



A NOTE ON SUMS OF A CLASS OF SERIES

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Abstract. The aim of this note is to provide sums of a unified class of series of the form

$$S_i(a) = \sum_{k=0}^{\infty} (-1)^k \binom{a-i}{k} \frac{1}{2^k (a+k+1)}$$

in the most general form for any $i \in \mathbb{Z}$. For each $\nu \in \mathbb{N}$, in four cases when $i = \pm 2\nu$ and $i = \pm(2\nu - 1)$, simple explicit expressions for $S_i(a)$ are obtained, e.g.

$$S_{2\nu}(a) = \frac{2^{2\nu-1-a}}{(a-2\nu+1)_\nu} \left[\frac{\sqrt{\pi} \Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}-\nu\right)} - P_{\nu-1}(a) \right],$$

where $P_\nu(a)$ is an algebraic polynomial in a of degree ν .

For $i = 1$ and $a = n \in \mathbb{N}$, we recover the well known sum of the series due to Vowe and Seiffert. Several other known results due to Srivastava and Kim *et al.* can be considered as special cases of our result.

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1. INTRODUCTION AND PRELIMINARIES

In 1812, Gauss [6] defined his famous infinite series as follows:

$$1 + \frac{a \cdot b}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots \quad (1.1)$$

The series (1.1) is denoted by the notation

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix}; z \right], \text{ or } {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix}; z \right], \text{ or } {}_2F_1[a, b; c; z],$$

or simply F and is popularly known as the Gauss's function or hypergeometric function.

The quantities a , b and c are known as the parameters (real or complex) of the series with $c \neq 0, -1, -2, \dots$ and z is termed as the variable of the series. This is

called the ‘Hypergeometric series’ because either $a = 1$ and $b = c$ or $b = 1$ and $a = c$, it reduces to the well known ‘Geometric series’.

In terms of Pochhammer’s symbol $(a)_n$, defined by

$$(a)_n = \begin{cases} a(a + 1) \cdots (a + n - 1), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{1.2}$$

the series (1.1) is represented as

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}.$$

Also, in term of Gamma function, $(a)_n$ can be represented as

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}.$$

Thus, from (1.1), we have

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}. \tag{1.3}$$

Here, we verify that the series (1.1)

- (i) is convergent for all values of z provided $|z| < 1$ and divergent when $|z| > 1$.
- (ii) is convergent for $z = 1$ provided $\text{Re}(c - a - b) > 0$ and divergent for $\text{Re}(c - a - b) \leq 0$.
- (iii) is absolutely convergent for $z = -1$ provided $\text{Re}(c - a - b) > 0$ and convergent but not absolutely for $-1 < \text{Re}(c - a - b) \leq 0$ and divergent for $\text{Re}(c - a - b) < -1$.

The limiting case of (1.1) is worth mentioning here. For this, if we replace z by z/b in (1.3) and take the limit as $b \rightarrow \infty$, then since

$$\frac{(b)_n}{b^n} z^n \rightarrow z^n,$$

we arrive at the following series which is in the literature known as the Kummer’s series or the confluent hypergeometric series [9]

$${}_1F_1 \left[\begin{matrix} a \\ c \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.4}$$

Further, it is interesting to mention here that almost all elementary functions of mathematics and mathematical physics are special cases or limiting cases of Gauss’s hypergeometric function or the confluent hypergeometric function, see [3, 15].

It is well known that whenever a ${}_2F_1$ hypergeometric function reduces to a gamma function, then the obtained result is very useful for application. Thus, for example,

Gauss summation theorem, Gauss second, Kummer, Vandermonde and Bailey’s theorems for the series ${}_2F_1$ play a vital role in the theory of hypergeometric function, especially in applied mathematics, mathematical physics and engineering.

