



## FIXED POINT OF CONTINUOUS MAPPINGS DEFINED ON AN ARBITRARY INTERVAL

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*Dedicated to Birol Gunduz who is one of the co-authors and passed away on the 3rd of April, 2019.*

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*Abstract.* In this work, we consider an iterative method given by Karaca and Yildirim [6] to approximate fixed point of continuous mappings defined on an arbitrary interval. Then, we give a necessary and sufficient condition for convergence theorem. We also compare the rate of convergence between the other iteration methods. Finally, we provide a numerical example which supports our theoretical results. Our findings improve corresponding results in the contemporary literature.

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### 1. INTRODUCTION AND PRELIMINARIES

Let  $E$  be a closed interval which is a subset of real line and let  $f : E \rightarrow E$  be a continuous function. Any  $p \in E$  is called a fixed point of  $f$  if  $f(p) = p$ . The set of all fixed points of  $f$  is denoted by  $F(f)$ . It is known that any linear or non-linear equation  $f(x) = 0$  can be turned into a fixed point problem such as:

$$g(x) = x \tag{1.1}$$

where  $g : E \rightarrow E$  is a contraction. In order to approximate fixed points of (1.1), Picard iteration can be applied. If  $g$  does not satisfy the contractive condition, then some other iteration methods can be applied to obtain a solution of (1.1). Thus, a question appears. Which iteration method should be used for approximating the fixed points of (1.1)?

In 1953, W.R. Mann defined an iteration called Mann iteration [7] to approximate fixed point of a non-linear mapping as follows: a sequence  $\{x_n\}$  defined by  $x_1 \in E$  and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f(x_n) \tag{1.2}$$

for all  $n \geq 1$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ . In 1991, D. Borwein and J. Borwein [2] proved a convergence theorem for a continuous function by using the iteration (1.2).

In 1974, Ishikawa [4] introduced an iteration process as follows: a sequence  $\{x_n\}$  defined by  $x_1 \in E$  and

$$\begin{cases} y_n &= (1 - \beta_n)x_n + \beta_n f(x_n) \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n f(y_n) \end{cases} \quad (1.3)$$

for all  $n \geq 1$  where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $[0, 1]$ . In 2006, Qing and Qihou [11] proved a convergence theorem of the sequence generated by the iteration (1.3) for continuous function.

In 2000, Noor [8] defined the following iterative scheme by  $x_1 \in E$  and

$$\begin{cases} z_n &= (1 - \gamma_n)x_n + \gamma_n f(x_n) \\ y_n &= (1 - \beta_n)x_n + \beta_n f(z_n) \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n f(y_n) \end{cases} \quad (1.4)$$

for all  $n \geq 1$ , where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences in  $[0, 1]$ . Clearly the Mann and the Ishikawa iterations are special cases of the Noor iteration.

Recently, Phuengrattana and Suantai [9] introduced SP-iteration as follows:  $x_1 \in E$  and

$$\begin{cases} z_n &= (1 - \gamma_n)x_n + \gamma_n f(x_n) \\ y_n &= (1 - \beta_n)z_n + \beta_n f(z_n) \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n f(y_n) \end{cases} \quad (1.5)$$

for all  $n \geq 1$ , where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences in  $[0, 1]$ . They proved a convergence theorem of the iteration (1.5) for continuous functions defined on an arbitrary interval in the real line.

In 2013, Kadioglu and Yildirim [5] defined the following iteration process:  $x_1 \in E$  and

$$\begin{cases} z_n &= (1 - a_n)x_n + a_n f(x_n) \\ y_n &= (1 - b_n - c_n)x_n + b_n f(z_n) + c_n f(x_n) \\ x_{n+1} &= (1 - \alpha_n - \beta_n)x_n + \alpha_n f(y_n) + \beta_n f(z_n) \end{cases} \quad (1.6)$$

for all  $n \geq 1$ , where  $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}$  and  $\{c_n\}$  are sequences in  $[0, 1]$ . They also showed that the iteration (1.6) converges to a fixed point of  $f$ . Moreover, they showed that the iteration of (1.6) is better than Mann, Ishikawa and Noor iteration processes in the sense of Rhoades [12]. In 2006, Cholamjiak and Baiya [3] proposed a new three-step iteration process for solving a fixed point problem for continuous

