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COINCIDENCE POINT RESULTS IN B-METRIC SPACES VIA C$_F$-s-SIMULATION FUNCTION

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Abstract. The notion of C_F-s-simulation function is introduced and the existence and uniqueness of coincidence point of two self mappings in the framework of b-metric spaces is investigated. An example with a corresponding numerical simulation is also provided to support the obtained result.

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1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is a widely used tool in mathematical analysis and its applications. Over the decades this field has intrigued researchers and has developed extensively. Numerous authors have generalized metric spaces and contraction principle. Bakhtin [3] and Czerwik [6] generalized the notion of metric space and introduced the concept of b-metric space. Many mathematicians have obtained fixed point and coincidence point results in various generalizations of metric spaces. Mleşniţe [11] and Falset and Mleşniţe [7] studied the existence, uniqueness and Ulam-Hyers stability for the coincidence point problem of a pair of single-valued mappings. Also, Petruşel et al. [12] investigated the existence and uniqueness of coincidence points of a pair of operators satisfying contraction and expansion type conditions in the setting of b-metric spaces.

Recently, Khojasteh [9] introduced the notion of simulation function and unified several known fixed point theorems in the setting of metric spaces. In fact, Hierro et al. [15] obtained coincidence point of two self mappings in the framework of metric spaces by using simulation functions. Also, Yamaod and Sintunavarat [17] studied the existence and uniqueness of fixed point of nonlinear mappings in the context of b-metric spaces involving s-simulation functions.

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Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and real numbers, respectively. The following terminologies and definitions will be used in the sequel:

Definition 1 ([6])**.** A b-metric on a non-empty set X is a function $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$ and a constant $s \geq 1$, the following conditions hold:

(b_1) $d(x,y) = 0$ if and only if $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space. The number s is called the coefficient of (X,d).

Definition 2 ([5])**.** Let (X,d) be a b-metric space. Then
(i) A sequence $\{x_n\} \subseteq X$ *converges* to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n,x) = 0$.
(ii) A sequence $\{x_n\} \subseteq X$ is called a *Cauchy sequence* if and only if

$$\lim_{n,m\to\infty} d(x_n,x_m) = 0.$$

(iii) A b-metric space (X,d) is said to be *complete* if every Cauchy sequence $\{x_n\} \subseteq X$ converges to a point $x \in X$ such that $\lim_{n,m\to\infty} d(x_n,x_m) = 0 = \lim_{n\to\infty} d(x_n,x)$.
(iv) A mapping $T : X \to X$ is said to be *b-continuous* if for $\{x_n\} \subseteq X$, $x_n \to x$ in (X,d) implies that $Tx_n \to Tx$ in (X,d).

Remark 1. In a b-metric space (X,d) a convergent sequence has a unique limit.

Let $T,S : X \to X$ be self mappings on a b-metric space (X,d). If $y = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S. If $Tx = Sx = x$ for some $x \in X$, then x is called a common fixed point of T and S. We say that the pair (T,S) is compatible if $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$ for every sequence $\{x_n\} \subseteq X$ such that the sequences $\{Tx_n\}$ and $\{Sx_n\}$ are convergent and have the same limit. We say that T and S are weakly compatible if T and S commute at their coincidence points. A sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Picard-Jungck sequence of the pair (T,S) (based at x_0) if $y_n = Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. If $T(X) \subseteq S(X)$ then there exists a Picard-Jungck sequence of (T,S) based at any point $x_0 \in X$. The following result of Abbas and Jungck [1] establishes the relationship between point of coincidence and common fixed point of T and S.

Proposition 1. *Let T and S be weakly compatible self mappings on a set X. If T and S have a unique point of coincidence $y = Tx = Sx$, then y is the unique common fixed point of T and S.*

Definition 3 ([2])**.** A mapping $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a *C-class function* if it satisfies the following conditions:

(i) F is continuous,
(ii) $F(v,u) \leq v$ for all $u, v \in [0,\infty)$,
(iii) $F(v,u) = v$ implies that either $u = 0$ or $v = 0$ for all $u, v \in [0,\infty)$.

Definition 4 ([10]). A mapping $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to satisfy *property* C_F, if there exists $C_F \geq 0$ such that
(i) $F(v,u) > C_F$ implies that $v > u$,
(ii) $F(u,u) \leq C_F$ for all $u \in [0,\infty)$.

