = \cdots = a_0 \text{ but } a_{n+k} = a_{n-k} \neq 0. \tag{3}$$

Hence

$$p(z) = \sum_{j=0}^{2n} a_j z^j = z^n \left[a_{n+k}\left(z^k + \frac{1}{z^k}\right) + \cdots + a_{n+1}\left(z + \frac{1}{z}\right) + a_n\right]. \tag{4}$$

Let T_j be the jth Chebyshev polynomial of the first kind, defined by

$$T_j(\cos x) = \cos jx \ (j = 0, 1, \ldots).$$

With $z + \frac{1}{z} = x$ we have $z^j + \frac{1}{z^j} = C_j(x) \ (j = 1, 2, \ldots)$ (see e.g. [6], p. 224) where

$$C_j(x) := 2T_j\left(\frac{x}{2}\right) \quad (x \in \mathbb{C},\ j = 1, 2, \ldots)$$

are the normalized Chebyshev polynomials of the first kind. For us it will be now more convenient to define C_0 by

$$C_0(x) := T_0(x) \quad (x \in \mathbb{C}).$$

Hence, from (4)

$$p(z) = z^n \sum_{j=0}^{k} a_{n+j} C_j(x) = a_{n+k}\, z^n \prod_{j=1}^{k} (x - \alpha_j) \tag{5}$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \ldots, k)$ are the zeros of the polynomial $\sum_{j=0}^{k} a_{n+j} C_j(x)$. Equation (5) remains true in the case when $k = 0$, i.e. $p(z) = a_n z^n$ if we agree that

$$\prod_{j=1}^{0} b_j := 1. \tag{6}$$

Going back to the variable z we get that

$$p(z) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} z \left(z + \frac{1}{z} - \alpha_j \right) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} (z^2 - \alpha_j\, z + 1).$$

With this we have justified

Proposition 1. *Every non-zero polynomial $p \in \mathcal{R}_{2n}$ has the decomposition*

$$p(z) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} (z^2 - \alpha_j\, z + 1) \tag{7}$$

where $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, $a_{n+k} \neq 0$ for some k with $0 \leq k \leq n$ and the convention (6) is adopted. If $p \in \mathcal{R}_{2n}$ is a reciprocal polynomial of degree $2n$, then (7) holds with $k = n$.

Definition 2. *The Chebyshev transform of a non-zero polynomial $p \in \mathcal{R}_{2n}$ having the decomposition (7) is defined by*

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^{k} (x - \alpha_j) \tag{8}$$

(with (6) adopted) while for the zero polynomial $p = o$ let

$$\mathcal{T}o(x) = 0. \tag{9}$$

It is clear that $\mathcal{T}$ maps $\mathcal{R}_{2n}$ into the set $\mathcal{P}_n$ of all polynomials of degree $\leq n$ with real coefficients.

Proposition 2. *The Chebyshev transform $\mathcal{T}$ is an isomorphism of the (real) vector space $\mathcal{R}_{2n}$ onto $\mathcal{P}_n$.*

Proof. (i) *$\mathcal{T}$ preserves the addition and the multiplication by a real constant.* Using (5) and (3) (to include also the zero coefficients into the sum) we can write $\mathcal{T}p$ into the form

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^{k} (x - \alpha_j) = \sum_{j=0}^{k} a_{n+j} C_j(x) = \sum_{j=0}^{n} a_{n+j} C_j(x)$$

and the last form of $\mathcal{T}p$ is valid also for the zero polynomial. Taking now another $q \in \mathcal{R}_{2n}$ with $q(z) = \sum_{j=0}^{2n} b_j z^j$ ($b_j = b_{2n-j}$ for $j = 0, \dots, n-1$) and constants $\alpha, \beta \in \mathbb{R}$ we have

$$(\alpha p + \beta q)(z) = \sum_{j=0}^{2n} (\alpha a_j + \beta b_j) z^j$$

thus

$$\begin{aligned} \mathcal{T}(\alpha p + \beta q)(x) &= \sum_{j=0}^{n} (\alpha a_{n+j} + \beta b_{n+j}) C_j(x) \\ &= \alpha \sum_{j=0}^{n} a_{n+j} C_j(x) + \beta \sum_{j=0}^{n} b_{n+j} C_j(x) = \alpha \left(\mathcal{T}p(x)\right) + \beta \left(\mathcal{T}q(x)\right). \end{aligned}$$

