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NEW BOUNDS FOR HERMITE-HADAMARD'S TRAPEZOID AND MID-POINT TYPE INEQUALITIES VIA FRACTIONAL INTEGRALS

M. ROSTAMIAN DELAVAR

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Abstract. Some trapezoid and mid-point type inequalities with new bounds for Hermite-Hadamard inequality related to Riemann-Liouville integrals of order $\alpha > 0$ are obtained. Also a refinement of Hermite-Hadamard inequality for nonnegative monotone convex functions is presented. Furthermore some applications in connection with special means are given.

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1. INTRODUCTION

The following inequality is known in literature as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f:[a;b]\to\mathbb{R}$ is convex on $[a,b]$. For historical information about inequality (1.1), see [6].
In inequality (1.1), we may deal with two issues:

(i) Estimation of the difference between left and middle terms which we call it *trapezoid type estimation*

(ii) Estimation of the difference between right and middle terms which we call it *mid-point (rectangle) type estimation*.

At first S. S. Dragomir et al, in [3], obtained the trapezoid type inequality related to (1.1) as well:

Theorem 1. *Let $f: I^\circ \subseteq \mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{8}\Big(|f'(a)|+|f'(b)|\Big). \tag{1.2}$$

The striped area shown in Figure 1, is equivalent to the difference between the area of trapezoid $abcd$ and the area under the graph of f which is estimated by $\frac{(b-a)^2}{8}\big(|f'(a)|+|f'(b)|\big)$. Also U. S. Kirmaci in [5], obtained the mid-point type

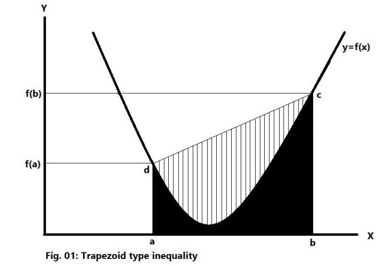

FIGURE 1.

inequality related to (1.1) as the following:

Theorem 2. *Consider I^* as the interior of interval $I \subset \mathbb{R}$. Let $f : I^* \to \mathbb{R}$ be a differentiable mapping on I^*, $a,b \in I^*$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\int_a^b f(x)dx-(b-a)f\Big(\frac{a+b}{2}\Big)\right| \le \frac{(b-a)^2}{8}\Big(|f'(a)|+|f'(b)|\Big). \tag{1.3}$$

In Figure 2, it is shown that the difference between the area under the graph of f and the area of rectangle $abcd$ can be estimated by $\frac{(b-a)^2}{8}\big(|f'(a)|+|f'(b)|\big)$. Recently In [8], the authors obtained Hermite-Hadamard's inequality related to frac-

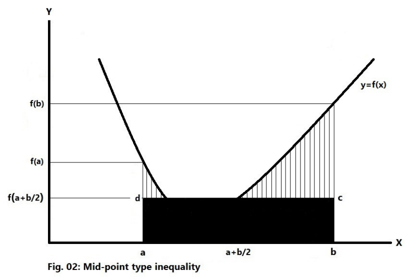

FIGURE 2.

tional integrals as the following:

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be convex function with $0 \le a < b$ and $f \in L[a,b]$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds.*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\big] \le \frac{f(a)+f(b)}{2}, \tag{1.4}$$

where $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *are the Riemann-Liouville integrals of order* $\alpha > 0$ *defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

such that

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt,$$

is Gamma function and $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

The trapezoid and mid-point type inequalities related to (1.4) have been obtained in [8] and [4] respectively.

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$*. If* $|f'|$ *is convex function on* $[a,b]$*, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| \leq \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\Big[|f'(a)|+|f'(b)|\Big]. \tag{1.5}$$

Theorem 5. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$*. If* $|f'|$ *is convex function on* $[a,b]$*, then the following inequality for Riemann-Liouville fractional integrals holds for* $0 < \alpha \leq 1$*:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+1}(\alpha+1)}\Big[|f'(a)|+|f'(b)|\Big]. \tag{1.6}$$

On the other hand in [7], we can find two results related to the convexity of a function as the following:

(a) Any convex function defined on a closed interval $[a,b]$ is bounded.

(b) If a real valued function defined on the interval I is convex, then it satisfies a Lipschitz condition on any closed interval $[a,b]$ (there is a constant K so that for any two points $x,y \in [a,b]$, $|f(x)-f(y)| \leq K|x-y|$) contained in the interior I° of I.

