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REVERSIBLE AND REFLEXIVE PROPERTIES FOR RINGS WITH INVOLUTION

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Abstract. In this note, we give a generalization for the class of *-IFP rings. Moreover, we introduce *-reversible and *-reflexive *-rings, which represent the involutive versions of reversible and reflexive rings and expose their properties. Nevertheless, the relation between these rings and those without involution are indicated. Moreover, a nontrivial generalization for *-reflexive *-rings is given. Finally, in *-reversible *-rings it is shown that each nilpotent element is *-nilpotent and Köthe's conjecture has a strong affirmative solution.

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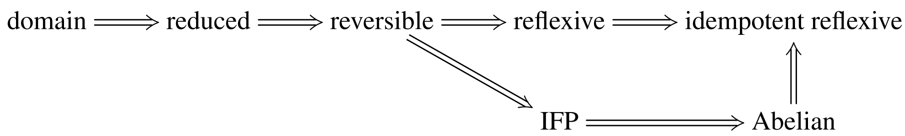
1. INTRODUCTION

All rings considered are associative with unity. A **-ring* R will denote a ring with involution and a self-adjoint ideal I of R; that is $I^* = I$, is called **-ideal*. A *projection* e of R is an idempotent satisfies $e^2 = e = e^*$. Recall from [7], an idempotent $e \in R$ is left (resp. right) semicentral in R if $eRe = Re$ (resp. $eRe = eR$). Equivalently, an idempotent $e \in R$ is left (resp. right) semicentral in R if eR (resp. Re) is an ideal of R. Moreover, if R is semiprime then every left (resp. right) semicentral idempotent is central. A semicentral projection is clearly central. A ring (resp. *-ring) R is said to be Abelian (resp. **-Abelian*) if all its idempotents (resp. projections) are central. R is *reduced* if it has no nonzero nilpotent elements. An involution * is called *proper* (resp. *semiproper*) if for every nonzero element a of R, $aa^* = 0$ (resp. $aRa^* = 0$) implies $a = 0$. Obviously, a proper involution is semiproper.

From [5], R is *semicommutative* or has *IFP* if the right annihilator $r(a) = \{x \in A | ax = 0\}$ of every element $a \in R$ is a two-sided ideal. In [1], the involutive version of IFP, that is **-IFP*, is given as the ring in which the right annihilator of each element of R is *-ideal. Clearly, each *-ring having $*$-IFP has also IFP.

Cohn [9] called a ring R *reversible* (or *completely reflexive*) if $ab = 0$ implies $ba = 0$ for every $a, b \in R$. Clearly, the class or reversible rings contains the reduced rings. Moreover, each reversible ring has IFP. Moreover, in [9, Theorem 2.2], Cohn proved that for reversible rings, Köthe's conjecture has an affirmative solution. Here, we give a strong affirmative solution for Köthe's conjecture for *-reversible *-rings and show that each nilpotent element is *-nilpotent.

In [13], Mason introduced a generalization of reversible rings; namely reflexive rings. A right ideal I of a ring R is said to be *reflexive* if $aRb \subseteq I$ implies $bRa \subseteq I$, for every $a,b \in R$. A ring R is called *reflexive* if 0 is a reflexive ideal. In [10], Kim and Baik defined an *idempotent reflexive* ideal as a right ideal I satisfying $aRe \subseteq I$ if and only if $eRa \subseteq I$ for $e^2 = e, a \in R$. R is an *idempotent reflexive ring* if 0 is an idempotent reflexive ideal. Obviously, the class of idempotent reflexive rings contains reflexive rings and Abelian rings.



A subring B of a *-ring R is said to be a **-biideal*,or self adjoint biideal, of R if $BRB \subseteq B$ and $B^* = B$.

Recall from [2], a nonzero element a of a *-ring R is a **-zero divisor* if $ab = 0$ and $a^*b = 0$ for some nonzero element $b \in R$. Obviously, a *-zero divisor element is a zero divisor, but the converse is not true (example 3 in [2]). A *-ring without *-zero divisors is said to be a **-domain*.

Recall from [3], an element a of a *-ring R is said to be *-nilpotent if there exist two positive integers m and n such that $a^m = 0$ and $(aa^*)^n = 0$. R is a *-reduced *-ring if it has no nonzero *-nilpotent elements; equivalently $a^2 = aa^* = 0$ implies $a = 0$ for every $a \in R$. A reduced (or *-domain) *-ring with proper involution is *-reduced. Moreover, every *-reduced *-ring is semiprime.

From [4], the **-right annihilator* of a nonempty subset S of a *-ring R is the self adjoint biideal $r_*(S) = \{x \in A | Sx = 0 = Sx^*\}$. Finally, $\mathbb{M}_n(R)$ will denote the full matrix ring of all $n \times n$ matrices over R.

2. *-RINGS WITH QUASI-*-IFP

In this section, we introduce the property of having quasi-$*$-IFP which generalizes that of having *-IFP introduced in [1].

Definition 1. A *-ring R is said to have *quasi-*-IFP* if for every $a \in R$, the *-right annihilator $r_*(a)$ is a *-ideal of R.

In view of $l_*(a) = r_*(a^*)$, we see that the *-left annihilator is also *-ideal. Thus the definition of quasi-*-IFP *-ring is left-right symmetric.