Some examples of C-class functions having property C_F are:
(i) $F(v,u) = v - u$, $C_F = r$, where $r \in [0,\infty)$,
(ii) $F(v,u) = \frac{v}{1+u}$, $C_F = 1, 2$,
(iii) $F(v,u) = \frac{kv}{1+u}$, $0 < k < 1$, $C_F = 1, k$.
For more examples of C-class functions having property C_F see [10, 14]. Liu et al. [10] generalized the simulation function introduced by Khojasteh et al. [9] using C-class function as follows:

Definition 5. A C_F-simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:
(i) $\zeta(0,0) = 0$,
(ii) $\zeta(u,v) < F(v,u)$ for all $u, v > 0$, where $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a C-class function satisfying property C_F,
(iii) if $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n > 0$ and $u_n < v_n$, then $\limsup_{n\to\infty} \zeta(u_n, v_n) < C_F$.

Yamaod and Sintunavarat [17] defined the concept of s-simulation function as follows:

Definition 6. Let $s \geq 1$ be a given real number. A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be an *s-simulation function* if it satisfies the following conditions:
(i) $\zeta(u,v) < v - u$ for all $u, v > 0$,
(ii) if $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \liminf_{n\to\infty} u_n \leq s\Big(\limsup_{n\to\infty} v_n\Big) \leq s^2\Big(\liminf_{n\to\infty} u_n\Big) < \infty$$

and

$$0 < \liminf_{n\to\infty} v_n \leq s\Big(\limsup_{n\to\infty} u_n\Big) \leq s^2\Big(\liminf_{n\to\infty} v_n\Big) < \infty,$$

then $\limsup_{n\to\infty} \zeta(u_n, v_n) < 0$.

If we take $s = 1$ then ζ is a simulation function in the sense of Khojasteh [9] if and only if ζ is an s-simulation function.

Liu et al. [10] and Radenović and Chandok [14] generalized the simulation function defined by Khojasteh et al. [9]. Motivated by them we have generalized the

s-simulation function introduced by Yamaod and Sintunavarat [17] using C-class functions having property C_F. We have generalized the fixed point results proved in [17] to the coincidence point case. We introduce the notion of C_F-s-simulation function. It is observed that every s-simulation function is a C_F-s-simulation function but the converse is not true in general. Our concept has broadened the family of s-simulation functions. The main objective of the paper is to establish the existence and uniqueness of point of coincidence of a pair of self mappings in the setting of b-metric spaces via C_F-s-simulation function, covering the case of commuting and compatible mappings. This approach enables us to study several coincidence point and fixed point problems from a common perspective. The purpose is to unify, generalize and improve several existing results in b-metric spaces. We underline that our approach has generalized the main results of [13, 17]. An example with a corresponding numerical simulation is also provided to demonstrate the utility of the results.

2. Main results

In this section, we establish the existence and uniqueness of coincidence point and common fixed point in the context of b-metric spaces. We begin with the following definition:

Definition 7. Let $s \geq 1$ be a given real number. A C_F-*s-simulation function* is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(i) $\zeta(u,v) < F(v,u)$ for all $u, v > 0$, where $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a C-class function satisfying property C_F,

(ii) if $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \liminf_{n\to\infty} u_n \leq s\Big(\limsup_{n\to\infty} v_n\Big) \leq s^2\Big(\liminf_{n\to\infty} u_n\Big) < \infty$$

and

$$0 < \liminf_{n\to\infty} v_n \leq s\Big(\limsup_{n\to\infty} u_n\Big) \leq s^2\Big(\liminf_{n\to\infty} v_n\Big) < \infty,$$

then $\limsup_{n\to\infty} \zeta(u_n, v_n) < C_F$.