functions on an arbitrary interval in the real line as follows:  $p_1 \in E$  and

$$\begin{cases} r_n &= (1 - a_n)p_n + a_n f(p_n) \\ q_n &= (1 - b_n - c_n)r_n + b_n f(r_n) + c_n f(p_n) \\ p_{n+1} &= (1 - \alpha_n - \beta_n)q_n + \alpha_n f(q_n) + \beta_n f(r_n) \end{cases} \quad (1.7)$$

where  $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}$  and  $\{c_n\}$  are sequences in  $[0, 1)$  with  $0 \leq b_n + c_n < 1$  and  $0 \leq \alpha_n + \beta_n < 1$ . They also showed that the iteration (1.7) is better than the iteration (1.6) in the sense of Rhoades [12].

On the other hand in 2014, Karaca and Yildirim [6] defined a new iteration as follows: Let  $u_1 \in E$  and the sequence  $\{u_n\}$  be defined by

$$\begin{cases} w_n &= (1 - a_n)u_n + a_n f(u_n) \\ v_n &= (1 - b_n)w_n + b_n f(w_n) \\ u_{n+1} &= f(v_n) \end{cases} \quad (1.8)$$

where  $\{a_n\}$  and  $\{b_n\}$  are sequence in  $(0, 1)$ . They showed that the iteration (1.8) process is faster than all of Picard [10], Mann [7], Ishikawa [4] and Agarwal et al. [1] processes. They also proved a convergence theorem for nonexpansive mappings in Banach spaces.

In this paper we prove some convergence theorems by using the (1.8) iteration for continuous function defined on an arbitrary closed interval in the real line. Secondly, we compare the rate of convergence of (1.7) iteration and (1.8) iteration in the sense of Rhoades [12].

## 2. CONVERGENCE THEOREMS

In this section, we propose convergence theorem for the iteration process defined by (1.8) for continuous functions on an arbitrary interval.

**Theorem 1.** *Let  $E$  be a closed interval in the real line and let  $f : E \rightarrow E$  be a continuous mapping. For  $x_1 \in E$ , let the iteration  $\{u_n\}$  be defined by (1.8), where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \beta_n = 0$  and  $\lim_{n \rightarrow \infty} |f(v_n) - v_n| = 0$ . Then  $\{u_n\}$  is bounded if and only if  $\{u_n\}$  converges to a fixed point of  $f$ .*

*Proof.* If  $\{u_n\}$  converges to a fixed point of  $f$  then it is obvious that  $\{u_n\}$  is bounded. Now, assume that  $\{u_n\}$  is bounded. Our goal is to show that  $\{u_n\}$  is convergent. Assume to get a contradiction that it is not. Then there exist  $a, b \in \mathbb{R}$ ,  $a = \liminf_{n \rightarrow \infty} u_n, b = \limsup_{n \rightarrow \infty} u_n$  and  $a < b$ . First, we show that if  $a < m < b$ , then  $f(m) = m$ . Suppose that  $f(m) \neq m$ . Then, without loss of generality, we suppose that  $f(m) - m > 0$ . Since  $f$  is a continuous mapping, there exists  $\delta \in (0, b - a)$  such that

$$f(x) - x > 0 \quad \text{for} \quad |x - m| \leq \delta. \quad (2.1)$$

By the hypothesis of  $\{u_n\}$ , we have  $\{u_n\}$  belongs to a bounded closed interval. The continuity of  $f$  implies that  $\{f(u_n)\}$  belongs to another bounded closed interval. So,  $\{f(u_n)\}$  is bounded and since

$$w_n = (1 - \beta_n)u_n + \beta_n f(u_n)$$

we get that  $\{w_n\}$  is bounded and so,  $\{f(w_n)\}$  is bounded. Similarly, since

$$v_n = (1 - \alpha_n)w_n + \alpha_n f(w_n),$$

we have  $\{v_n\}$  and  $\{f(v_n)\}$  are bounded. It follows by (1.8) that

$$\begin{aligned} u_{n+1} - u_n &= f(v_n) - v_n + \alpha_n(f(w_n) - w_n) + \beta_n(f(u_n) - u_n), \\ v_n - u_n &= \beta_n(f(u_n) - u_n) + \alpha_n(f(w_n) - w_n), \\ w_n - u_n &= \beta_n(f(u_n) - u_n). \end{aligned}$$