Clearly, every *-ring R having *-IFP has also quasi-*-IFP, since $r(a)$ is *-ideal implies $r_*(a) = r(a)$ for all $a \in A$. However, the converse is not true as shown by the following example.

Example 1. Consider the $*$-ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field and the adjoint of matrices is the involution. Since $r\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ is not an ideal of R, then R does not have IFP and consequently does not have *-IFP. Moreover, R has quasi-*-IFP since the *-right annihilator of every nonzero noninvertible element of R takes the form $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ which is a *-ideal of R.

The following are some equivalents for a *-ring to have quasi-*-IFP.

Proposition 1. *For a *-ring R, the following conditions are equivalent:*

(1) *R has quasi-*-IFP.*
(2) *$r_*(S)$ is a *-ideal of R for every subset S of R.*
(3) *$l_*(S)$ is a *-ideal of R for every subset S of R.*
(4) *For every $a,b \in R$, $ab = ab^* = 0$ implies $aRb = 0$ (consequently $aRb^* = 0$)*

Proof. (1)⇒(2): For every $S \subseteq R$, $r_*(S) = \bigcap_{s\in S} r_*(s)$ being the intersection of *-ideals is also a *-ideal.

(2)⇒(3): From (2), $l_*(S) = r_*(S^*)$ is a *-ideal of R.

(3)⇒(4): $ab = ab^* = 0$ implies $b^*a^* = ba^* = 0$ and consequently $b, b^* \in l_*(a^*)$ which is a *-ideal of R. Hence $bR, b^*R \subseteq l_*(a^*)$ from which $bRa^* = b^*Ra^* = 0$ and therefore $aRb = aRb^* = 0$.

(4)⇒(1): Let $x \in r_*(a)$, which is a self-adjoint biideal of R, then $ax = ax^* = 0$ implies $aRx = aRx^* = 0$, form the assumption. Hence $Rx \subseteq r_*(a)$ which means that $r_*(a)$ is a left ideal of R. Therefore $r_*(a)$ is a *-ideal due to its self-adjointness. □

The following results show that quasi-*-IFP implies *-Abelian while the converse is not true.

Proposition 2. *Every *-ring with quasi-*-IFP is *-Abelian.*

Proof. Let e be a projection in R, then $(1-e)e = (1-e)e^* = 0$ implies $(1-e)Re = 0$, from Proposition 1. Hence e is a left semicentral projection and consequently is central. □

Moreover, The next example shows that the converse of Proposition 2 is not true.

Example 2. Let F be a field of characteristic 2 and consider the *-ring $R = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in F \right\}$, with involution defined as

$$\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & a_{34} & a_{24} & a_{14} \\ 0 & a & a_{23} & a_{13} \\ 0 & 0 & a & a_{12} \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Since for the matrices $x = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, we have $xy = 0 = xy^*$, while

$$xzy = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$\begin{pmatrix} 0 & 0 & 0 & a_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$, for every $z \in R$ with $a_{23} \neq 0$, it follows that R does not have quasi-$*$-IFP, by Proposition 1. Moreover R is $*$-Abelian since for any projection $e = \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}$, $e^2 = e^* = e$ implies $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{33} = 0$ and $a^2 = a$, so that R has no nontrivial projections.

Next, we answer the question of when a *-ring with quasi-*-IFP is *-reduced.

Proposition 3. *Let R be a semiprime *-ring having quasi-*-IFP, then R is *-reduced.*

Proof. Let R be a semiprime *-ring having quasi-*-IFP. Set $a^2 = aa^* = 0$ for some $a \in R$, then $aRa = aRa^* = 0$, from Proposition 1. Since R is semiprime, then $a = 0$ and R is *-reduced. □

Finally, one can easily show that the class of *-rings having quasi-*-IFP is closed under direct sums (with changeless involution) and under taking *-subrings.

Proposition 4. *The class of *-rings having quasi-*-IFP is closed under direct sums and under taking *-subrings.*

3. *-REVERSIBLE *-RINGS

Definition 2. An ideal I of a *-ring R is called **-reversible* if $ab, ab^* \in I$ implies $ba \in I$, for every $a, b \in R$.

It is obvious that if I is *-reversible then $ab, ab^* \in I$ implies also $b^*a \in I$, for every $a, b \in R$.

We note the following:

- A one-sided *-reversible ideal must be two-sided ideal.
- The *-reversible ideal may not be self adjoint according to the following example.

Example 3. Let R be the *-ring in Example 1. The ideal $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is *-reversible but not self-adjoint

Definition 3. A *-ring R is said to be **-reversible* if 0 is a *-reversible ideal of R; that is $ab = ab^* = 0$ implies $ba = 0$ (consequently $b^*a = 0$), for every $a, b \in R$.

Example 4. Every *-domain is a *-reversible *-ring.

It is clear that every reversible ring with involution is *-reversible. But the converse is not always true as shown by the next example.

Example 5. Let R be the *-ring in Example 1. R is not reversible since the matrices $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ satisfy $\alpha\beta = 0$ while $\beta\alpha \neq 0$. Moreover, it easy to check that R is *-reversible.

The following are some equivalents for a *-ring to be *-reversible.