(ii) $\mathcal{T}$ *maps onto* $\mathcal{P}_n$. Every polynomial $\tilde{r} \in \mathcal{P}_n$ can uniquely be written as a (real) linear combination of $C_0, C_1, \dots, C_n$ in the form $\tilde{r}(x) = \sum_{j=0}^{n} A_{n+j} C_j(x)$ ($A_{n+j} \in \mathbb{R}$). With $r(z) := \sum_{j=0}^{2n} A_j z^j$ where $A_j := A_{2n-j}$ for $j = 0, \dots, n-1$ we have $r \in \mathcal{R}_{2n}$ and

$$\mathcal{T}r = \tilde{r}$$

proving our claim.
(iii) $\mathcal{T}$ *is one-to-one.* Namely, if $\mathcal{T}p = \mathcal{T}q$ for $p, q \in \mathcal{R}_{2n}$, then $\mathcal{T}p - \mathcal{T}q = \mathcal{T}(p-q) = o$ hence, by (8), (9) $p - q = o$, $p = q$. ☐

Lemma 1. (i) *Let p be a real reciprocal polynomial of degree $2n$. Then all zeros of p are on the unit circle if and only if all zeros of its Chebyshev transform $\mathcal{T}p$ are in the closed interval $[-2, 2]$.*

(ii) *Moreover, if all zeros α_j of $\mathcal{T}p$ are in $[-2, 2]$, written as $\alpha_j = 2\cos u_j$ with $u_j \in [0, \pi]$ $(j = 1, 2, \dots, n)$, then all zeros of p are given by*

$$e^{\pm i u_j} \quad (j = 1, 2, \dots, n).$$

The multiplicity of $\alpha_j \neq \pm 2$ is the same as the multiplicities of e^{iu_j} and e^{-iu_j} $(j = 1, 2, \dots, n)$ while in the case of $\alpha_j = \pm 2$ the multiplicities of the corresponding zeros $e^{iu_j} = \pm 1$ of p are doubled.

Proof. (i)*Necessity.* Suppose that all zeros of p are on the unit circle. They can be arranged in conjugate pairs $(\beta_1, \bar{\beta}_1)$, $(\beta_2, \bar{\beta}_2) \dots (\beta_n, \bar{\beta}_n)$. By assumption $|\beta_j|^2 = \beta_j \bar{\beta}_j = 1$, $\bar{\beta}_j = \dfrac{1}{\beta_j}$ $(j = 1, \dots n)$, hence

$$p(z) = a_{2n} \prod_{j=1}^{n} (z - \beta_j)(z - \bar{\beta}_j) = a_{2n} \prod_{j=1}^{n} \left(z^2 - (\beta_j + \bar{\beta}_j) z + 1\right)$$

and

$$\mathcal{T}p(x) = a_{2n} \prod_{j=1}^{n} \left(x - (\beta_j + \bar{\beta}_j)\right).$$

It is clear that $|\beta_j + \bar{\beta}_j| = |2\mathrm{Re}\,(\beta_j)| \leq 2|\beta_j| = 2$.

(i) *Sufficiency.* Assume that the Chebyshev transform has the form

$$\mathcal{T}p(x) = a_{2n}\prod_{j=1}^{n}(x-\alpha_j)$$

where $a_{2n} \neq 0$ and $\alpha_j \in [-2,2]$ $(j = 1,\dots,n)$. Then

$$p(z) = a_{2n}\prod_{j=1}^{n}\left(z^2-\alpha_j z+1\right).$$

Since $\alpha_j \in [-2,2]$ we have $z^2-\alpha_j z+1 = (z-\beta_j)(z-\bar{\beta}_j)$ with $\beta_j\bar{\beta}_j = 1 = |\beta_j|^2$ proving that all zeros $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2 \dots \beta_n, \bar{\beta}_n$ of p are on the unit circle.
(ii) We have $\alpha_j = 2\cos u_j = \beta_j + \bar{\beta}_j$. Writing β_j as e^{iq_j} (here we may suppose that $0 \le q_j \le \pi$) we obtain that $2\cos u_j = e^{iq_j} + e^{-iq_j} = 2\cos q_j$ hence $u_j = q_j$ $(j = 1,2,\dots,n)$. The statement concerning the multiplicities is obvious. □