Let $\mathscr{Z}_{F_s}$ be the family of all C_F-s-simulation functions. Every s-simulation function is a C_F-s-simulation function but the converse may not be true in general. This can be illustrated by taking $F(v,u) = v - u$ and $C_F = 0$ in Example 3.3 of [15] in which $k \in \mathbb{R}$ be such that $k < 1$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as

$$\zeta(u,v) = \begin{cases} 2(v-u), & \text{if } v < u, \\ kv - u, & \text{otherwise.} \end{cases}$$

Definition 8. Let (X,d) be a b-metric space and $T, S : X \to X$ be self mappings. Then T is called a $(\mathscr{Z}_{F_s}, S)$-contraction if there exists $\zeta \in \mathscr{Z}_{F_s}$ such that

$$\zeta(d(Tx,Ty), d(Sx,Sy)) \geq C_F \tag{2.1}$$

for all $x, y \in X$ with $Sx \neq Sy$.

Theorem 1. *Let (X,d) be a b-metric space with coefficient $s \geq 1$, $T, S : X \to X$ be self mappings and T be a $(\mathscr{Z}_{F_s}, S)$-contraction. Assume that $T(X) \subseteq S(X)$ and atleast one of the following conditions hold:*
(i) $(T(X),d)$ or $(S(X),d)$ is complete,
(ii) (X,d) is complete, S is b-continuous and (T,S) is compatible,
(iii) (X,d) is complete, S is b-continuous and T and S are commuting.
Then T and S have a unique point of coincidence.

Proof. Since $T(X) \subseteq S(X)$, there exists a Picard-Jungck sequence $\{x_n\}$ such that $y_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N} \cup \{0\}$. If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then $Tx_{n_0} = Sx_{n_0+1} = y_{n_0} = y_{n_0+1} = Tx_{n_0+1} = Sx_{n_0+2}$. This implies that T and S have a point of coincidence. Therefore, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Putting $x = x_{n+1}$ and $y = x_{n+2}$ in (2.1) we get,

$$\begin{aligned} C_F &\leq \zeta(d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1})) \\ &< F(d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})). \end{aligned}$$

By (i) of Definition 4, $d(y_n, y_{n+1}) > d(y_{n+1}, y_{n+2})$. Then $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of non-negative real numbers therefore, it is convergent. Let $\lim_{n\to\infty} d(y_n, y_{n+1}) = L \geq 0$. Suppose that $L > 0$ then $0 < L \leq sL \leq s^2 L < \infty$. This implies that

$$\begin{aligned} 0 < \liminf_{n\to\infty} d(y_{n+1}, y_{n+2}) &\leq s\Big(\limsup_{n\to\infty} d(y_n, y_{n+1})\Big) \\ &\leq s^2\Big(\liminf_{n\to\infty} d(y_{n+1}, y_{n+2})\Big) < \infty \end{aligned}$$

and

$$\begin{aligned} 0 < \liminf_{n\to\infty} d(y_n, y_{n+1}) &\leq s\Big(\limsup_{n\to\infty} d(y_{n+1}, y_{n+2})\Big) \\ &\leq s^2\Big(\liminf_{n\to\infty} d(y_n, y_{n+1})\Big) < \infty. \end{aligned}$$

Using (ii) of Definition 7 we have $C_F \leq \zeta(d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1})) < C_F$, a contradiction. Therefore, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Now we prove that $\{y_n\}$ is a Cauchy sequence in (X,d). Assume that $\{y_n\}$ is not Cauchy in (X,d). Then there exists $\epsilon_0 > 0$ for which we can find two subsequences $\{y_{n_i}\}$ and $\{y_{m_i}\}$ of $\{y_n\}$ such that n_i is the smallest integer for which

$$n_i > m_i > i, \qquad d(y_{m_i}, y_{n_i}) \geq \epsilon_0. \tag{2.2}$$

This means

$$d(y_{m_i}, y_{n_i - 1}) < \epsilon_0. \tag{2.3}$$

Since $\epsilon_0 \leq d(y_{m_i}, y_{n_i}) \leq sd(y_{m_i}, y_{n_i-1}) + sd(y_{n_i-1}, y_{n_i}) < s\epsilon_0 + sd(y_{n_i-1}, y_{n_i})$. Taking limit superior as $i \to \infty$ we get,

$$\epsilon_0 \leq \limsup_{i\to\infty} d(y_{m_i}, y_{n_i}) \leq s\epsilon_0. \tag{2.4}$$

Similarly, we have

$$\epsilon_0 \leq \liminf_{i\to\infty} d(y_{m_i}, y_{n_i}) \leq s\epsilon_0. \tag{2.5}$$