Proposition 5. *For a *-ring R, the following statements are equivalent.*

(i) *R is *-reversible.*
(ii) *$r_*(S) = l_*(S)$ for every subset S of R.*
(iii) *$r_*(a) = l_*(a)$ for every element $a \in R$.*
(iv) *For any two nonempty subsets A and B of R , $AB = AB^* = 0$ implies $BA = 0$ (consequently $B^*A = 0$) .*

Proof. $(i) \Rightarrow (ii)$: Let $x \in r_*(S)$, then $sx = sx^* = 0$ for every $s \in S$. Since R is *-reversible, we have $xs = x^*s = 0$ for every $s \in S$. Hence, $xS = x^*S$ implies $x \in l_*(S)$ and we get $r_*(S) \subseteq l_*(S)$. Similarly, $l_*(S) \subseteq r_*(S)$ and $r_*(S) = l_*(S)$ follows.

$(ii) \Rightarrow (iii)$ is direct by considering S as the singleton set $\{a\}$.

$(iii) \Rightarrow (iv)$: Set $AB = AB^* = 0$ for some nonempty subsets A and B of R. Then $ab = ab^* = 0$ for every $a \in A$ and $b \in B$, and hence $b \in r_*(a) = l_*(a)$ from the condition. Therefore $ba = b^*a = 0 = 0$ for every $a \in A$ and $b \in B$ which implies $BA = B^*A = 0$.

$(iv) \Rightarrow (i)$ is direct by considering A and B as the singleton sets containing a and b, respectively. ☐

The question when does a *-reversible *-ring become reversible has been answered in the following proposition.

Proposition 6. *Let R be a *-reversible *-ring and either*

(1) *R has *-IFP, or*
(2) ** is proper.*

Then, R is reversible.

Proof. (1) Let R have *-IFP and $ab = 0$ for some $a, b \in R$. Then, by [1, Proposition 7], $aRb^* = 0$ and hence $ab^* = 0$. The *-reversibility of R implies $ba = 0$ and R is reversible.

(2) Let the involution * be proper and $ab = 0$ for some $a, b \in R$. Then $a(bb^*) = a(bb^*)^* = 0$ and hence $bb^*a = 0$ from the *-reversibility of R. Now $(a^*b)(a^*b)^* = a^*bb^*a = 0$ implies $a^*b = b^*a = 0$, since * is proper. Finally, by the *-reversibility of R, $b^*aa^* = 0$ implies $aa^*b^* = 0$ and $(ba)(ba)^* = baa^*b^* = 0$ implies $ba = 0$. Hence R is reversible. □

Now, we see that each *-reversible *-ring has quasi-*-IFP.

Proposition 7. *Every *-reversible *-ring has quasi-*-IFP.*

Proof. Let $ab = ab^* = 0$ for some elements a, b of a *-reversible *-ring R. Using the *-reversibility of R, we have $ba = b^*a = 0$ which implies $bar = b^*ar = 0$. Again, by the *-reversibility of R, $arb = arb^* = 0$ for every $r \in R$. Therefore $aRb = aRb^* = 0$ which means that R has quasi-*-IFP, by Proposition 1. □

From Propositions 7 and 2, we get the following.

Corollary 1. *Every *-reversible *-ring is *-Abelian.*

However, the next example shows that the converse of the previous proposition and its corollary is not always true.

Example 6. Let D be a commutative domain. Then the ring

$$R = \left\{ \begin{pmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in D \right\}$$

has IFP, by [11, Proposition 1.2]. Define an involution * on R as $\begin{pmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & c & -d \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$. One can easily check that R has quasi-*-IFP and hence is *-Abelian. But R is not *-reversible since the elements $\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ of R satisfy $\alpha\beta = \alpha\beta^* = 0$ but $\beta\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$

Moreover, if the involution * is proper then the properties IFP, *-IFP, quasi-*-IFP, *-reversibility and reducedness are identical as shown in the following result.

Proposition 8. *Let R be a *-ring and the involution * is proper. Then the following conditions are equivalent:*

(1) *R is *-reversible*
(2) *R has quasi-*-IFP.*
(3) *R has IFP.*
(4) *R has *-IFP.*
(5) *R is reduced.*

Proof. (3),(4) and (5) are equivalent from [1, Proposition 9].
(1)⇒(2) is direct from Proposition 7.
(2)⇒(3): Let $ab = 0$ for some $a, b \in R$. Then $a(bb^*) = a(bb^*)^* = 0$ implies $aRbb^* = 0$ from the quasi-*-IFP of R. Now $(arb)(arb)^* = arbb^*r^*a^* = 0$ implies $arb = 0$ fore every $r \in R$ since * is proper. Therefore $aRb = 0$ and so R has IFP.
(5)⇒(1): Let $ab = ab^* = 0$ for some $a, b \in R$, then $(ba)^2 = baba = 0$ and $(b^*a)^2 = b^*ab^*a = 0$. Hence, $ba = b^*a = 0$ from the reducedness of R and so R is *-reversible. □

Next, we discuss the converse of Example 4; that is when a *-reversible *-ring is *-domain.