By the assumption of the theorem (1), we get  $|u_{n+1} - u_n| \rightarrow 0, |v_n - u_n| \rightarrow 0$  and  $|w_n - u_n| \rightarrow 0$ . Thus, there exists  $N$  such that

$$|u_{n+1} - u_n| < \frac{\delta}{3}, \quad |v_n - u_n| < \frac{\delta}{3}, \quad |w_n - u_n| < \frac{\delta}{3} \quad (2.2)$$

for all  $n > N$ . Since  $b = \limsup_{n \rightarrow \infty} u_n > m$ , there exists  $k_1 > N$  such that  $u_{n_{k_1}} > m$ . Let  $k = n_{k_1}$ , then  $u_k > m$ . For  $u_k$ , there exist only two cases;

**Case 1:**  $u_k \geq m + \frac{\delta}{3}$ , then by (2.2) we have  $u_{k+1} - u_k > -\frac{\delta}{3}$ , then  $u_{k+1} > u_k - \frac{\delta}{3} \geq m$ , so  $u_{k+1} > m$ .

**Case 2:**  $m < u_k < m + \frac{\delta}{3}$ , then by (2.2), we have  $m - \frac{\delta}{3} < v_k < m + \frac{2\delta}{3}$  and  $m - \frac{\delta}{3} < w_k < m + \frac{2\delta}{3}$ . So, we have  $|u_k - m| < \frac{\delta}{3}, |v_k - m| < \frac{2\delta}{3} < \delta$  and  $|w_k - m| < \frac{2\delta}{3} < \delta$  Using (2.1) we get

$$f(u_k) - u_k > 0, \quad f(v_k) - v_k > 0, \quad f(w_k) - w_k > 0$$

By iteration (1.8), we have

$$u_{k+1} = u_k + f(v_k) - v_k + \alpha_k(f(w_k) - w_k) + \beta_k(f(u_k) - u_k)$$

By Case 1 and Case 2, we can conclude that  $u_{k+1} > m$ . By using the above argument, we obtain  $u_{k+2} > m, u_{k+3} > m, u_{k+4} > m \dots$

Thus, we get  $u_n > m$  for all  $n = k = n_{k_1}$ . So,  $a = \liminf_{n \rightarrow \infty} u_n \geq m$  which is a contradiction with  $a < m$ . Thus,  $f(m) = m$ .

For the sequence  $\{u_n\}$ , we consider the following two cases,

**Case 1':** There exists  $u_m$  such that  $a < u_m < b$ . Then,  $f(u_m) = u_m$ . Thus,

$$\begin{aligned} w_m &= (1 - \beta_m)u_m + \beta_m f(u_m) = u_m, \\ v_m &= (1 - \alpha_m)u_m + \alpha_m f(w_m) = u_m. \end{aligned}$$

$$u_{m+1} = f(v_m) = f(u_m) = u_m.$$

By induction, we obtain  $u_m = u_{m+1} = u_{m+2} = u_{m+3} = \dots$

So  $u_n \rightarrow u_m$ . This implies that  $u_m = a$  and  $u_n \rightarrow a$ , which contradicts to our assumption.

**Case 2':** For all  $n$ ,  $u_n \leq a$  or  $u_n \geq b$ . Because  $b - a > 0$  and  $|u_{n+1} - u_n| \rightarrow 0$ , there exists  $N_0$  such that  $|u_{n+1} - u_n| < \frac{b-a}{3}$  for all  $n > N_0$ . This implies that either  $u_n \leq a$  for all  $n > N_0$  or  $u_n \geq b$  for all  $n > N_0$ . If  $u_n \leq a$  for  $n > N_0$ , then  $b = \limsup_{n \rightarrow \infty} u_n \leq a$ , which is a contradiction with  $a < b$ . Hence, we have  $\{u_n\}$  is convergent.