However, in this note, we are interested in the following Bailey’s summation theorem [1,2] viz.

$${}_2F_1 \left[\begin{matrix} a, 1-a \\ b \end{matrix} ; \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}b + \frac{1}{2}a)\Gamma(\frac{1}{2}b - \frac{1}{2}a + \frac{1}{2})}. \tag{1.5}$$

During 1992–96, Lavoie *et al.* [10–12] have successfully investigated the generalizations of the above mentioned classical summation theorems and also obtained the generalization of the classical summation theorems for the series ${}_3F_2$ with unit argument such as those of Watson, Dixon and Whipple.

In particular, they have generalized Bailey’s summation theorem (1.5) by obtaining a single formula containing eleven results in the form

$${}_2F_1 \left[\begin{matrix} a, 1-a+i \\ b \end{matrix} ; \frac{1}{2} \right] \tag{1.6}$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. Later, Rakha and Rathie [17] and Kim *et al.* [8] have further generalized and extended respectively, the above mentioned classical summation theorems in the most general form.

One of such kind of summation theorems will be used in this note in order to provide sums of a unified class of the series of the form

$$\sum_{k=0}^{\infty} (-1)^k \binom{a-i}{k} \frac{1}{2^k(a+k+1)} \tag{1.7}$$

in the most general form for any $i \in \mathbb{Z}$. Otherwise, the first result of this type was appeared in 1987, when Vowe and Seiffert [20] evaluated the following very interesting and useful sum, viz.

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{2^k(n+k+1)} = \frac{2^n(n-1)!n!}{(2n)!} - \frac{1}{2^n n} \quad (n \in \mathbb{N}) \tag{1.8}$$

by identifying it with an Eulerian integral of the type

$$\int_0^1 \left(1 - \frac{t}{2}\right)^{n-1} t^n dt. \tag{1.9}$$

In recent years, the sum (1.8) has gained a fair amount of attention. In 1988, Srivastava [18] established the generalization of (1.8) in the form

$$\sum_{k=0}^{\infty} (-1)^k \binom{a-1}{k} \frac{1}{2^k(a+k+1)} = \frac{2^a \Gamma(a)\Gamma(a+1)}{\Gamma(2a+1)} - \frac{1}{2^a a} \tag{1.10}$$

by employing the classical Bailey’s summation theorem (1.5) and also discussed a few more series of this type.

In 1989, Srivastava [19] gave a basic (or q -) extension of (1.10).

In 1999, Choi *et al.* [4] have re-derived the result (1.10) and obtained the following two results closely related to (1.10) by utilizing certain contiguous function relations for ${}_2F_1$ due to Gauss [6, 16],

$$\sum_{k=0}^{\infty} (-1)^k \binom{a}{k} \frac{k}{2^k (a+k)(a+k+1)} = \frac{2^{a-1} \{\Gamma(a+1)\}^2}{\Gamma(2a+2)} - \frac{1}{2^{a+1}} \tag{1.11}$$

and

$$\sum_{k=0}^{\infty} (-1)^k \binom{a-2}{k} \frac{k}{2^k (a+k)(a+k+1)} = \frac{3 \cdot 2^a \{\Gamma(a+1)\}^2}{(a-1)\Gamma(2a+1)} - \frac{a+2}{(a-1)^{2a-1}}. \tag{1.12}$$

In 2009, Prodinger [14] established a more general sum of the form

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{2^k (m+k)} \quad (m, k \in \mathbb{N})$$

and obtained as special cases, the results (1.11) and (1.12) for $a = n$ by completely elementary tools.

In 2010, Dahlberg *et al.* [5] have presented an elegant and elementary proof of the identities (1.11) and (1.12) for $a = n$ using Zeilberger algorithm and the Wilf-Zeilberger proof style.

In 2012, Kim *et al.* [7] generalized the sum (1.10) (and of course, Vowe and Seiffert’s sum (1.8)) and obtained the explicit expressions of (1.7) for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ by utilizing the generalizations of the Bailey’s summation formulas (1.6) obtained earlier by Lavoie *et al.* [11] and deduced a large number of very interesting results including (1.8) and (1.10).