Motivated by above works and results, we obtain some trapezoid and mid-point type inequalities related to (1.4) where the convexity condition for the absolute value of the derivative of considered function is replaced by boundedness and a Lipschitzian condition for the derivative. In fact we obtain new bounds for the left side of inequalities (1.5) and (1.6) which give some refinements for these inequalities. Also by the use of fractional integrals we present a refinement of Hermite-Hadamard inequality for nonnegative monotone convex functions. Finally we give some applications of our results in connection with special means.

2. TRAPEZOID AND MID-POINT TYPE INEQUALITIES

In this section, we obtain some Hermite-Hadamard's trapezoid and mid-point type inequalities via fractional integrals where the derivative of considered function is bounded and satisfies a Lipschitz condition. The following result has been obtained in [8] and we use it to obtain trapezoid type inequalities.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integral holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]$$
$$=\frac{b-a}{2}\int_0^1\left[t^{\alpha}-(1-t)^{\alpha}\right]f'\big(ta+(1-t)b\big)dt.$$

In the following theorem we consider that the derivative of considered function is bounded.

Theorem 6. *Suppose that $f:I\to\mathbb{R}$ is a differentiable function on I°. Consider $a,b\in I^{\circ}$ with $a<b$ such that $f'\in L[a,b]$. If there exist constants $l<L$ such that $-\infty<l\le f'(x)\le L<\infty \quad for\ all\ x\in[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\Big]\right| \tag{2.1}$$
$$\le\frac{(b-a)(L-l)}{2(\alpha+1)}\Big(1-\frac{1}{2^{\alpha}}\Big).$$

Proof. From Lemma 1 we have

$$J=\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\right|$$
$$=\frac{b-a}{2}\int_0^1 h(t)\Big[f'\big(ta+(1-t)b\big)-\frac{l+L}{2}+\frac{l+L}{2}\Big]dt,$$

where $h(t)=t^{\alpha}-(1-t)^{\alpha}$, for all $t\in[0,1]$. Since

$$\int_0^1 h(t)dt=0,$$

$$\int_0^1|h(t)|dt=\frac{2}{\alpha+1}(1-\frac{1}{2^{\alpha}}),$$

and

$$\Big|f'\big(ta+(1-t)b\big)-\frac{l+L}{2}\Big|\leq\frac{L-l}{2},$$

then

$$|J|\leq\frac{b-a}{2}\int_0^1|h(t)|\Big|f'\big(ta+(1-t)b\big)-\frac{l+L}{2}\Big|dt$$
$$\leq\frac{(b-a)(L-l)}{4}\int_0^1|h(t)|dt=\frac{(b-a)(L-l)}{2(\alpha+1)}(1-\frac{1}{2^\alpha}).$$

□

Remark 1. If $|f'|$ is convex on $[a,b]$, then there exist l,L with

$$0\leq l=2\Big|f'(\frac{a+b}{2})\Big|-L\leq|f'(x)|\leq L=\max\{|f'(a)|,|f'(b)|\}<\infty$$

for all $x\in[a,b]$ which implies that $L-l\leq|f'(a)|+|f'(b)|$, since

$$\begin{cases}|f'(b)|>|f'(a)|\to L=|f'(b)|\ and\ -l\leq l\leq|f'(a)|\to L-l\leq|f'(a)|+|f'(b)|,\\ |f'(a)|>|f'(b)|\to L=|f'(a)|\ and\ -l\leq l\leq|f'(b)|\to L-l\leq|f'(a)|+|f'(b)|.\end{cases}$$

So (2.1) gives a refinement for (1.5).

In the following theorem we consider that the derivative of considered function satisfies a Lipschitz condition.

Theorem 7. *Suppose that $f:I\to\mathbb{R}$ is a differentiable function on I°. Consider $a,b\in I^\circ$ with $a<b$ such that $f'\in L[a,b]$ and satisfies a Lipschitz condition for some $K>0$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\Big]\right|$$
$$\leq\frac{K\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}.$$

Proof. From Lemma 1 we have

$$J=\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\big]\right|$$
$$=\frac{b-a}{2}\int_0^1h(t)\Big[f'\big(ta+(1-t)b\big)-f'\Big(\frac{a+b}{2}\Big)+f'\Big(\frac{a+b}{2}\Big)\Big]dt$$
$$=\frac{b-a}{2}\int_0^1h(t)\Big[f'\big(ta+(1-t)b\big)-f'\Big(\frac{a+b}{2}\Big)\Big]dt,$$

where $h(t)=t^\alpha-(1-t)^\alpha$, for all $t\in[0,1]$ and

$$\int_0^1h(t)f'\Big(\frac{a+b}{2}\Big)dt=0.$$

Since f' satisfies a Lipschitz condition for some $K > 0$, then

$$|J| \le \frac{K(b-a)^2}{2}\int_0^1 |h(t)||t-\frac{1}{2}|dt.$$

The definition of $h(t)$ implies that

$$|J| \le \frac{K(b-a)^2}{2}\int_0^1 \big(t^\alpha-(1-t)^\alpha\big)(t-\frac{1}{2})dt = \frac{K\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}.$$