3. Results and proofs

Theorem 1. *All zeros of the (real reciprocal) polynomial*

$$h_m(z) = l(z^m + z^{m-1} + \cdots + z + 1) + \sum_{k=1}^{\left[\frac{m}{2}\right]} a_k\left(z^{m-k}+z^k\right) \quad (z \in \mathbb{C}) \tag{10}$$

of degree m where $l, a_1, \dots, a_{\left[\frac{m}{2}\right]} \in \mathbb{R}$, $l \neq 0$, $m \in \mathbb{N}$, $m \ge 2$, are on the unit circle if

$$|l| \ge 2\sum_{k=1}^{\left[\frac{m}{2}\right]} |a_k|. \tag{11}$$

Moreover, if (11) is satisfied, then for even $m = 2n$ all zeros of h_m can be given as

$$e^{iu_j},\ e^{-iu_j} \quad (j = 1,2,\dots,n)$$

where

$$\frac{j-\frac{1}{2}}{m+1}2\pi < u_j < \frac{j+\frac{1}{2}}{m+1}2\pi \quad (j = 1,2,\dots,n-1)$$

$$\frac{n-\frac{1}{2}}{m+1}2\pi < u_n \le \pi.$$

In the last inequality $u_n \le \pi$, we have equality if and only if

$$|l| = 2\sum_{k=1}^{\left[\frac{m}{2}\right]} |a_k| \text{ and } \operatorname{sgn} l = \operatorname{sgn}(-1)^{k+1}a_k \text{ for all } k = 1,2,\dots,n \text{ with } a_k \neq 0. \tag{12}$$

If (12) holds, then $-1 = e^{i\pi} = e^{-i\pi}$ is a double zero of h_m and all other zeros are single. For odd $m = 2n+1$ all zeros of h_m are single, they can be given as

$$-1,\ e^{iu_j},\ e^{-iu_j} \quad (j = 1,2,\dots,n)$$

where

$$\frac{j-\frac{1}{2}}{m+1}2\pi < u_j < \frac{j+\frac{1}{2}}{m+1}2\pi \quad (j=1,2,\dots,n).$$

Remark 1. The statement concerning the location of the zeros of h_m can also be formulated as follows.

If (11) is satisfied, then all the zeros e^{iu_j} $(j=1,2,\dots,m)$ of h_m can be arranged such that

$$\left|\epsilon_j - e^{iu_j}\right| < \frac{\pi}{m+1} \quad (j=1,\dots,m)$$

where, as in the introduction, ϵ_j are the $(m+1)$st roots of unity, except 1.

Namely, for even $m=2n$, let u_j $(j=1,2,\dots,n)$ be the same as in Theorem 1 and $u_{n+j} := 2\pi - u_{n+1-j}$ $(j=1,2,\dots,n)$. If (12) does not hold, then all zeros of h_m are single. If (12) holds, then $u_n = u_{n+1} = \pi$ and $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero and all other zeros are single.

For odd $m=2n+1$ let u_j $(j=1,2,\dots,n)$ be the same as in Theorem 1, $u_{n+1} := \pi$ and $u_{n+1+j} := 2\pi - u_{n+1-j}$ $(j=1,2,\dots,n)$. The number $-1 = e^{iu_{n+1}}$ is always a zero and all zeros are single.

Proof. The basic idea of our proof is the following. Assume that (11) holds and let

$$x_j = 2\cos\frac{j+\frac{1}{2}}{m+1}2\pi \quad (j=0,\dots,\left[\frac{m}{2}\right]).$$

If $m=2n$ is an even number, we show that $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j=0,1,\dots,n-1)$ and $\mathcal{T}h_{2n}(x_n)=0$ if (12) holds, otherwise $\operatorname{sgn}\mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j=0,\dots,n)$.
If $m=2n+1$ is odd, then $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$ with a suitable reciprocal polynomial $\bar{h}_{2n}$ from $\mathcal{R}_{2n}$. We show that $\operatorname{sgn}\mathcal{T}\bar{h}_{2n}(x_j) = \operatorname{sgn} l \operatorname{sgn}(-1)^j$ $(j=0,1,\dots,n)$.

Applying Lemma 1 completes the proof.