Finally, we show that  $\{u_n\}$  converges to a fixed point of  $f$ . Let  $u_n \rightarrow p$  and suppose that  $f(p) \neq p$ . By the continuity of  $f$ , we have  $\{f(u_n)\}$  is bounded. From

$$w_n = (1 - \beta_n)u_n + \beta_n f(u_n)$$

and  $\beta_n \rightarrow 0$ , we obtain  $w_n \rightarrow p$ . Similarly, by

$$v_n = (1 - \alpha_n)w_n + \alpha_n f(w_n)$$

and  $\alpha_n \rightarrow 0$ , it follows that  $v_n \rightarrow p$ . Let  $p_k = f(v_k) - u_k$ . By the continuity of  $f$ , we have  $\lim_{k \rightarrow \infty} p_k = \lim_{k \rightarrow \infty} (f(v_k) - u_k) = f(p) - p \neq 0$ . Put  $w = f(p) - p$ . Then  $w \neq 0$ . By the iteration (1.8), we have

$$u_{n+1} - u_n = f(v_n) - u_n$$

it follows that

$$u_n = u_1 + \sum_{k=1}^{n-1} p_k \tag{2.3}$$

By  $p_k \rightarrow w \neq 0$ , we have that  $\sum_{k=1}^{\infty} p_k$  is divergent, which is a contradiction with  $u_n \rightarrow p$ . Thus,  $f(p) = p$ . That is,  $\{u_n\}$  converges to a fixed point of  $f$ .  $\square$

### 3. RATE OF CONVERGENCE

In this section, we give conclusions on the rate of convergence of the iteration (1.8) and the iteration (1.7). We use some useful definition and lemmas to do this.

**Definition 1** ([12]). Let  $E$  be a closed interval in the real line and let  $f : E \rightarrow E$  be a continuous mapping. Suppose that  $\{u_n\}$  and  $\{x_n\}$  are two iterations which converge to a fixed point of  $p$  of  $f$ . We say that  $\{u_n\}$  is better than  $\{x_n\}$  if

$$|u_n - p| \leq |x_n - p|, \quad \forall n \geq 1$$

**Lemma 1** ([3, Lemma 3.3]). Let  $E$  be a closed interval in the real line and let  $f : E \rightarrow E$  be a continuous and non-decreasing mapping. Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  be sequences in  $[0, 1)$  with  $0 \leq b_n + c_n < 1$  and  $\alpha_n + \beta_n < 1$ . Let  $\{p_n\}$  be defined by (1.7). Then the followings hold.

- (1) If  $f(p_1) < p_1$ , then  $f(p_n) < p_n$  for all  $n \geq 1$  and  $\{p_n\}$  is non-increasing

(2) If  $f(p_1) > p_1$ , then  $f(p_n) > p_n$  for all  $n \geq 1$  and  $\{u_n\}$  is non-decreasing

**Lemma 2.** Let  $E$  be a closed interval in the real line and let  $f : E \rightarrow E$  be a continuous and non-decreasing mapping. Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0, 1)$ . Let  $\{u_n\}$  be defined by (1.8). Then the followings hold.

- (1) If  $f(u_1) < u_1$ , then  $f(u_n) < u_n$  for all  $n \geq 1$  and  $\{u_n\}$  is non-increasing  
 (2) If  $f(u_1) > u_1$ , then  $f(u_n) > u_n$  for all  $n \geq 1$  and  $\{u_n\}$  is non-decreasing

*Proof. (1):* Let  $f(u_1) < u_1$ . Then  $f(u_1) < w_1 \leq u_1$ . Since  $f$  is non-decreasing, we have  $f(w_1) \leq f(u_1) < w_1$ . On the other hand, we have  $f(w_1) < v_1 \leq w_1$ . Since  $f$  is non-decreasing we get  $f(v_1) \leq f(w_1) \leq v_1$ . It follows that  $f(v_1) = u_2 \leq v_1$ , then  $f(u_2) \leq f(v_1) = u_2$ . Hence, we have  $f(u_2) < u_2$ . By induction, we conclude that  $f(u_n) < u_n$  for all  $n \geq 1$ . This implies  $f(u_n) < w_n \leq u_n$  for all  $n \geq 1$ . Since  $f$  is non-decreasing, we have  $f(w_n) \leq f(u_n) < w_n \leq u_n$  for all  $n \geq 1$ . Thus,  $f(w_n) < v_n \leq w_n$  for all  $n \geq 1$ . Then  $f(v_n) \leq f(w_n) \leq u_n$  for all  $n \geq 1$ . Hence, we have  $u_{n+1} \leq u_n$  for all  $n \geq 1$ , that is  $\{u_n\}$  is non-increasing