Proposition 9. *A *-ring is a *-domain if and only if R is *-prime and *-reversible.*

Proof. First, Suppose that R is a *-domain, hence R is obviously *-reversible. Let $IJ = 0$ for some *-ideals I and J of R, then $ab = ab^* = 0$ for every $a \in I$ and $b \in J$. Hence, either $a = 0$ or $b = 0$ which implies $I = 0$ or $J = 0$ and so R is *-prime. Conversely, let R be both *-prime and *-reversible and $ab = a^*b = 0$ for some $0 \neq a, b \in R$. We have $r^*b^*a^* = r^*b^*a = 0$ for every $r \in R$ and so $a^*r^*b^* = ar^*b^* = 0$ for every $r \in R$ from the *-reversibility of R, which gives $bRa = bRa^* = 0$. Since R is *-prime and $a \neq 0$, we get $b = 0$, by [[6], Proposition 5.4], and so R has no *-zero divisors; that is a *-domain. □

As a consequence, we get Proposition 4 in [3] as a corollary.

Corollary 2 ([3], Proposition 4)**.** *If R is a reduced *-prime *-ring, then R is *-domain.*

For a *-ring R, the trivial extension of R, denoted by $T(R,R)$, is the ring $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R\right\}$. One can define the componentwise involution $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a^* & b^* \\ 0 & a^* \end{pmatrix}$ to make $T(R,R)$ a *-ring.

Proposition 10. *Let R be a *-reduced *-ring. If R is *-reversible, then $T(R,R)$ is a *-reversible *-ring.*

Proof. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha^* & \beta^* \\ 0 & \alpha^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $a\alpha = a\alpha^* = 0$ and $a\beta + b\alpha = a\beta^* + b\alpha^* = 0$. Since R is *-reversible then $\alpha a = \alpha^* a = 0$. By the *-reversibility of R, it is easy to see that $aR\alpha = 0$. Now $0 = a\beta + b\alpha = \alpha(a\beta + b\alpha) = \alpha b\alpha$ and $0 = a\beta^* + b\alpha^* = a\beta^*\alpha + b\alpha^*\alpha = b\alpha^*\alpha$. Hence $(b\alpha)^2 = b\alpha b\alpha = 0$ and $(b\alpha)(b\alpha)^* = b\alpha\alpha^* b^* = 0$. Then $b\alpha = 0$ because R is *-reduced and therefore $a\beta = 0$. Similarly, one can show that $b\alpha^* = 0$ and $a\beta^* = 0$. Using the *-reversibility of R again we get $\alpha b = \alpha^* b = \beta a = \beta^* a = 0$ which implies $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ 0 & \alpha^* \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $T(R,R)$ is a *-reversible *-ring. □

Furthermore, one can easily show that the class of *-reversible *-rings is closed under direct sums (using changeless involution) and taking *-subrings.

Proposition 11. *The class of *-reversible *-rings is closed under direct sums and under taking *-subrings.*

4. *-REFLEXIVE *-RINGS

In this section, we introduce the involute version of reflexive ideals and rings defined by Mason [13] and study the relation between these rings and the *-reversible rings introduced in the previous section.

Definition 4. A ideal I of a *-ring R is called **-reflexive* if for every $a,b \in R$, $aRb, aRb^* \subseteq I$ implies $bRa \subseteq I$ (consequently $b^*Ra \subseteq I$). A *-ring R is said to be **-reflexive* if 0 is a *-reflexive ideal of R.

By the way, the ideal in the previous definition can not be one sided since for every $a \in I$ satisfying $aR \subseteq I$ implies $Ra \subseteq I$ by taking $b = 1$. Also, this ideal need not be self-adjoint by Example 3.

Example 7. Every *-reduced *-ring is *-reflexive.

It is evident that every reflexive *-ring is *-reflexive. However, the next example shows that the converse is not true.

Example 8. Let D be a commutative domain and $R = \{\begin{pmatrix} \alpha & \beta & \delta \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{pmatrix} | \alpha,\beta,\gamma,\delta \in D\}$. R is not reflexive according to [12, Example 2.3]. Define the involution $*$: $\begin{pmatrix} \alpha & \beta & \delta \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{pmatrix} \to \begin{pmatrix} \alpha & \gamma & \delta \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix}$. It is easy to check that R is $*$-reversible and in particular is $*$-reflexive.

Lemma 1. *Let R be a ring with semiproper involution *. Then $aRb = 0$ implies $aRb^* = bRa = b^*Ra = 0$.*

Proof.

$$(arb^*)R(arb^*)^* = arb^*Rbr^*a^* \subseteq aRbr^*a^* = 0,$$
$$\text{for every } r \in R \text{ implies } aRb^* = 0,$$

$$(bra)R(bra)^* = braRa^*r^*b^* \subseteq braRb^* = 0, \text{ for every } r \in R \text{ implies } bRa = 0$$

and

$$(b^*ra)R(b^*ra)^* = braRa^*r^*b \subseteq braRb = 0, \text{ for every } r \in R \text{ implies } b^*Ra = 0.$$

□

Corollary 3. *Every *-ring with semiproper involution is reflexive (and hence *-reflexive).*

The converse of the previous corollary is not necessary true as shown in the next example.