Case 1: $m=2n$.
With the notation $v_j(z) = z^j + z^{j-1} + \dots + 1 = \frac{z^{j+1}-1}{z-1}$, $e_j(z) = z^j$, $w_j(z) = z^j + 1$ $(j=0,1,\dots)$ we have

$$h_{2n}(z) = l v_{2n}(z) + \sum_{k=1}^{n} a_k e_k(z)\cdot w_{2n-2k}(z),$$

$$\mathcal{T}h_{2n}(x) = l\mathcal{T}v_{2n}(x) + \sum_{k=1}^{n} a_k \mathcal{T}(e_k\cdot w_{2n-2k})(x).$$

The zeros of v_{2n} are the $(2n+1)$st roots of unity, except 1: $e^{\frac{2j\pi i}{2n+1}}$ $(j = 1, 2, \ldots, 2n)$. They can be arranged into conjugate pairs: $\left(e^{\frac{2j\pi i}{2n+1}}, e^{\frac{2(2n+1-j)\pi i}{2n+1}}\right) = \left(e^{\frac{2j\pi i}{2n+1}}, e^{-\frac{2j\pi i}{2n+1}}\right)$ $(j = 1, \ldots, n)$, thus

$$\begin{aligned} v_{2n}(z) = \prod_{j=1}^{2n}\left(z - e^{\frac{2j\pi i}{2n+1}}\right) &= \prod_{j=1}^{n}\left(z - e^{\frac{2j\pi i}{2n+1}}\right)\left(z - e^{-\frac{2j\pi i}{2n+1}}\right) \\ &= \prod_{j=1}^{n}\left(z^2 - 2\cos\tfrac{2j\pi}{2n+1}z + 1\right), \\ \mathcal{T}v_{2n}(x) &= \prod_{j=1}^{n}\left(x - 2\cos\tfrac{2j\pi}{2n+1}\right). \end{aligned}$$

Similarly, for each $0 \le k \le n$ the zeros of w_{2n-2k} are the $(2n-2k)$ st roots of $-1:$ $e^{\frac{(2j-1)\pi i}{2n-2k}}$ $(j = 1, \ldots, 2n-2k)$. They can be arranged into conjugate pairs

$$\left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{\frac{(2(2n-2k+1-j)-1)\pi i}{2n-2k}}\right) = \left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{-\frac{(2j-1)\pi i}{2n-2k}}\right) \ (j = 1, \ldots, n-k).$$

Therefore

$$\begin{aligned} w_{2n-2k}(z) &= \prod_{j=1}^{2n-2k}\left(z - e^{\frac{(2j-1)\pi i}{2n-2k}}\right) = \prod_{j=1}^{n-k}\left(z^2 - 2\cos\tfrac{(2j-1)\pi}{2n-2k}z + 1\right), \\ \mathcal{T}(e_k w_{2n-2k})(x) &= \prod_{j=1}^{n-k}\left(x - 2\cos\tfrac{(2j-1)\pi}{2n-2k}\right). \end{aligned}$$

Denote by U_n the nth Chebyshev polynomial of the second kind (see for example in [6]), defined by

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x} \quad (n = 0, 1, \ldots).$$

We claim that

$$\mathcal{T}v_{2n}(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right), \tag{13}$$

$$\mathcal{T}(e_k \cdot w_{2n-2k})(x) = 2T_{n-k}\left(\frac{x}{2}\right). \tag{14}$$

To justify the first identity we note that

$$U_n(\cos y) + U_{n-1}(\cos y) = \frac{\sin(n+1)y + \sin ny}{\sin y} = 2\frac{\sin\frac{(2n+1)y}{2}\cos\frac{y}{2}}{\sin y} = \frac{\sin\frac{(2n+1)y}{2}}{\sin\frac{y}{2}}. \tag{15}$$

The right hand side is zero if and only if $y = \frac{2j\pi}{2n+1}$ $(j \in \mathbb{Z} \setminus \{0\})$ hence all zeros of $U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)$ are $2\cos\frac{2j\pi}{2n+1}$ $(j = 1, \ldots n)$. Since both sides of (13) are monics which have the same zeros, they are identical.

The zeros of T_p can be calculated easily from their definition, for $p \in \mathbb{N}$ they are

$$\cos\frac{(2j-1)\pi}{2p} \quad (j = 1, \ldots p).$$

Thus for $k < n$ the zeros of the monic $2T_{n-k}\left(\frac{x}{2}\right)$ are $2\cos\frac{(2j-1)\pi}{2n-2k}$ $(j = 1, \ldots, n-k)$. They are the same as the zeros of $\mathcal{T}(e_k \cdot w_{2n-2k})$, hence (14) holds. It also holds for $k = n$ since then both sides of (14) are equal to 2.