In our present investigation, we will use the generalization of the Bailey’s summation theorem (1.5) in the following two forms [17]:

$$\begin{aligned} {}_2F_1 \left[a, \begin{matrix} 1-a+i \\ b \end{matrix}; \frac{1}{2} \right] &= \frac{2^{1+i-b} \Gamma(\frac{1}{2}) \Gamma(b) \Gamma(a-i)}{\Gamma(a) \Gamma(\frac{1}{2}b - \frac{1}{2}a) \Gamma(\frac{1}{2}b - \frac{1}{2}a + \frac{1}{2})} \\ &\times \sum_{r=0}^i \binom{i}{r} \frac{(-1)^r \Gamma(\frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}r)}{\Gamma(\frac{1}{2}b + \frac{1}{2}a + \frac{1}{2}r - i)} \end{aligned} \tag{1.13}$$

for $i = 0, 1, 2, \dots$ and

$${}_2F_1 \left[a, \begin{matrix} 1-a-i \\ b \end{matrix}; \frac{1}{2} \right] = \frac{2^{1-i-b} \Gamma(\frac{1}{2}) \Gamma(b)}{\Gamma(\frac{1}{2}b - \frac{1}{2}a) \Gamma(\frac{1}{2}b - \frac{1}{2}a + \frac{1}{2})}$$

$$\times \sum_{r=0}^i \binom{i}{r} \frac{\Gamma(\frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}r)}{\Gamma(\frac{1}{2}b + \frac{1}{2}a + \frac{1}{2}r)} \tag{1.14}$$

for $i = 0, 1, 2, \dots$, as well as a general formula (containing (1.13) and (1.14)), obtained recently by Milovanović *et al.* [13],

$$\begin{aligned} & {}_2F_1 \left[\begin{matrix} a, 1-a+i \\ b \end{matrix} ; \frac{1}{2} \right] \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-i-1}\Gamma(1-a+\frac{1}{2}(i+|i|))} \left\{ \frac{C_i(a,b)}{\Gamma(\frac{1}{2}b - \frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2}a - \lfloor \frac{1+i}{2} \rfloor)} \right. \\ & \quad \left. + \frac{D_i(a,b)}{\Gamma(\frac{1}{2}b - \frac{1}{2}a)\Gamma(\frac{1}{2}b + \frac{1}{2}a - \frac{1}{2} - \lfloor \frac{i}{2} \rfloor)} \right\}, \tag{1.15} \end{aligned}$$

with coefficients $C_i(a, b)$ and $D_i(a, b)$ given by

$$\begin{aligned} C_{2\nu}(a, b) &= \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} - (2\nu-j)\right)_{\nu-j}, \\ D_{2\nu}(a, b) &= -\sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} - (2\nu-j)\right)_{\nu-j-1}, \\ C_{2\nu+1}(a, b) &= -\sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} - (2\nu-j+1)\right)_{\nu-j}, \\ D_{2\nu+1}(a, b) &= \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} - (2\nu-j+1)\right)_{\nu-j}, \\ C_{-2\nu}(a, b) &= \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} + j\right)_{\nu-j}, \\ D_{-2\nu}(a, b) &= \sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} + j\right)_{\nu-j-1}, \\ C_{-(2\nu+1)}(a, b) &= \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} + j\right)_{\nu-j}, \end{aligned}$$

$$D_{-(2\nu+1)}(a, b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} + j\right)_{\nu-j}.$$

Clearly, for $i = 0$, the results (1.13), (1.14) and (1.15) reduce to the Bailey’s summation theorem (1.5).

The main objective of this note is to provide sums of a unified class of the series of the form

$$S_i(a) = \sum_{k=0}^{\infty} (-1)^k \binom{a-i}{k} \frac{1}{2^k(a+k+1)} \tag{1.16}$$

in the most general form for any $i \in \mathbb{Z}$. The results obtained earlier by Vowe and Seiffert [20] and Srivastava [18] follows as special cases of our main findings.

2. MAIN RESULTS

The following theorem gives a result on sums of a unified class of series.