□

The following lemma has been proved in [4] and we use it to obtain mid-point type inequalities.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following identity for Riemann-Liouville fractional integrals holds:*

$$f\Big(\frac{a+b}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+}f\big(b)+J^\alpha_{b-}f(a)\Big]=\frac{b-a}{2}\sum_{k=1}^{4}I_k,$$

where

$$I_1=\int_0^{\frac{1}{2}} t^\alpha f'\big(tb+(1-t)a\big)dt, \qquad I_2=\int_0^{\frac{1}{2}}(-t^\alpha)f'\big(ta+(1-t)b\big)dt,$$

$$I_3=\int_{\frac{1}{2}}^1 (t^\alpha-1)f'\big(tb+(1-t)a\big)dt, \qquad I_4=\int_{\frac{1}{2}}^1(1-t^\alpha)f'\big(ta+(1-t)b\big)dt.$$

If we consider the boundedness of the derivative of considered function we get the following mid-point type inequality.

Theorem 8. *Suppose that $f:I\to\mathbb{R}$ is a differentiable function on I°. Consider $a,b\in I^\circ$ with $a<b$ such that $f'\in L[a,b]$. If there exist constants $l<L$ such that $-\infty<l\le f'(x)\le L<\infty \quad for\ all\ x\in[a,b]$, then*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+}f\big(b)+J^\alpha_{b-}f(a)\Big]\right|\le\frac{M(b-a)}{2(\alpha+1)}\Big[\alpha-1+\big(\frac{1}{2}\big)^{\alpha-1}\Big],$$

where $M=\max\{-l,L\}$.

Proof. If we consider

$$I_1=\int_0^{\frac{1}{2}} t^\alpha\Big[f'\big(ta+(1-t)b\big)-\frac{l+L}{2}+\frac{l+L}{2}\Big]dt,$$

then

$$\left|I_1-\frac{l+L}{2}\int_0^{\frac{1}{2}}t^\alpha\right|\le\int_0^{\frac{1}{2}}t^\alpha\left|f'\big(ta+(1-t)b\big)-\frac{l+L}{2}\right|dt$$

$$\leq \frac{L-l}{2}\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big),$$

which implies that

$$\big|I_1\big| \leq \Big[\frac{L-l}{2}+\Big|\frac{l+L}{2}\Big|\Big]\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big).$$

Moreover it is not hard to see that

$$M=\max\{-l,L\}=\Big[\frac{L-l}{2}+\Big|\frac{l+L}{2}\Big|\Big].$$

So

$$\big|I_1\big| \leq M\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big).$$

Similarly we can obtain that

$$\big|I_2\big| \leq M\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big).$$

It follows that

$$\big|I_3\big| \leq M\Big[\frac{1}{2}-\frac{1}{\alpha+1}\Big(1-(\frac{1}{2})^{\alpha+1}\Big)\Big],$$

and

$$\big|I_4\big| \leq M\Big[\frac{1}{2}-\frac{1}{\alpha+1}\Big(1-(\frac{1}{2})^{\alpha+1}\Big)\Big],$$

Now by adding all of above inequalities we get

$$\sum_{i=1}^{4}|I_i| \leq \frac{M}{\alpha+1}\Big(\alpha-1+(\frac{1}{2})^{\alpha-1}\Big),$$

which implies the desired result. □

Remark 2. In proof of Theorem 8 if we consider $0<\alpha\leq 1$, then by the use of the fact that

$$|t_1^\alpha - t_2^\alpha| \leq |t_1-t_2|^\alpha,$$

for any $t_1,t_2\in[0,1]$, we obtain that

$$\Big|f\Big(\frac{a+b}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\Big]\Big| \leq \frac{M(b-a)}{2^\alpha(\alpha+1)}. \tag{2.2}$$

where $M=\max\{-l,L\}$. Furthermore in the case that

$$2M \leq |f'(a)|+|f'(b)|,$$

inequality (2.2) gives a refinement for (1.6).