(2): Following the line of (1), we show the desired result.  $\square$

**Lemma 3.** Let  $E$  be a closed interval in the real line and let  $f : E \rightarrow E$  be a continuous and non-decreasing mapping. Let  $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0, 1)$ . For  $u_1 = p_1 \in E$ , let  $\{u_n\}$  and  $\{p_n\}$  be the sequences defined by (1.8) and (1.7), respectively. Then, the following are satisfied:

- (1) If  $f(p_1) < p_1$  then  $u_n \leq p_n$  for all  $n \geq 1$ .  
 (2) If  $f(p_1) > p_1$  then  $u_n \geq p_n$  for all  $n \geq 1$ .

*Proof. (1):* Let  $f(p_1) < p_1$ . Then  $f(u_1) < u_1$  since  $u_1 = p_1$ . From the iteration (1.8) we get,  $f(u_1) < w_1 \leq u_1$ . Since  $f$  is non-decreasing, we obtain  $f(w_1) \leq f(u_1) < w_1 \leq u_1$ . Hence,  $f(w_1) < v_1 \leq w_1$ .

Using the iteration (1.7) and the iteration (1.8), we obtain the following estimation:

$$w_1 - r_1 = (1 - a_1)(u_1 - p_1) + a_1(f(u_1) - f(p_1)) = 0.$$

So,  $w_1 = r_1$ , and also

$$v_1 - q_1 = (1 - b_1)(w_1 - r_1) + b_1(f(w_1) - f(r_1)) + c_1(r_1 - f(p_1)) \geq 0.$$

Since  $f$  is non-decreasing, we have  $f(q_1) \leq f(v_1)$ . On the other hand, if  $f(p_1) < p_1$ . Then from the iteration (1.7), we get  $f(p_1) < r_1 < p_1$ . Since  $f$  is non-decreasing, we get  $f(r_1) < f(p_1) < r_1$ . By the iteration (1.7) we also conclude  $f(r_1) < q_1 < r_1$ . Since  $f$  is non-decreasing, we get  $f(q_1) < f(r_1) < q_1$ . Next, by using the above result which is  $w_1 = r_1$  in the inequality of  $f(w_1) < v_1 < w_1$ , we get  $f(r_1) < v_1 < r_1$ . Since  $f$  is non-decreasing, we get  $f(v_1) < f(r_1) < v_1$  and so it follows that  $f(v_1) < f(r_1) < q_1$ . By using the above arguments, anyone can easily see that

$$u_2 - p_1 = f(v_1) - q_1 + \alpha_1(q_1 - f(q_1)) + \beta_1(q_1 - f(r_1)) \leq 0.$$

So,  $u_2 \leq p_2$ . Assume that  $u_k \leq p_k$ . Thus  $f(u_k) \leq f(p_k)$ . From Lemma 1(i) and Lemma 2(i), we get  $f(p_k) < p_k$  and  $f(u_k) < u_k$ . It follows that  $f(u_k) < w_k < u_k$  and  $f(w_k) < f(u_k) < w_k$ . Hence,

$$w_k - r_k = (1 - a_k)(u_k - p_k) + a_k(f(u_k) - f(p_k)) \leq 0.$$

So,  $w_k \leq r_k$ . Since  $f(w_k) \leq f(r_k)$ ,

$$v_k - q_k = (1 - b_k)(w_k - r_k) + b_k(f(w_k) - f(r_k)) + c_k(r_k - f(p_k)) \leq 0.$$

So  $v_k \leq q_k$ , which yields  $f(v_k) \leq f(q_k)$ . This shows that

$$u_{k+1} - p_{k+1} = f(v_k) - q_k + \alpha_k(q_k - f(q_k)) + \beta_k(q_k - f(r_k)) \leq 0,$$

which gives,  $u_{k+1} \leq w_{k+1}$ . By induction, we conclude that  $u_n \leq p_n$  for all  $n \geq 1$ .