Example 9. If F is a field, then the ring $R = F \oplus F^{op}$, with the exchange involution * defined by $(a,b)^* = (b,a)$ for all $a,b \in R$, is obviously a reflexive and hence *-reflexive but * is not semiproper. Indeed, the element $0 \neq \alpha = (0,a)$ for some nonzero element a of F satisfies $\alpha R \alpha^* = 0$.

In the following proposition, we state some equivalent definitions for a *-ring to be *-reflexive .

Proposition 12. *For a *-ring R, the following statements are equivalent :*

(i) *R is *-reflexive.*
(ii) *$r_*(aR) = l_*(Ra)$ for every $a \in R$.*
(iii) *For any two nonempty subsets A and B of R , $ARB = ARB^* = 0$ implies $BRA = B^*RA = 0$.*

Proof. $(i) \Rightarrow (ii)$: Let $x \in r_*(aR)$, then $aRx = aRx^* = 0$. Hence $xRa = x^*Ra = 0$, by the *-reflexivity of R, implies $x \in l_*(Ra)$ and so $r_*(aR) \subseteq l_*(Ra)$. Similarly, $l_*(aR) \subseteq r_*(Ra)$ and we get $r_*(aR) = l_*(Ra)$.

$(ii) \Rightarrow (iii)$: Set $ARB = ARB^* = 0$ for some subsets A and B of R. Then $aRb = aRb^* = 0$ for every $a \in A$ and $b \in B$, and hence $b \in r_*(aR) \subseteq l_*(Ra)$ from the condition. Therefore $bRa = b^*Ra = 0$ for every $a \in A$ and $b \in B$ which implies $BRA = b^*RA = 0$.

$(iii) \Rightarrow (i)$ is direct by considering A and B as the singleton sets containing a and b, respectively.. □

The following proposition and example show that the class of *-reflexive *-rings generalizes strictly that of *-reversible *-rings.

Proposition 13. *Every *-reversible *-ring is *-reflexive*

Proof. Let $aRb = aRb^* = 0$, then $ab = ab^* = 0$ implies $rab = rab^* = 0$, for every $r \in R$. So that $bra = b^*ra$ for every $r \in R$, from the *-reversibility of R. Thus $bRa = b^*Ra = 0$ and hence R is *-reflexive. □

Example 10. Let $n > 2$ be an integer and $p \leq n$ be a prime number. The *-ring $R = \mathbb{M}_n(\mathbb{Z}_p)$, where * is the transpose involution, is prime and hence reflexive (in particular *-reflexive). Moreover, R is not *-reversible. Indeed , the nonzero elements

$$\alpha = e_{12} + e_{13} + \cdots + e_{1n},$$
$$\beta = e_{11} + e_{12} + \cdots + e_{1(n-1)} + 2e_{1n}$$

of R, where e_{ij} is the square matrix of order n with 1 in the (i,j)-position and 0 elsewhere, satisfy $\alpha\beta = \alpha\beta^* = 0$, while $\beta\alpha \neq 0$ and $\beta^*\alpha \neq 0$.

The question when a *-reflexive *-ring is *-reversible is answered in the following proposition.

Proposition 14. *A *-ring R is *-reversible if and only if R has quasi-*-IFP and *-reflexive.*

Proof. The necessity is obvious. For sufficiency, let $ab = ab^* = 0$ for some $a, b \in R$. Since R has quasi-*-IFP, then $aRb = aRb^* = 0$. The *-reflexivity of R implies $bRa = b^*Ra = 0$. Hence $ba = b^*a = 0$ and R is *-reversible. □

In the next result we discuss when a principal right ideal generated by a projection in a *-reflexive *-ring is *-reflexive.

Proposition 15. *Let e be a projection of a *-reflexive *-ring R. Then e is central if and only if eR is a *-reflexive *-ideal.*

Proof. Let e be central and $aRb, aRb^* \subset eR$, then $arb = earb$ and $arb^* = earb^*$ for every $r \in R$. Hence $(1-e)aRb = (1-e)aRb^* = 0$ and consequently $(1-e)bRa = (1-e)b^*Ra = 0$, since R is *-reflexive and e is central. Hence $bRa, b^*Ra \subseteq eR$ and eR is *-reflexive ideal. The converse implication is clear since eR is a *-ideal and so e is central. □

Now, we show that *-reflexive property is extended to the *-corner.

Proposition 16. *Let R be a *-reflexive *-ring, then the *-corner eRe for every projection e of R is also *-reflexive.*

Proof. Let R be *-reflexive and $a = exe, b = eye \in eRe$ such that $a(eRe)b = a(eRe)b^* = 0$. Then $exeReye = exeRey^*e = 0$ implies $eyeRexe = ey^*eRexe = 0$, since R is *-reflexive. Therefore $b(eRe)a = b^*(eRe)a = 0$ and so eRe is *-reflexive. □

Next, we illustrate by example that *-reflexivity is not closed under taking *-subrings.

Example 11. The ring $R = \mathbb{M}_2(\mathbb{Z}_2)$ is prime and hence reflexive. The upper triangular matrix ring $S = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ over $\mathbb{Z}_2$ is a *-subring of R under the involution * defined as $\begin{pmatrix} a & b \\ d & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ -d & a \end{pmatrix}$. R is clearly *-reflexive but S is not, since the elements $\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of R satisfy $\alpha R\beta = \alpha R\beta^* = 0$ but $\beta R\alpha = \beta^* R\alpha = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} \neq 0$

We end this section by showing that the *-reflexivity is restricted from the full matrix ring to its underlying ring.