Next we evaluate $\mathcal{T}h_{2n}$ at the points

$$x_j = 2\cos\frac{j+\frac{1}{2}}{m+1}2\pi \quad (j = 0, \ldots, n)$$

of the interval $[-2, 2]$. Since $x_j = 2\cos y_j$ with $y_j = \frac{j+\frac{1}{2}}{2n+1}2\pi$ we have by (13), (14)

$$\begin{aligned}\mathcal{T}h_{2n}(x_j) &= l\left(U_n\left(\tfrac{x_j}{2}\right) + U_{n-1}\left(\tfrac{x_j}{2}\right)\right) + \sum_{k=1}^{n} 2a_k T_{n-k}\left(\tfrac{x_j}{2}\right) \\ &= 2\left[\frac{\frac{l}{2}\sin\frac{2n+1}{2}y_j}{\sin\frac{1}{2}y_j} + \sum_{k=1}^{n} a_k\cos(n-k)y_j\right] = 2\left[\frac{\frac{l}{2}(-1)^j}{\sin\frac{y_j}{2}} + \sum_{k=1}^{n} a_k\cos(n-k)y_j\right].\end{aligned}$$

If $j = 0, 1, \ldots, n-1$, then $0 < \sin\dfrac{y_j}{2} < 1$, $\displaystyle\sum_{k=1}^{n}|a_k\cos(n-k)y_j| \leq \sum_{k=1}^{n}|a_k|$ and by (11) the sign of the expression in the bracket is $(-1)^j\ \mathrm{sgn}\, l$.

If $j = n$, then $y_n = \pi$ and the expression in the bracket is

$$\frac{l}{2}(-1)^n + \sum_{k=1}^{n} a_k(-1)^{n-k} = (-1)^n\left(\frac{l}{2} + \sum_{k=1}^{n} a_k(-1)^k\right).$$

Its sign is $(-1)^n\ \mathrm{sgn}\, l$ if in (11) strict inequality holds or if in (11) we have equality and at least for one k $(1 \leq k \leq n)$ we have $\mathrm{sgn}\, l = \mathrm{sgn}\,(-1)^k a_k$. If we have equality in (11) and $\mathrm{sgn}\, l = \mathrm{sgn}\,(-1)^{k+1}a_k$ for all $k = 1, \ldots, n$ such that $a_k \neq 0$, then the expression in the bracket is zero.

Thus either $\mathrm{sgn}\,\mathcal{T}h_{2n}(x_j) = \mathrm{sgn}\,(-1)^j\ \mathrm{sgn}\, l$ $(j = 0, \ldots, n)$ or $\mathrm{sgn}\,\mathcal{T}h_{2n}(x_j) = \mathrm{sgn}\,(-1)^j\ \mathrm{sgn}\, l$ $(j = 0, 1, \ldots, n-1)$ and $\mathcal{T}h_{2n}(x_n) = 0$. In both cases $\mathcal{T}h_{2n}$ has n distinct zeros in the interval $[-2, 2]$. Writing these in the form $2\cos u_j$ with $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \pi$ and applying Lemma 1 we can complete the proof in the first case.

Case 2: $m = 2n + 1$.
We have $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$ with

$$\bar{h}_{2n}(z) = l\bar{v}_{2n}(z) + \sum_{k=1}^{n} a_k z^k \bar{w}_{2n-2k}(z)$$

where

$$\begin{aligned}\bar{v}_{2n}(z) &= z^{2n} + z^{2n-2} + \cdots + z^2 + 1 = v_n(z^2), \\ \bar{w}_{2n-2k}(z) &= \frac{w_{2n+1-2k}(z)}{z+1} = \frac{z^{2n+1-2k}+1}{z+1}.\end{aligned}$$

Using the factorization of v_n we get

$$\bar{v}_{2n}(z) = v_n(z^2) = \prod_{j=1}^{n}\left(z^2 - e^{\frac{2j\pi i}{n+1}}\right) = \prod_{j=1}^{n}\left(z - e^{\frac{j\pi i}{n+1}}\right)\left(z - e^{\frac{j\pi i}{2n+1}-\pi i}\right).$$