(2): From Lemma 1(ii) and Lemma 2(ii) and the same proof as in (i), we can show that  $u_n \geq p_n$  for all  $n \geq 1$ . □

**Theorem 2.** Let  $E$  be a closed interval in the real line and let  $f : E \rightarrow E$  be a continuous and non-decreasing mapping such that  $F(f)$  is non-empty and bounded. Let  $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0, 1)$ . For  $u_1 = p_1 \in E$ , let  $\{u_n\}$  and  $\{p_n\}$  be the sequences defined by (1.8) and (1.7), respectively, and converge to  $p \in F(f)$ . Then, the iteration (1.8) is better than (1.7).

*Proof.* Put  $L = \inf\{p \in F : p = f(p)\}$  and  $U = \sup\{p \in E : p = f(p)\}$ .

For  $u_1$  there are three cases;

**Case 1:**  $u_1 = p_1 > U$ . By [9, Proposition 3.5], we get  $f(u_1) < u_1$  and  $f(p_1) < p_1$ . Using Lemma 3(1), we get that  $u_n \leq p_n$  for all  $n \geq 1$ .

Following the line of the proof of [9, Theorem 3.7], we have  $U \leq u_n$  for all  $n \geq 1$ . Then we have  $0 \leq u_n - p \leq p_n - p$ , so  $|u_n - p| \leq |x_n - p|$  for all  $n \geq 1$ . We can see that the iteration (1.8) is better than the iteration (1.7).

**case 2:**  $u_1 = p_1 < U$ . By [9, Proposition 3.5], we get  $f(u_1) > u_1$  and  $f(p_1) > p_1$ . Using Lemma 3(2), we get that  $u_n \geq p_n$  for all  $n \geq 1$ .

Following the line of the proof of [9, Theorem 3.7], we get  $u_n \leq L$  for all  $n \geq 1$ . So,  $|u_n - p| \leq |x_n - p|$  for all  $n \geq 1$ . We can see that the iteration (1.8) is better than the iteration (1.7).

**Case 3:**  $L \leq u_1 = p_1 \leq U$ . Suppose that  $f(u_1) \neq u_1$ . If  $f(u_1) < u_1$ , we have by Lemma 1(1) that  $\{u_n\}$  is non-decreasing with limit  $p$ . By Lemma 3(1), we have  $p \leq u_n \leq p_n$  for all  $n \geq 1$ . It follows that  $|u_n - p| \leq |p_n - p|$  for all  $n \geq 1$ . Hence we have that the iteration (1.8) is better than the iteration (1.7). If  $f(u_1) > u_1$ , we have by Lemma 1(2) that  $\{u_n\}$  is nondecreasing with limit  $p$ . By Lemma 3(2), we have  $p \geq u_n \geq p_n$  for all  $n \geq 1$ . It follows that  $|u_n - p| \leq |p_n - p|$  for all  $n \geq 1$ . Hence, we have that the iteration (1.8) is better than the iteration (1.7). □

Now, we give a numerical example to compare the rates of convergence of the iteration (1.8) and the iteration (1.7).

*Example 1.* Let  $f : [0, 3] \rightarrow [0, 3]$  be defined by  $f(x) = \frac{x^2+6}{7}$ . It is obvious that  $f$  is continuous and non-decreasing with a fixed point  $p = 1$ . Set the initial point  $u_1 = x_1 = 3.0$  and control sequences be defined by  $\alpha_n = \beta_n = a_n = \frac{1}{n^2+1}$ ,  $b_n = \frac{2}{7}$  and  $c_n = \frac{1}{3}$ .

*Remark 1.* Since Karaca and Yildirim showed that the iteration (1.6) is better than the Mann, the Ishikawa and the Noor iteration under the same control conditions in Table (1) in [5] and since the iteration (1.8) is better than the iteration (1.7) ; from the example above, we see that the iteration (1.6) is also better than the Mann, the Ishikawa and the Noor iteration (see Table 1).

TABLE 1. Comparison of rates of convergence between the Mann, the Ishikawa, the Noor, the iteration (1.7) and the iteration (1.8) for the given function in Example 1.