Proposition 17. *If $\mathbb{M}_n(R)$ is a *-reflexive *-ring for some $n \geq 1$ and with the transpose involution *, then R is also a *-reflexive *-ring .*

Proof. let $\mathbb{M}_n(R)$ be a *-reflexive *-ring for some $n \geq 1$. Since $R \cong e_{11}\mathbb{M}_n(R)e_{11}$, as *-rings, then R is *-reflexive, by Proposition 16. □

5. Projection *-reflexive rings

In this last section, we give another generalization for the class of *-reflexive rings; that is projection *-reflexive *-rings.

In [10], Kim defines an idempotent reflexive ring R as the ring satisfying $aRe = 0$ if and only if $eRa = 0$ for every idempotent $e, a \in R$.

Definition 5. An ideal I of a *-ring R satisfies $aRe \subseteq I$ if and only if $eRa \subseteq I$ for every projection $e, a \in R$, is called *projection *-reflexive*. A *-ring R is called *projection *-reflexive* if 0 is a projection *-reflexive ideal.

The ideal I of the previous definition can not be one-sided ideal, because if I is a right ideal then $aR1 \subseteq I$ for every $a \in I$ implies $1Ra \subseteq I$, since 1 is a projection. Moreover, the ideal I in the definition need not be self-adjoint; indeed, for a field F the *-ring $F \bigoplus F$ with the exchange involution, possesses the non self-adjoint projection *-reflexive ideal $(0, F)$.

It is evident from the definition that *-reflexive and idempotent reflexive *-rings are projection *-reflexive. Accordingly, we raise the following two questions.

- Is there a projection *-reflexive *-ring which is not idempotent reflexive?
- Is there a projection *-reflexive *-ring which is not *-reflexive?

The answers of these questions are in the following example.

Example 12. The *-ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ over a field F with the involution * defined by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$, is projection *-reflexive because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are the only projections of R. Clearly, R is not idempotent reflexive, since the idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ of R satisfies

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \neq 0.$$

Moreover, R is not *-reflexive, since

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* =$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$$

The proof of the following proposition, which gives an equivalent definition for projection *-reflexive *-rings, is straightforward.

Proposition 18. *A *-ring R is projection *-reflexive if and only if for any nonempty subset A and any projection e of R , $ARe = 0$ implies $eRA = 0$.*

Obviously, every *-Abelian *-ring is projection *-reflexive and consequently every *-ring having quasi-*-IFP is also projection *-reflexive, by Proposition 2. However, the converse of this statement needs additional condition, as in the next proposition.

Proposition 19. *A *-ring R is *-Abelian if and only if R is projection *-reflexive and satisfies $eR(1-e)Re = 0$ for every projection e of R.*

Proof. The necessity is obvious, For sufficiency, let e be an arbitrary projection of the projection *-reflexive *-ring R and $eR(1-e)Re = 0$. By Proposition 18, we have $eReR(1-e) = 0$ and taking involution gives $(1-e)ReRe = 0$. Hence, $(1-e)Re = 0$ which implies that e is semicentral, from [Lemma 1.1, [8]], and hence it is central. Thus R is *-Abelian □

In the next result we show when a projection in a projection *-reflexive *-ring is central.

Proposition 20. *Let R be a projection *-reflexive *-ring and e is a projection of R. Then the following are equivalent:*

(i) *e is central.*
(ii) *eR is a projection-*-reflexive *-ideal.*

Proof. $(i) \Rightarrow (ii)$: Assume that $aRf \subset eR$ for some projection f of R. So that $arf = earf$ for every $r \in R$ and hence $(1-e)aRf = 0$. Therefore $fR(1-e)a = 0 = (1-e)fRa$, since R is projection *-reflexive, and consequently $fRa = efRa \subseteq eR$. Hence eR is a projection-*-reflexive ideal.

$(ii) \Rightarrow (i)$: is clear since eR is a *-ideal and so e is central. □

Corollary 4. *If every principal *-ideal of R is projection *-reflexive, then R is *-Abelian.*

Finally, Since the only projections of the *-corner eRe is the projection e, then eRe is projection *-reflexive if R is projection *-reflexive.

Proposition 21. *Let R be a projection *-reflexive *-ring, then the *-corner eRe, for every projection e of R, is also projection *-reflexive.*

6. *-NILPOTENCY IN *-REVERSIBLE *-RINGS

According to [3], in a *-ring R every *-nilpotent element is nilpotent but the converse is not always true as shown in [3, Example 2.2]. In the next, we give a sufficient condition that makes a nilpotent element *-nilpotent.

Proposition 22. *In a *-reversible *-ring R, every nilpotent element is *-nilpotent.*

Proof. Let a be a nilpotent element of a *-reversible *-ring R. Hence $a^n = 0$, for some positive integer n, and multiplying by a^* form right, we get $a^{n-1}(aa^*) = 0$. From the *-reversibility of R, we have $(aa^*)a^{n-1} = 0$. Multiply again by a^* form right and apply the *-reversible property, we get $(aa^*)a^{n-2} = 0$. Continuing this process, we get $(aa^*)^n = 0$ and a is *-nilpotent. □

However, the *-reversibility condition in the previous proposition is sufficient but not necessary as clear from Example 6. Indeed, the elements of the *-ideal $\begin{pmatrix} 0 & D & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}$ are precisely all the nilpotent (which also *-nilpotent) elements of the ring R.