Putting $x = x_{m_i}$ and $y = x_{n_i}$ in (2.1) we get,

$$C_F \leq \zeta(d(y_{m_i}, y_{n_i}), d(y_{m_i-1}, y_{n_i-1})) < F(d(y_{m_i-1}, y_{n_i-1}), d(y_{m_i}, y_{n_i})).$$

By (i) of Definition 4 we have $d(y_{m_i}, y_{n_i}) < d(y_{m_i-1}, y_{n_i-1})$. Therefore, $\epsilon_0 \leq d(y_{m_i}, y_{n_i}) < d(y_{m_i-1}, y_{n_i-1}) \leq sd(y_{m_i-1}, y_{m_i}) + sd(y_{m_i}, y_{n_i-1}) < sd(y_{m_i-1}, y_{m_i}) + s\epsilon_0$. Taking limit superior as $i \to \infty$ we get,

$$\epsilon_0 \leq \limsup_{i\to\infty} d(y_{m_i-1}, y_{n_i-1}) \leq s\epsilon_0. \tag{2.6}$$

Similarly, we have

$$\epsilon_0 \leq \liminf_{i\to\infty} d(y_{m_i-1}, y_{n_i-1}) \leq s\epsilon_0. \tag{2.7}$$

Using (2.4), (2.5), (2.6) and (2.7) we have

$$\begin{aligned} 0 < \liminf_{i\to\infty} d(y_{m_i}, y_{n_i}) &\leq s\epsilon_0 \leq s\Big(\limsup_{i\to\infty} d(y_{m_i-1}, y_{n_i-1})\Big) \\ &\leq s^2\epsilon_0 \leq s^2\Big(\liminf_{i\to\infty} d(y_{m_i}, y_{n_i})\Big) < \infty \end{aligned}$$

and

$$\begin{aligned} 0 < \liminf_{i\to\infty} d(y_{m_i-1}, y_{n_i-1}) &\leq s\epsilon_0 \leq s\Big(\limsup_{i\to\infty} d(y_{m_i}, y_{n_i})\Big) \\ &\leq s^2\epsilon_0 \leq s^2\Big(\liminf_{i\to\infty} d(y_{m_i-1}, y_{n_i-1})\Big) < \infty. \end{aligned}$$

By (ii) of Definition 7 we have $C_F \leq \zeta(d(y_{m_i}, y_{n_i}), d(y_{m_i-1}, y_{n_i-1})) < C_F$, a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in (X, d).

Suppose that (i) holds. Assume that $(S(X), d)$ (or $(T(X), d)$) is complete. Then there exists $w \in S(X)$ such that $\lim_{n\to\infty} d(Sx_n, w) = 0$. Since $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, $\lim_{n\to\infty} d(Tx_n, w) = 0$. Let $z \in X$ such that $Sz = w$. We shall show that z is a coincidence point of T and S. We have $C_F \leq \zeta(d(Tx_n, Tz), d(Sx_n, Sz)) < F(d(Sx_n, Sz), d(Tx_n, Tz))$. Therefore, $d(Tx_n, Tz) < d(Sx_n, Sz)$ which implies that $\lim_{n\to\infty} d(Tx_n, Tz) = 0$. Since limit of a convergent sequence in a b-metric space is unique, $Tz = Sz = w$. Thus, w is a point of coincidence of T and S.

Suppose that (ii) holds. Since (X,d) is complete, there exists $w' \in X$ such that $\lim_{n\to\infty} d(Sx_n, w') = 0$. Therefore, $\lim_{n\to\infty} d(Tx_n, w') = 0$. As S is b-continuous, $\lim_{n\to\infty} d(STx_n, Sw') = 0$ and $\lim_{n\to\infty} d(SSx_n, Sw') = 0$. We have

$$C_F \le \zeta(d(TSx_n, Tw'), d(SSx_n, Sw')) < F(d(SSx_n, Sw'), d(TSx_n, Tw')).$$

Therefore, $d(TSx_n, Tw') < d(SSx_n, Sw')$ which gives $\lim_{n\to\infty} d(TSx_n, Tw') = 0$. Consider

$$\begin{aligned} d(Tw', Sw') &\le sd(Tw', TSx_n) + sd(TSx_n, Sw') \\ &\le sd(Tw', TSx_n) + s^2 d(TSx_n, STx_n) + s^2 d(STx_n, Sw'). \end{aligned}$$

Taking limit as $n \to \infty$ and using (T,S) is compatible we have $Tw' = Sw'$. Therefore, w' is a coincidence point of T and S.