$n$	Mann	Ishikawa	Noor	Iteration (1.7)	Iteration (1.8)
1	3.000000000000	3.000000000000	3.000000000000	3.000000000000	3.000000000000
2	2.5714285714286	2.3888439436791	2.4008746355685	1.7313063612876	1.5401673015252
3	2.4174927113703	2.2254517072870	2.2376492818701	1.5181740350893	1.1412984225699
4	2.3449473117980	2.1550012729794	2.1669386104705	1.4419390309085	1.0370942630072
5	2.3036376203066	2.1162688700155	2.1279937754439	1.4035834760126	1.0098984194011
6	2.2771611529634	2.0918680685218	2.1034379658412	1.3806435160226	1.0026876820299
7	2.2588033724454	2.0751132884023	2.0865685584221	1.3654218100568	1.0007395163945
8	2.2453478554976	2.0629069049125	2.0742749466098	1.3545983144137	1.0002053712816
9	2.2350712664948	2.0536220205337	2.0649218163292	1.3465132450409	1.0000574007962
10	2.2269702990746	2.0463236735674	2.0575687715750	1.3402467219105	1.0000161162114
11	2.2204223565279	2.0404368413529	2.0516371936686	1.3352485577363	1.0000045397692
12	2.2150210906564	2.0355886384227	2.0467517462118	1.3311697883865	1.0000012819360
13	2.2104902244470	2.0315267588978	2.0426584045758	1.3277785155870	1.0000003626680
14	2.2066354643304	2.0280744244732	2.0391791535626	1.3249146769522	1.0000001027505
15	2.2033162415763	2.0251041121328	2.0361855592762	1.3224642602428	1.0000000291448
16	2.2004283710133	2.0225215278982	2.0335826431041	1.3203438702731	1.0000000082746
17	2.1978930104902	2.0202554519681	2.0312986612631	1.3184911072883	1.0000000023511
18	2.1956494058656	2.0182510914141	2.0292784132917	1.3168583563639	1.0000000006684
19	2.1936499906887	2.0164656045780	2.0274787359825	1.3154086566764	1.0000000001901
20	2.1918569955612	2.0148650125129	2.0258653933936	1.3141128814885	1.0000000000541
21	2.1902400515356	2.0134220207870	2.0244108832959	1.3129477682783	1.0000000000154
22	2.1887744636632	2.0121144533649	2.0230928598647	1.3118945144684	1.0000000000044
23	2.1874399459628	2.0109241069485	2.0218929797399	1.3109377580364	1.0000000000013
24	2.1862196801664	2.0098358997613	2.0207960445974	1.3100648253675	1.0000000000004
25	2.1850996055759	2.0088372301401	2.0197893550185	1.3092651680701	1.0000000000001
26	2.1840678764676	2.0079174869986	2.0188622173221	1.3085299356161	1.0000000000000

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 1 visually.

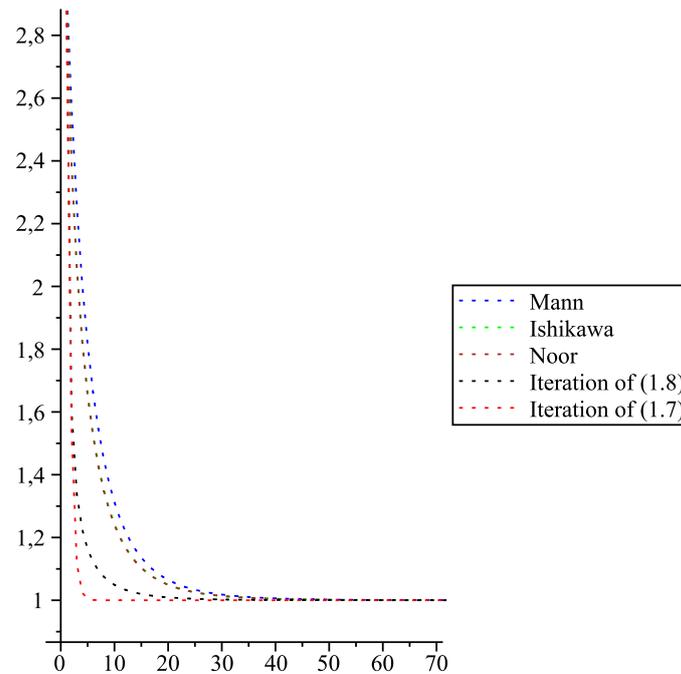


FIGURE 1. Behaviour of the iterations given in Example 1.

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