Theorem 1. *For $i = 0, 1, 2, \dots$, the following results hold true.*

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k \binom{a-i}{k} \frac{1}{2^k(a+k+1)} \\ = \frac{\Gamma(a+1-i)}{2^{a-i+1}} \sum_{r=0}^i (-1)^r \binom{i}{r} \frac{\Gamma(\frac{1}{2}r + \frac{1}{2})}{\Gamma(a + \frac{3}{2} + \frac{1}{2}r - i)} \end{aligned} \tag{2.1}$$

and

$$\sum_{k=0}^{\infty} (-1)^k \binom{a+i}{k} \frac{1}{2^k(a+k+1)} = \frac{\Gamma(a+1)}{2^{a+i+1}} \sum_{r=0}^i \binom{i}{r} \frac{\Gamma(\frac{1}{2}r + \frac{1}{2})}{\Gamma(a + \frac{3}{2} + \frac{1}{2}r)} \tag{2.2}$$

Proof. The proofs of our results are quite straight forward. For this, in order to establish the result (2.1), we consider (1.16) for $i \in \mathbb{N}_0$.

If we use the elementary identities

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad \Gamma(\alpha-n) = \frac{(-1)^n \Gamma(\alpha)}{(1-\alpha)_n},$$

then, after some simplification, the left-hand side in (2.1) becomes

$$S_i(a) = \frac{1}{a+1} \sum_{k=0}^{\infty} \frac{(-a+i)_k (a+1)_k}{(a+2)_k} \cdot \frac{1}{k!2^k}$$

Summing up the series with the help of (1.3), we get

$$S_i(a) = \frac{1}{a+1} {}_2F_1 \left[\begin{matrix} -a+i, a+1 \\ a+2 \end{matrix}; \frac{1}{2} \right]. \tag{2.3}$$

We now observe here that the right-hand side of (2.1) can be verified with the result (1.13) to ${}_2F_1$ above and some calculation.

In exactly the same manner, the second result (2.2) for $S_{-i}(a)$, $i \in \mathbb{N}$, can be proven with the help of the known result (1.14). \square

Alternatively, the results given in previous theorem can be expressed in the following explicit form:

Theorem 2. For each $v \in \mathbb{N}$ the sum (1.16) can be expressed as

$$S_{2v}(a) = \frac{2^{2v-1-a}}{(a-2v+1)_v} \left[\frac{\sqrt{\pi} \Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}-v\right)} - P_{v-1}(a) \right] \tag{2.4}$$

and

$$S_{2v-1}(a) = \frac{2^{2v-2-a}}{(a-2v+2)_v} \left[\frac{\sqrt{\pi} \Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}-v\right)} - Q_{v-1}(a) \right], \tag{2.5}$$

where P_v and Q_v are polynomials of degree v , defined by the recurrence relations

$$\left. \begin{aligned} P_v(a) &= 2Q_v(a-1) - (a-2v-1)P_{v-1}(a-1), \\ Q_v(a) &= 2(a-v)P_{v-1}(a-1) - (a-2v)Q_{v-1}(a-1), \end{aligned} \right\} \tag{2.6}$$

with $P_0(a) = 2$ and $Q_0(a) = 1$.

For negative indices we have

$$S_{-2v}(a) = \frac{2^{-a-2v-1}}{(a+1)_v} \left[\frac{\sqrt{\pi} \Gamma(a+2v+1)}{\Gamma\left(a+\frac{3}{2}+v\right)} + R_{v-1}(a) \right] \tag{2.7}$$

and

$$S_{-(2v-1)}(a) = \frac{2^{-a-2v}}{(a+1)_v} \left[\frac{\sqrt{\pi} \Gamma(a+2v)}{\Gamma\left(a+\frac{1}{2}+v\right)} + T_{v-1}(a) \right], \tag{2.8}$$

where R_v and T_v are polynomials of degree v , defined by the recurrence relations

$$\left. \begin{aligned} T_v(a) &= 2(a+v+1)R_{v-1}(a) - (a+1)T_{v-1}(a+1), \\ R_v(a) &= 2T_v(a) - (a+1)R_{v-1}(a+1), \end{aligned} \right\} \tag{2.9}$$

with $R_0(a) = 2$ and $T_0(a) = 1$.