Arranging the zeros of $\bar{v}_{2n}$ into conjugate pairs $\left(e^{\frac{j\pi i}{n+1}}, e^{-\frac{j\pi i}{n+1}}\right)$ $(j = 1, \ldots, n)$ we have

$$\bar{v}_{2n}(z) = \prod_{j=1}^{n}\left(z - e^{\frac{j\pi i}{n+1}}\right)\left(z - e^{-\frac{j\pi i}{n+1}}\right) = \prod_{j=1}^{n}\left(z^2 - 2\cos\frac{2j\pi}{2n+1}z + 1\right)$$

therefore

$$\mathcal{T}\bar{v}_{2n}(x) = \prod_{j=1}^{n}\left(x - 2\cos\frac{j\pi}{n+1}\right).$$

We can easily calculate the zeros of $\bar{w}_{2n-2k}$ (we omit this elementary calculation) and obtain the factorization

$$\bar{w}_{2n-2k}(z) = \prod_{j=1}^{n-k}\left(z - e^{\frac{(2j-1)\pi i}{2n-2k+1}}\right)\left(z - e^{-\frac{(2j-1)\pi i}{2n-2k+1}}\right) = \prod_{j=1}^{n-k}\left(z^2 - 2\cos\frac{(2j-1)\pi}{2n-2k+1}z + 1\right)$$

therefore

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = \prod_{j=1}^{n-k}\left(x - 2\cos\frac{(2j-1)\pi}{2n-2k+1}\right).$$

Next we show that

$$\mathcal{T}\bar{v}_{2n}(x) = U_n\left(\frac{x}{2}\right), \tag{16}$$

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right). \tag{17}$$

where we have to adopt the convention

$$U_{-1}(x) = 0 \quad (x \in \mathbb{C}). \tag{18}$$

The first identity follows from the fact that the zeros of both sides are the same. To justify the second we note that

$$\begin{aligned} U_{n-k}(\cos y) - U_{n-k-1}(\cos y) &= \frac{\sin(n-k+1)y - \sin(n-k)y}{\sin y} \\ &= \frac{2\cos\frac{(2n-2k+1)y}{2}\sin\frac{y}{2}}{\sin y} = \frac{\cos\frac{(2n-2k+1)y}{2}}{\cos\frac{y}{2}} \end{aligned}$$

for all $k = 0, \ldots, n$ provided that the convention (18) is adopted.

If $k = n$, then both sides of (17) are equal to 1 thus (17) holds. For $k < n$ the right hand side of (17) is zero if and only if $y = \frac{(2j-1)\pi}{2n-2k+1}$ $(j \in \mathbb{Z})$ hence all zeros of $U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right)$ are $2\cos\frac{(2j-1)\pi}{2n-2k+1}$ $(j = 1, \ldots, n-k)$, they are the same as the zeros of $\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})$ proving (17).

By the linearity of the Chebyshev transform and by (16),(17) we have

$$\mathcal{T}\bar{h}_{2n}(x) = l\mathcal{T}\bar{v}_{2n}(x) + \sum_{k=1}^{n} a_k \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = l\; U_n\left(\frac{x}{2}\right) + \sum_{k=1}^{n} a_k \left[U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right)\right].$$

Next we evaluate $\mathcal{T}\bar{h}_{2n}$ at the points

$$\bar{x}_j = x_j = 2\cos\frac{j+\frac{1}{2}}{2n+2}2\pi \quad (j = 0, \dots, n)$$

of the interval $[-2, 2]$. Since $\bar{x}_j = 2\cos\bar{y}_j$ with $\bar{y}_j = \frac{j+\frac{1}{2}}{2n+2}2\pi$ we have

$$\mathcal{T}\bar{h}_{2n}(\bar{x}_j) = 2\left[\frac{l}{2}\frac{\sin(n+1)\bar{y}_j}{\sin\bar{y}_j} + \frac{\sum_{k=1}^{n} a_k \cos\frac{2n-2k+1}{2}\bar{y}_j}{2\cos\frac{\bar{y}_j}{2}}\right]$$

$$= 2\left[\frac{l}{2}\frac{(-1)^j}{\sin\bar{y}_j} + \sum_{k=1}^{n} a_k \frac{\cos\frac{2n-2k+1}{2}\bar{y}_j}{2\cos\frac{\bar{y}_j}{2}}\right] = 2\frac{\frac{l}{2}(-1)^j + \sum_{k=1}^{n} a_k \sin\frac{\bar{y}_j}{2}\cos\frac{2n-2k+1}{2}\bar{y}_j}{\sin\bar{y}_j}.$$