Finally, suppose that (iii) holds. Since T and S are commuting then $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$. The proof is similar to the case when (ii) holds. Let w_1 and w_2 be two distinct point of coincidence of T and S. Then there exists $z_1,\ z_2 \in X$ such that $w_1 = Tz_1 = Sz_1$ and $w_2 = Tz_2 = Sz_2$. We have $C_F \le \zeta(d(Tz_1, Tz_2), d(Sz_1, Sz_2)) < F(d(w_1, w_2), d(w_1, w_2))$. Using (i) of Definition 4 we have $d(w_1, w_2) < d(w_1, w_2)$, a contradiction. Hence, T and S have a unique point of coincidence. □

The following example illustrates the efficiency of Theorem 1 by establishing the existence and uniqueness of the solution of a nonlinear equation.

Example 1. Let $X = [0,\infty)$ and $d : X \times X \to [0,\infty)$ be defined as

$$d(x,y) = \begin{cases} (x+y)^2, & \text{if } x \ne y, \\ 0, & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Then (X,d) is a complete b-metric space with coefficient $s = 2$. Define $T, S : X \to X$ as $Tx = 2x$ and $Sx = e^x + 6x - 1$. Take $\zeta(u,v) = \frac{1}{3}(v - 7u)$, $F(v,u) = v - u$ and $C_F = 0$. Consider

$$\begin{aligned} \zeta(d(Tx,Ty), d(Sx,Sy)) &= \frac{1}{3}\{(e^x + 6x + e^y + 6y - 2)^2 - 7(2x+2y)^2\} \\ &\ge \frac{1}{3}\{(6x+6y)^2 - 28(x+y)^2\} > 0. \end{aligned}$$

Therefore, T is a $(\mathscr{Z}_{F_s}, S)$-contraction. Also, we observe that $T(X) \subseteq S(X)$ and both $(T(X), d)$ and $(S(X), d)$ are complete. Hence, by Theorem 1 T and S have a unique coincidence point 0. For an initial point $x_0 = 0.2, 0.5, 1, 1.5$, the Picard-Jungck iterations are listed below. Also, the behavior of the process is shown by a graph.

TABLE 1. Picard-Jungck iterations

y_i	$x_0 = 0.2$	$x_0 = 0.5$	$x_0 = 1.0$	$x_0 = 1.5$
y_0	0.4000000000	1.0000000000	2.0000000000	3.0000000000
y_1	0.1138141813	0.2827202137	0.5591451924	0.8288366299
y_2	0.0324804548	0.0805423810	0.1588304694	0.2347627942
y_3	0.0092770515	0.0229931547	0.0453062682	0.0669133785
y_4	0.0026503351	0.0065679304	0.0129386562	0.0191050307
y_5	0.0007572181	0.0018764257	0.0036962706	0.0054575155
y_6	0.0002163463	0.0005361113	0.0010560375	0.0015592033
y_7	0.0000618131	0.0001531738	0.0003017217	0.0004454795
y_8	0.0000176608	0.0000437638	0.0000862059	0.0001272792
y_9	0.0000050459	0.0000125039	0.0000246302	0.0000363654
y_{10}	0.0000014417	0.0000035725	0.0000070372	0.0000103901
y_{11}	0.0000004119	0.0000010207	0.0000020106	0.0000029686
y_{12}	0.0000001176	0.0000002916	0.0000005744	0.0000008481
y_{13}	0.0000000336	0.0000000833	0.0000001641	0.0000002423
y_{14}	0.0000000096	0.0000000238	0.0000000468	0.0000000692
y_{15}	0.0000000027	0.0000000068	0.0000000133	0.0000000197
y_{16}	0.0000000007	0.0000000019	0.0000000038	0.0000000056
y_{17}	0.0000000002	0.0000000005	0.0000000010	0.0000000016
y_{18}	0.0000000000	0.0000000001	0.0000000003	0.0000000004
y_{19}	0.0000000000	0.0000000000	0.0000000000	0.0000000001
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