The expressions (2.6) and (2.9) hold also for $v = 0$ if we take $P_{-1}(a) = R_{-1}(a) = 0$.

In proving Theorem 2.2 we need the following auxiliary result:

Lemma 1. For each $\nu \in \mathbb{N}_0$ the sum (1.16) satisfies the following recurrence relations

$$\left. \begin{aligned} S_{2\nu+1}(a) &= 2[S_{2\nu}(a-1) - S_{2\nu-1}(a-1)], \\ S_{2\nu+2}(a) &= 2[S_{2\nu+1}(a-1) - S_{2\nu}(a-1)], \\ S_{-(2\nu+1)}(a) &= S_{-2\nu}(a) - \frac{1}{2}S_{-(2\nu-1)}(a+1), \\ S_{-(2\nu+2)}(a) &= S_{-(2\nu+1)}(a) - \frac{1}{2}S_{-2\nu}(a+1). \end{aligned} \right\} \quad (2.10)$$

Proof. Starting from (1.16) for $i = 2\nu + 1$ and using the identity

$$\binom{a-2\nu-1}{k} = \binom{a-2\nu}{k+1} - \binom{a-2\nu-1}{k+1},$$

we can easily prove the first recurrence relation in (2.10). In a similar way we get the other ones. We omit the details. \square

Proof of Theorem 2. In order to prove explicit expressions (2.4), (2.5), (2.7), and (2.8), we apply (1.15) to (2.3) for $i = 2\nu, 2\nu - 1, -2\nu$, and $-(2\nu - 1)$, respectively, and use the corresponding coefficients $C_i(a, b)$ and $D_i(a, b)$.

Finally, using Lemma 1 we obtain the recurrence relations (2.6) and (2.9) for the polynomials P_ν and Q_ν and R_ν and T_ν , respectively. \square

Remark 1. The polynomials P_ν and Q_ν which satisfy the recurrence relations (2.6) are given, for $\nu = 0, 1, \dots, 9$, by

$$\begin{aligned} P_0(a) &= 2, \quad P_1(a) = 4(a-1), \quad P_2(a) = 2(3a^2 - 11a + 12), \\ P_3(a) &= 8(a-3)(a^2 - 5a + 10), \\ P_4(a) &= 2(5a^4 - 70a^3 + 427a^2 - 1322a + 1680), \\ P_5(a) &= 4(a-5)(3a^4 - 50a^3 + 401a^2 - 1698a + 3024), \\ P_6(a) &= 2(7a^6 - 217a^5 + 3227a^4 - 28967a^3 + 159750a^2 \\ &\quad - 496680a + 665280), \\ P_7(a) &= 16(a-7)(a^6 - 35a^5 + 623a^4 - 6889a^3 + 47256a^2 \\ &\quad - 183516a + 308880), \\ P_8(a) &= 6(3a^8 - 164a^7 + 4494a^6 - 79280a^5 + 953387a^4 - 7765436a^3 \\ &\quad + 40967236a^2 - 126332400a + 172972800), \\ P_9(a) &= 4(a-9)(5a^8 - 300a^7 + 9402a^6 - 193944a^5 + 2747829a^4 \\ &\quad - 26422764a^3 + 164805772a^2 - 602206800a + 980179200), \end{aligned}$$