Corollary 5. *Every *-reduced *-reversible *-ring is reduced.*

By the definition of nilpotency, an element is nilpotent if and only if a power of it is also nilpotent. This is not the case for *-nilpotent elements as shown in the following examples.

Example 13. In the *-ring $R = \mathbb{M}_2(\mathbb{C})$ of 2×2 matrices with complex entries and transpose involution $*$, the element $a = \begin{pmatrix} \frac{\sqrt{3}+\imath}{2} & 1 \\ 1 & \frac{\sqrt{3}-\imath}{2} \end{pmatrix}$ satisfies $(aa^*)^6 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ which can not tend to zero ever with any power. Thus a is not $*$-nilpotent, while $(a^3(a^3)^*)^1 = (a^3)^2 = 0$ which means that a^3 is *-nilpotent.

In the next, a sufficient condition is given to make *-nilpotency transfers between the element and its powers.

Lemma 2. *In a *- reversible *-ring R, the element a is *-nilpotent if and only if a^2 is *-nilpotent.*

Proof. Let a be a *-nilpotent element of R, then $a^n = (aa^*)^m = 0$, for some positive integers m and n. Now, $0 = (aa^*)^m = a^*(aa^*)^m = a^*(aa^*)^{m-1}(aa^*)$ and from the *-reversibility of R, we get $0 = (aa^*)a^*(aa^*)^{m-1} = a(a^*)^2(aa^*)^{m-1}$. Multiply the last equation by a from right to get $a(a^*)^2a(a^*a)^{m-2}(a^*a) = 0$ and applying the *-reversible property again, we get $a^*a^2(a^*)^2a(a^*a)^{m-2} = 0 = a^*a^2(a^*)^2(aa^*)^{m-2}a$. Multiply again by a^* from right and apply the *-reversibility, we get $a(a^*)^2a^2(a^*)^2(aa^*)^{m-2} = 0$. Continuing, we get $(a^2(a^*)^2)^m = 0$ and a^2 is *-nilpotent.

For sufficiency, if a^2 is *-nilpotent; that is $(a^2)^n = 0 = (a^2(a^*)^2)^m$ for some positive integers m and n, we get by the same procedure as above $(a^*a)^4m = 0$ and a is *-nilpotent. □

Proposition 23. *In a *- reversible *-ring R, the element a is *-nilpotent if and only if a^k is also *-nilpotnet for every positive integer k.*

Proof. The sufficient condition is clear. For the necessity, let a be a *-nilpotent element of R, then $a^l = (aa^*)^n = 0$ for some positive integers l and n. We use induction on k to show that $a^k(a^*)^k$ is nilpotent. The case $k = 2$ is clear from Lemma 2. Now, we have to show that $a^{k+1}(a^*)^{k+1}$ is also nilpotent if $a^k(a^*)^k$ is nilpotent. Now, if $0 = (a^k(a^*)^k)^m = a^k(a^*)^k(a^k(a^*)^k)^{m-1}$, multiply by $(a^*)^{k+1}a$ from left and apply the *-reversibility, we get $(a^*)^k(a^k(a^*)^k)^{m-1}(a^*)^{k+1}a^{k+1} = 0$. Multiply by a^* from left and take involution of both sides, we obtain $(a^*)^{k+1}a^{k+1}(a^k(a^*)^k)^{m-1}a^{k+1} = 0$. The *-reversibility of R gives $a^k(a^*)^k(a^k(a^*)^k)^{m-2}a^{k+1}(a^*)^{k+1}a^{k+1} = 0$. Multiplying by $(a^*)^{k+1}a$ from left gives $(a^*)^{k+1}a^{k+1}(a^*)^k(a^k(a^*)^k)^{m-2}a^{k+1}(a^*)^{k+1}a^{k+1} = 0$ and the *-reversibility of R gives $(a^*)^k(a^k(a^*)^k)^{m-2}a^{k+1}((a^*)^{k+1}a^{k+1})^2 = 0$. Multiply again by $(a^*)^{k+1}$, we get $(a^*)^k(a^k(a^*)^k)^{m-2}(a^{k+1}(a^*)^{k+1})^3 = 0$. Continuing, we get $(a^*)^k(a^{k+1}(a^*)^{k+1})^{2m-1} = 0$ and multiplication by $a^{k+1}a^*$ gives $(a^{k+1}(a^*)^{k+1})^{2m} = 0$. □

Conjecture 1 (Köthe's conjecture)**.** *If a ring has a non-zero nil right ideal, then it has a nonzero nil ideal, is still unsolved.*

In [9, Theorem 2.2], Cohn proved that for reversible rings, Köthe's conjecture has an affirmative solution. In the next, we have a strong affirmative solution for *-reversible *-rings.