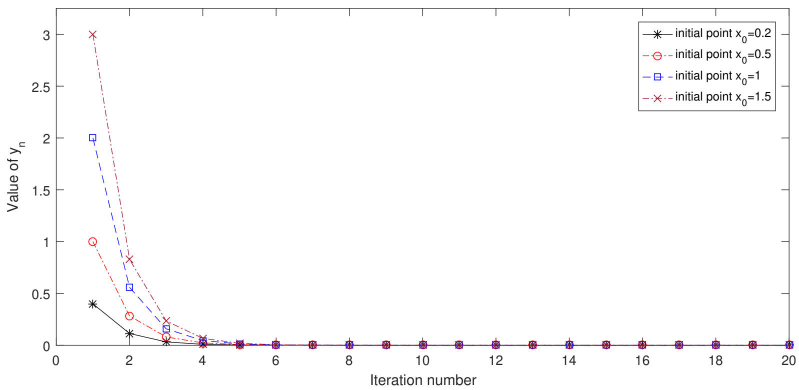

FIGURE 1. Behavior of iteration process

In the following results we see that several existing results in the literature can be obtained via the C_F-s-simulation function. We observe that the main result of [17, Theorem 4.4] can be easily deduced from Theorem 1.

Corollary 1. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$\zeta(d(Tx,Ty),d(x,y)) \geq 0$$

for all $x,y \in X$, where $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is an s-simulation function. Then T has a unique fixed point in X.

Proof. Take $F(v,u) = v-u$, $C_F = 0$ and $S = I$, where $I : X \to X$ is the identity mapping in Theorem 1. Then the results follows. □

The well-known Banach contraction principle in the framework of b-metric spaces [8, Theorem 3.3] can be deduced as follows:

Corollary 2. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq \lambda d(x,y)$$

for all $x,y \in X$, where $\lambda \in \left[0,\frac{1}{2}\right)$. Then T has a unique fixed point in X.

Proof. Define the mappings $\zeta_1, F : [0,\infty)\times[0,\infty) \to \mathbb{R}$ by $\zeta_1(u,v) = \lambda v - u$ and $F(v,u) = v-u$. Take $C_F = 0$ and S be the identity mapping on X then $\zeta_1 \in \mathscr{Z}_{F_s}$. The desired result follows by taking $\zeta = \zeta_1$ in Theorem 1. □

Corollary 3 (Rhoades Type)**.** *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y))$$

for all $x,y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function such that $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X.

Proof. Define the mappings $\zeta_2, F : [0,\infty)\times[0,\infty) \to \mathbb{R}$ by $\zeta_2(u,v) = v - \phi(v) - su$ and $F(v,u) = v-u$. Take $C_F = 0$ and S be the identity mapping on X then $\zeta_2 \in \mathscr{Z}_{F_s}$. Taking $\zeta = \zeta_2$ in Theorem 1 we get that T has a unique fixed point in X. □

Berinde [4] introduced the notion of b-comparison function as follows:

Definition 9. Let $s \geq 1$ be a given real number. A function $\psi : [0,\infty) \to [0,\infty)$ is called a b-comparison function if it satisfies

(i) ψ is monotonically increasing,

(ii) there exists $n_0 \in \mathbb{N}$, $\kappa \in (0,1)$ and a convergent series of non-negative terms $\sum_{n=1}^{\infty} a_n$ such that for all $n \geq n_0$ and $t > 0$ we have

$$s^{n+1}\psi^{n+1}(t) \leq \kappa s^n \psi^n(t) + a_n.$$

Remark 2 ([13]). If ψ is a b-comparison function, then $\psi(t) < t$ for all $t > 0$.

Păcurar in [13] obtained fixed point results of contraction mappings involving b-comparison function. We observe that the main result of [13, Theorem 4] can be inferred in the following way:

Corollary 4. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq \psi(d(x,y))$$

for all $x,y \in X$, where ψ is a b-comparison function. Then T has a unique fixed point in X.

Proof. Define the mappings $\zeta_3, F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_3(u,v) = \psi(v) - su$ and $F(v,u) = v - u$. Take $C_F = 0$ and S be the identity mapping on X then $\zeta_3 \in \mathscr{Z}_{F_s}$. Taking $\zeta = \zeta_3$ in Theorem 1 we get the desired result. □

The following theorem is a direct consequence of Theorem 1 and Proposition 1.