and

$$\begin{aligned}
 Q_0(a) &= 1, \quad Q_1(a) = 3a - 2, \quad Q_2(a) = 5a^2 - 15a + 12, \\
 Q_3(a) &= 7a^3 - 49a^2 + 126a - 120, \\
 Q_4(a) &= 3(3a^4 - 38a^3 + 201a^2 - 526a + 560), \\
 Q_5(a) &= 11a^5 - 220a^4 + 1969a^3 - 9812a^2 + 26532a - 30240, \\
 Q_6(a) &= 13a^6 - 377a^5 + 5109a^4 - 41119a^3 + 202046a^2 \\
 &\quad - 558792a + 665280, \\
 Q_7(a) &= 15a^7 - 595a^6 + 11361a^5 - 134185a^4 + 1032120a^3 - 5031580a^2 \\
 &\quad + 14102064a - 17297280, \\
 Q_8(a) &= 17a^8 - 884a^7 + 22610a^6 - 367880a^5 + 4060433a^4 - 30316916a^3 \\
 &\quad + 146566860a^2 - 414018000a + 518918400, \\
 Q_9(a) &= 19a^9 - 1254a^8 + 41382a^7 - 886692a^6 + 13256091a^5 \\
 &\quad - 139709166a^4 + 1017764108a^3 - 4878321288a^2 \\
 &\quad + 13847306400a - 17643225600,
 \end{aligned}$$

respectively.

Remark 2. The polynomials R_ν and T_ν which satisfy the recurrence relations (2.9) are given, for $\nu = 0, 1, \dots, 9$, by

$$\begin{aligned}
 R_0(a) &= 2, \quad R_1(a) = 4(a + 3), \quad R_2(a) = 2(3a^2 + 25a + 54), \\
 R_3(a) &= 8(a + 5)(a^2 + 11a + 34), \\
 R_4(a) &= 2(5a^4 + 130a^3 + 1327a^2 + 6218a + 11160), \\
 R_5(a) &= 4(a + 7)(3a^4 + 94a^3 + 1193a^2 + 7062a + 16200), \\
 R_6(a) &= 2(7a^6 + 371a^5 + 8617a^4 + 110585a^3 + 817288a^2 \\
 &\quad + 3270524a + 5504688), \\
 R_7(a) &= 16(a + 9)(a^6 + 61a^5 + 1663a^4 + 25303a^3 + 222952a^2 \\
 &\quad + 1067812a + 2158800), \\
 R_8(a) &= 6(3a^8 + 268a^7 + 11046a^6 + 269992a^5 + 4228307a^4 \\
 &\quad + 43076572a^3 + 277195204a^2 + 1026169008a + 1668885120), \\
 R_9(a) &= 4(a + 11)(5a^8 + 500a^7 + 23402a^6 + 654296a^5 + 11765429a^4
 \end{aligned}$$

$$+ 137947556a^3 + 1023509532a^2 + 4376435280a + 8236080000),$$

and

$$T_0(a) = 1, \quad T_1(a) = 3a + 7, \quad T_2(a) = 5a^2 + 35a + 62,$$

$$T_3(a) = 7a^3 + 98a^2 + 469a + 762,$$

$$T_4(a) = 3(3a^4 + 70a^3 + 633a^2 + 2606a + 4088),$$

$$T_5(a) = 11a^5 + 385a^4 + 5599a^3 + 41855a^2 + 159390a + 245640,$$

$$T_6(a) = 13a^6 + 637a^5 + 13559a^4 + 158639a^3 + 1065636a^2 \\ + 3868332a + 5897520,$$

$$T_7(a) = 15a^7 + 980a^6 + 28686a^5 + 481640a^4 + 4958895a^3 \\ + 31072820a^2 + 109138164a + 165145680,$$

$$T_8(a) = 17a^8 + 1428a^7 + 54978a^6 + 1250520a^5 + 18186753a^4 \\ + 171842052a^3 + 1024655212a^2 + 3512174160a + 5284782720,$$

$$T_9(a) = 19a^9 + 1995a^8 + 97698a^7 + 2888646a^6 + 56212203a^5 \\ + 740755755a^4 + 6574624112a^3 + 37758440004a^2 \\ + 126997604208a + 190253266560,$$

respectively.

3. CONCLUDING REMARK

In this note, an attempt has been made to provide a unified sum of the series of the form

$$\sum_{k=0}^{\infty} (-1)^k \binom{a-i}{k} \frac{1}{2^k (a+k+1)}$$

in the most general form for any $i = 0, \pm 1, \pm 2, \dots$

We believe that the results obtained in this note may be potentially useful in combinatorics, applied mathematics, mathematical physics and engineering.

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