Proposition 24. *Every *-reversible *-ring which is not *-reduced, contains a nonzero nilpotent ideal.*

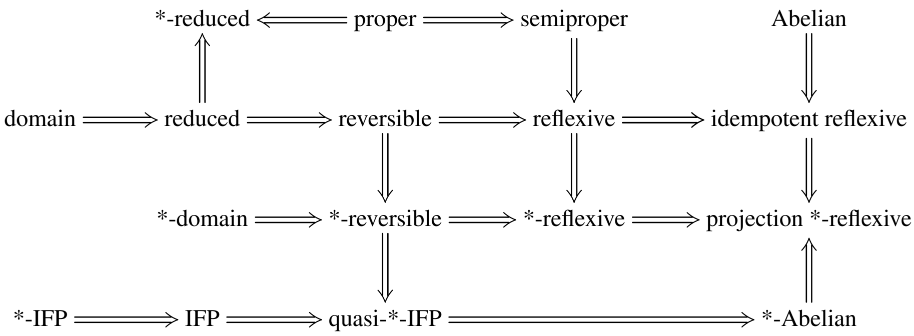
Proof. If R is not *-reduced and *-reversible *-ring, then R contains a nonzero *-nilpotent element, say a. So that $a^m = (aa^*)^n = 0$, for some positive integers m and n. If $n = 1$, we have $a^m = aa^* = 0$ which implies $r_1 a^m = r_1 a^{m-1} a^* = 0$ for every $r_1 \in R$. From the *-reversibility of R, we get $a r_1 a^{m-1} = 0$. Again $r_2 a r_1 a^{m-1} = r_2 a r_1 a^{m-2} a^* = 0$ implies $a r_2 a r_1 a^{m-2} = 0$ for every $r_1, r_2 \in R$. Continuing, we get $(RaR)^m = 0$; that is the ideal generated by a is a nonzero nilpotent ideal. If $n > 1$, we have $aa^* \neq 0$. Since $(aa^*)^n = 0$, then $r_1(aa^*)^n = 0$ gives $(aa^*)r_1(aa^*)^{n-1} = 0$ due to the self-adjointness of aa^* and using the *-reversible property. As before, we get $(Raa^*R)^n = 0$; that is the *-ideal generated by aa^* is a nonzero nilpotent ideal. □

Corollary 6. *In a *-reversible *-ring R, if R has a non-zero nil right ideal, then it has a nonzero nil ideal.*

Corollary 7. *Each semiprime *-reversible *-ring is *-reduced.*

CONCLUSION

We can now sate the following implications in the class of rings with involution.



REFERENCES

[1] U. A. Aburawash and M. Saad, "On biregular, IFP and quasi-Baer $*$-rings," *East-West J. Math.*, vol. 16, no. 2, pp. 182–192, 2014.

[2] U. A. Aburawash and K. B. Sola, "$*$-zero divisors and $*$-prime ideals," *East-West J. Math.*, vol. 12, no. 1, pp. 27–31, 2010.

[3] U. A. Aburawash and M. Saad, "∗-Baer property for rings with involution," *Studia Sci. Math. Hungar*, vol. 53, no. 2, pp. 243–255, 2016, doi: 10.1556/012.2016.53.2.1338.
[4] K. I. Beidar, L. Márki, R. Mlitz, and R. Wiegandt, "Primitive involution rings," *Acta Mathematica Hungarica*, vol. 109, no. 4, pp. 357–368, 2005, doi: 10.1007/S10474-005-0253-4.
[5] H. E. Bell, "Near-rings in which each element is a power of itself," *Bull. Austral. Math. Soc.*, vol. 2, pp. 363–368, 1970, doi: 10.1017/S0004972700042052.
[6] G. F. Birkenmeier and N. J. Groenewald, "Prime ideals in rings with involution," *Quaest. Math.*, vol. 20, no. 4, pp. 591–603, 1997, doi: 10.1080/16073606.1997.9632228.
[7] G. F. Birkenmeier, "Idempotents and completely semiprime ideals," *Comm. Algebra*, vol. 11, pp. 567–58, 1983, doi: 10.1080/00927878308822865.
[8] G. F. Birkenmeier, J. Y. Kim, and J. K. Park, "Quasi-Baer ring extensions and biregular rings," *Bull. Austral. Math. Soc.*, vol. 61, no. 1, pp. 39–52, 2000, doi: 10.1017/S0004972700022000.
[9] P. M. Cohn, "Reversible rings," *London Math. Soc.*, vol. 31, pp. 641–648, 1999, doi: 10.1112/S0024609399006116.
[10] J. Y. Kim and J. U. Baik, "On idempotent reflexive rings," *Kyungpook Math. J.*, vol. 46, pp. 597–601, 2006.
[11] N. K. Kima and Y. Lee, "Extensions of reversible rings," *J. Pure App. Algebra*, vol. 185, pp. 207–223, 2003, doi: 10.1016/S0022-4049(03)00109-9.
[12] T. K. Kwak and Y. Lee, "Reflexive property of rings," *Comm. Algebra*, vol. 40, no. 4, pp. 1576–1594, 2012, doi: 10.1080/00927872.2011.554474.
[13] G. Mason, "Reflexive ideals," *Comm. Algebra*, vol. 9, pp. 1709–1724, 1981, doi: 10.1080/00927878108822678.

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