Theorem 2. *Let (X,d) be a b-metric space with coefficient $s \geq 1$, $T,S : X \to X$ be self mappings and T be a $(\mathscr{Z}_{F_s}, S)$-contraction. Assume that $T(X) \subseteq S(X)$ and atleast one of the following conditions hold:*
(i) $(T(X),d)$ or $(S(X),d)$ is complete,
(ii) (X,d) is complete, S is b-continuous and (T,S) is compatible,
(iii) (X,d) is complete, S is b-continuous and T and S are commuting.
Moreover, assume that T and S are weakly compatible. Then T and S have a unique common fixed point in X.

In the sequel, we generalize several known results in the context of b-metric spaces via C_F-s-simulation functions.

Theorem 3. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $T,S : X \to X$ be self mappings. Suppose that $\zeta \in \mathscr{Z}_{F_s}$ and satisfies*

$$\zeta\Big(d(Tx,Ty), \max\Big\{d(Sx,Sy), d(Tx,Sx), d(Ty,Sy), \frac{d(Tx,Sy)+d(Sx,Ty)}{2s}\Big\}\Big) \geq C_F \tag{2.8}$$

for all $x,y \in X$ with $Sx \neq Sy$. Assume that $T(X) \subseteq S(X)$ and atleast one of the following conditions hold:
(i) $(T(X),d)$ or $(S(X),d)$ is complete,

(ii) (X,d) *is complete,* S *is b-continuous and* (T,S) *is compatible,*
(iii) (X,d) *is complete,* S *is b-continuous and* T *and* S *are commuting.*
Then T *and* S *have a unique point of coincidence. Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point in* X.

Proof. Proceeding similar to Theorem 1 we get a Picard-Jungck sequence $\{x_n\}$ such that $y_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N} \cup \{0\}$. Putting $x = x_{n+1}$ and $y = x_{n+2}$ in (2.8) we get,

$$
\begin{aligned}
C_F &\leq \zeta\Big(d(y_{n+1}, y_{n+2}), \max\Big\{d(y_n, y_{n+1}), d(y_{n+2}, y_{n+1}), \frac{d(y_n, y_{n+2})}{2s}\Big\}\Big)\\
&< F\Big(\max\Big\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \frac{d(y_n, y_{n+2})}{2s}\Big\}, d(y_{n+1}, y_{n+2})\Big).
\end{aligned}
$$

Therefore, $d(y_{n+1}, y_{n+2}) < \max\Big\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \frac{d(y_n,y_{n+2})}{2s}\Big\}$.
If $d(y_{n+1}, y_{n+2}) < d(y_{n+1}, y_{n+2})$, a contradiction. If $d(y_{n+1}, y_{n+2}) < \frac{d(y_n,y_{n+2})}{2s} \leq \frac{d(y_n,y_{n+1})+(y_{n+1},y_{n+2})}{2}$, then $d(y_{n+1}, y_{n+2}) < d(y_n, y_{n+1})$. Following the lines in the proof of Theorem 1 and Theorem 2 we get the desired result. □

Corollary 5. *Let* (X,d) *be a b-metric space with coefficient* $s \geq 1$ *and* $T,S : X \to X$ *be self mappings satisfying*

$$
s^3 d(Tx,Ty) \leq k \max\Big\{d(Sx,Sy), d(Tx,Sx), d(Ty,Sy), \frac{d(Tx,Sy)+d(Sx,Ty)}{2s}\Big\}
$$

for all $x, y \in X$*, where* $k \in [0,1)$. *Assume that* $T(X) \subseteq S(X)$ *and atleast one of the following conditions hold:*
(i) $(T(X),d)$ *or* $(S(X),d)$ *is complete,*
(ii) (X,d) *is complete,* S *is b-continuous and* (T,S) *is compatible,*
(iii) (X,d) *is complete,* S *is b-continuous and* T *and* S *are commuting.*
Then T *and* S *have a unique point of coincidence. Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point in* X.