Since $\bar{y}_j \in]0, \pi[$ we have $\sin\bar{y}_j > 0,\ 0 < \sin\frac{\bar{y}_j}{2} < 1,\ |\cos\frac{2n-2k+1}{2}\bar{y}_j| \le 1$ for all $k = 1, \dots, n$ therefore the sign of the expression in the bracket is $\operatorname{sgn} l \operatorname{sgn}(-1)^j$. Thus $\operatorname{sgn}\left(\mathcal{T}\bar{h}_{2n}(x_j)\right) = \operatorname{sgn} l \operatorname{sgn}(-1)^j$ $(j = 0, 1, \dots, n)$ proving that $\mathcal{T}\bar{h}_{2n}$ has n different zeros in $[-2, 2]$. Writing these zeros in the form $2\cos u_j$ with $0 \le u_1 \le u_2 \le \dots \le u_n \le \pi$ and applying Lemma 1 the proof is completed in the second case as well.

□

We can formulate Theorem 1 in a more symmetric way. This formulation explains, in a certain way, the appearance of the factor 2 in (11).

Theorem 2. *All zeros of the reciprocal polynomial*

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C}) \tag{19}$$

of degree $m \ge 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \ne 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, \left[\frac{m}{2}\right]$) are on the unit circle, provided that

$$|A_m| \ge \sum_{k=1}^{m-1} |A_k - A_m|. \tag{20}$$

If (20) holds, then all zeros e^{iu_j} $(j = 1, 2, \dots, m)$ of P_m can be arranged such that

$$\left|\epsilon_j - e^{iu_j}\right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m).$$

If $m=2n+1$ is odd, then $-1=e^{iu_{n+1}}$ is always a zero and all zeros of P_m are single.
If $m=2n$ is even

$$\left\{\begin{array}{lcl} |A_{2n}| & = & \sum_{k=1}^{2n-1} |A_k - A_{2n}| \quad \textit{and} \\ \operatorname{sgn} A_{2n} & = & \operatorname{sgn} (-1)^{k+1}(A_k - A_{2n}) \textit{ for all } k=1,2,\dots,n \textit{ with } A_k - A_{2n} \neq 0 \end{array}\right. \tag{21}$$

holds, then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_m and all other zeros are single. Otherwise (i.e. if $m=2n$, (21) does not hold) all zeros of P_m are single.

Proof. Comparing the coefficients of z^j in h_m and P_m we see that for even $m=2n$

$$A_{2n} = A_0 = l, A_{2n-1} = A_1 = l + a_1, \dots, A_{n+1} = A_{n-1} = l + a_{n-1}, A_n = l + 2a_n$$

thus $l = A_{2n}, a_k = A_{2n-k} - A_{2n} = A_k - A_{2n}$ for $k = 1,2,\dots,n-1$ and $2a_n = A_n - A_{2n}$. Therefore the condition (11)

$$|l| \geq 2\sum_{k=1}^{n} |a_k|$$

can be written as

$$|A_{2n}| \geq 2\sum_{k=1}^{n-1} |A_k - A_{2n}| + |A_n - A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}|$$

which is the same as (20).

For odd $m = 2n+1$ the comparison of the coefficients gives that

$$A_{2n+1} = A_0 = l, A_{2n} = A_1 = l + a_1, \dots, A_{n+1} = A_n = l + a_n$$

thus $l = A_{2n+1}, a_k = A_{2n+1-k} - A_{2n+1} = A_k - A_{2n+1}$ for $k = 1,2,\dots,n$ and (11) can be written as

$$|A_{2n+1}| \geq 2\sum_{k=1}^{n} |A_k - A_{2n+1}| = \sum_{k=1}^{n} \left(|A_k - A_{2n+1}| + |A_{2n+1-k} - A_{2n+1}|\right) = \sum_{k=1}^{2n} |A_k - A_{2n+1}|$$

proving (20). The statement concerning the location of the zeros follows from Remark 1. □

4. Necessary and sufficient conditions

If the degree m of P_m is small we can easily obtain necessary and sufficient conditions for all zeros of P_m to be on the unit circle.