Proof. Define the mappings $\zeta_4, F : [0,\infty)\times[0,\infty) \to \mathbb{R}$ by $\zeta_4(u,v) = kv - s^3 u$ and $F(v,u) = v - u$. Take $C_F = 0$ then $\zeta_4 \in \mathscr{Z}_{F_s}$. By taking $\zeta = \zeta_4$ in Theorem 3 we get the result. □

Indeed, a result of Yamaod and Sintunavarat [16, Corollary 3.6] can be obtained by considering S to be the identity mapping on X in the above result.

The following theorem can be proved on the similar lines of Theorem 1.

Theorem 4. *Let* (X,d) *be a b-metric space with coefficient* $s \geq 1$, $T,S : X \to X$ *be self mappings and* $G : [0,\infty) \to [0,\infty)$ *be a mapping satisfying* $G(0) = 0$ *and* $0 < G(t) \leq t$ *for all* $t > 0$. *Suppose that* $\zeta \in \mathscr{Z}_{F_s}$ *and satisfies*

$$
\zeta(d(Tx,Ty), G(d(Sx,Sy))) \geq C_F
$$

for all $x, y \in X$ with $Sx \neq Sy$. Assume that $T(X) \subseteq S(X)$ and at least one of the following conditions hold:

(i) $(T(X), d)$ or $(S(X), d)$ is complete,

(ii) (X, d) is complete, S is b-continuous and (T, S) is compatible,

(iii) (X, d) is complete, S is b-continuous and T and S are commuting.

Then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Theorem 5. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $T, S : X \to X$ be self mappings. Suppose that $\zeta \in \mathscr{Z}_{F_s}$ and satisfies*

$$\begin{aligned}\zeta(d(Tx, Ty), \lambda \max\{&d(Sx, Sy), d(Tx, Sx), d(Ty, Sy),\\ &d(Tx, Sy), d(Ty, Sx)\}) \geq C_F\end{aligned} \tag{2.9}$$

for all $x, y \in X$ with $Sx \neq Sy$ and $\lambda \in \left(0, \frac{1}{2s}\right)$. Assume that $T(X) \subseteq S(X)$ and at least one of the following conditions hold:

(i) $(T(X), d)$ or $(S(X), d)$ is complete,

(ii) (X, d) is complete, S is b-continuous and (T, S) is compatible,

(iii) (X, d) is complete, S is b-continuous and T and S are commuting.

Then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof. Following the lines in the proof of Theorem 1 we get a Picard-Jungck sequence $\{x_n\}$ such that $y_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N} \cup \{0\}$. Putting $x = x_{n+1}$ and $y = x_{n+2}$ in (2.9) we get,

$$\begin{aligned}C_F &\leq \zeta(d(y_{n+1}, y_{n+2}), \lambda \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\})\\ &< F(\lambda \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\}, d(y_{n+1}, y_{n+2})).\end{aligned}$$

Therefore, $d(y_{n+1}, y_{n+2}) < \lambda \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\}$. If $d(y_{n+1}, y_{n+2}) < \lambda d(y_{n+1}, y_{n+2}) < d(y_{n+1}, y_{n+2})$, a contradiction.
If $d(y_{n+1}, y_{n+2}) < \lambda d(y_n, y_{n+2}) \leq \lambda s d(y_n, y_{n+1}) + \lambda s d(y_{n+1}, y_{n+2})$. Then $d(y_{n+1}, y_{n+2}) \leq \frac{\lambda s}{1-\lambda s} d(y_n, y_{n+1}) < d(y_n, y_{n+1})$. Proceeding as in the proof of Theorem 1 we establish the existence of coincidence point of T and S. Let w_1 and w_2 be two distinct point of coincidence of T and S. Then there exists z_1, $z_2 \in X$ such that $w_1 = Tz_1 = Sz_1$ and $w_2 = Tz_2 = Sz_2$. We have

$$\begin{aligned}C_F &\leq \zeta(d(Tz_1, Tz_2), \lambda \max\{d(Sz_1, Sz_2),\\ &\qquad d(Tz_1, Sz_1), d(Tz_2, Sz_2), d(Tz_1, Sz_2), d(Tz_2, Sz_1)\})\\ &= \zeta(d(w_1, w_2), \lambda d(w_1, w_2)).\end{aligned}$$

Therefore, $d(w_1, w_2) < d(w_1, w_2)$, a contradiction. Hence, T and S have a unique point of coincidence. By Theorem 2, it follows that T and S have a unique common fixed point. □

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