If $m=2$, then $P_2(z) = A_2 z^2 + A_1 z + A_2 = z\left(A_2(z+\frac{1}{z}) + A_1\right)$ hence $\mathcal{T}P_2(x) = A_2 x + A_1$. The only zero of $\mathcal{T}P_2$ is in $[-2,2]$ if and only if

$$|A_2| \geq \frac{1}{2}|A_1|. \tag{22}$$

This is the criteria for P_2 to have all zeros on the unit circle.

If $m=3$, then $P_3(z) = A_3 z^3 + A_2 z^2 + A_2 z + A_3 = (z+1)\left(A_3 z^2 + (A_2 - A_3)z + A_3\right)$. By (22) the zeros of P_3 are on the unit circle if and only if

$$|A_3| \geq \frac{1}{2}|A_2 - A_3|. \tag{23}$$

If $m=4$, then $P_4(z)=A_4z^4+A_3z^3+A_2z^2+A_3z+A_4=z^2\left(A_4(z^2+\frac{1}{z^2})+A_3(z+\frac{1}{z})+A_2\right)$ hence with $x=z+\frac{1}{z}$ we get that $\mathcal{T}P_4(x)=A_4(x^2-2)+A_3x+A_2$. By Lemma 1 all zeros of P_4 are on the unit circle if and only if the discriminant of $\mathcal{T}P_4$ is non-negative :

$$A_3^2-4A_4(A_2-2A_4)\geq 0 \tag{24}$$

and

$$-2\leq x_1,\quad x_2\leq 2 \tag{25}$$

hold where $x_1\leq x_2$ are the real zeros of $\mathcal{T}P_4$. A simple calculation shows that (24) and (25) are equivalent to

$$2\sqrt{\max\{A_2A_4-2A_4^2,0\}}\leq|A_3|\leq\min\{4|A_4|,|A_4|+\frac{1}{2}A_2\operatorname{sgn}A_4\}. \tag{26}$$

This is the criterion for P_4 to have all of its zeros on the unit circle.

For $m=2$ (22) holds if and only if

$$A_1\in[-2|A_2|,2|A_2|]$$

while (20) gives only the smaller interval

$$A_1\in[A_2-|A_2|,A_2+|A_2|]\,.$$

This shows that (20) for $m=2$ is not necessary. The situation is similar for $m=3$.

For $m=4$ the necessary and sufficient condition (26) is non-linear in the coefficients, while our sufficient condition (20) is linear for all $m\geq 2$. In some special cases we get necessary and sufficient conditions.

Corollary 1. *All zeros of the polynomial*

$$l(z^m+z^{m-1}+\cdots+z+1)+\left(z^k+z^{m-k}\right)\quad(z\in\mathbb{C})$$

where m,k are fixed non-negative integers with $m\geq 2$, $1\leq k\leq\left[\frac{m}{2}\right]$ and l is a fixed positive number, are on the unit circle for all $m\geq 2$ and for all $k=1,2,\dots,\left[\frac{m}{2}\right]$ if and only if

$$l\geq 2.$$

Namely, taking $m=2,\ k=1$ by (22) all zeros of the resulting polynomial $lz^2+(l+2)z+l$ are on the unit circle if and only if $l\notin\left(-\frac{2}{3},2\right)$ therefore $l\geq 2$. On the other hand if $l\geq 2$, then by Theorem 1 all zeros of the polynomial $l(z^m+z^{m-1}+\cdots+z+1)+\left(z^k+z^{m-k}\right)$ are on the unit circle.

Remark 2. A preliminary version of some parts of this paper was reported in [5].

A. Schinzel [7] generalized Theorem 2 to the case of self-inversive polynomials over $\mathbb{C}$, i.e. polynomials $P_m(z)=\sum_{k=0}^{n}A_kz^k$ for which $A_k\in\mathbb{C},A_m\neq 0,\epsilon\bar{A}_k=A_{m-k}$ for all $k=0,\dots,m$ with a fixed $\epsilon\in\mathbb{C},|\epsilon|=1$. He proved that all zeros of P_m are on the unit circle, provided that

$$|A_m|\geq\inf\sum_{k=0}^{m}|cA_k-d^{m-k}A_m|,$$

where the infimum is taken over all $c,d\in\mathbb{C}$ and $|d|=1$.

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