If the derivative of considered function satisfies a Lipschitz condition, then the following inequality holds.

Theorem 9. *Suppose that $f : I \to \mathbb{R}$ is a differentiable function on I°. Consider $a,b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$ and satisfies a Lipschitz condition for some $K > 0$. Then*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[J^\alpha_{a+}f\big(b\big)+J^\alpha_{b-}f(a)\Big]\right| \leq \frac{b-a}{2^\alpha(\alpha+1)}\left[\frac{K(b-a)}{\alpha+2}+\left|f'\Big(\frac{a+b}{2}\Big)\right|\right],$$

for $0 < \alpha \leq 1$.

Proof. With some calculations similar to the proof of Theorem 7 we deduce that $|I_1| = |I_2| = |I_3| = |I_4| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\Big[\frac{K(b-a)}{\alpha+2}+\big|f'(\frac{a+b}{2})\big|\Big]$, which along with Lemma 2 imply the result. Note that in $|I_3|$ and $|I_4|$ we used the fact that

$$|t_1^\alpha - t_2^\alpha| \leq |t_1-t_2|^\alpha,$$

for any $t_1,t_2 \in [0,1]$ and $0 < \alpha \leq 1$. □

Corollary 1. *If we consider $\alpha = 1$ in*
(i) Theorem 6, then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)(L-l)}{8},$$

If there exist constants $l < L$ such that $-\infty < l \leq f'(x) \leq L < \infty \quad for\ all\ x \in [a,b]$.
(ii) Theorem 7, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{K(b-a)^2}{12},$$

if f' satisfies a Lipschitz condition for some $K > 0$ on $[a,b]$. Comparing this inequality with inequality (2.6) in [2], shows that the existence of a Lipschitz condition for f' gives a better estimation rather than the existence of a Lipschitz condition for f.
(iii) Theorem 8, then we have

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{M(b-a)}{4},$$

if there exist constants $l < L$ such that $-\infty < l \leq f'(x) \leq L < \infty \quad for\ all\ x \in [a,b]$ and $M = \max\{-l, L\}$.
(iv) Theorem 9, then we deduce that

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{K(b-a)^2}{12}+\frac{1}{4}\left|f'\Big(\frac{a+b}{2}\Big)\right|.$$

if f' satisfies a Lipschitz condition for some $K > 0$ on $[a,b]$.

3. Refinement of Hermite-Hadamard inequality

In this section we give a refinement of Hermite-Hadamard inequality and some new inequalities in connection with fractional integrals related to the nonnegative monotone convex functions.

Theorem 10. *Let $f : [a,b] \to \mathbb{R}$ be a convex function.*
(i) For $\alpha \geq 1$, the following refinement of Hermite-Hadamard inequality holds if f is nonnegative and increasing.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3.1}$$

(ii) For any $\alpha > 0$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \leq \frac{\alpha}{b-a}\int_a^b f(x)dx. \tag{3.2}$$

(iii) If $\alpha \geq 1$ and f is nonnegative and increasing, then

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]dx \leq \frac{\alpha}{b-a}\int_a^b f(x)dx. \tag{3.3}$$

Proof. From convexity of f we have

$$f\big(tb+(1-t)a\big) \leq tf(b) + (1-t)f(a),$$

and

$$f\big(ta+(1-t)b\big) \leq tf(a) + (1-t)f(b).$$

By adding these inequalities we get

$$f\big(tb+(1-t)a\big) + f\big(ta+(1-t)b\big) \leq f(a) + f(b). \tag{3.4}$$

Multiplying both sides of (3.4) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$ and using Theorem 1 in [1], we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}dt \cdot \int_0^1 f\big(tb+(1-t)a\big)dt + \int_0^1 t^{\alpha-1}dt \cdot \int_0^1 f\big(ta+(1-t)b\big)dt \\
&\leq \int_0^1 t^{\alpha-1} f\big(tb+(1-t)a\big)dt + \int_0^1 t^{\alpha-1} f\big(ta+(1-t)b\big)dt \\
&\leq \big[f(a)+f(b)\big]\int_0^1 t^{\alpha-1}dt.
\end{aligned}$$

Using the changes of variable $x = tb + (1-t)a$ and $x = ta + (1-t)b$, respectively, in above integrals we have

$$\frac{1}{b-a}\int_0^1 t^{\alpha-1}dt.\left[\int_a^b f(x)dx + \int_a^b f(a+b-x)dx\right]$$
$$\leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b)-}^{\alpha} f(a)\right] \leq [f(a)+f(b)]\int_0^1 t^{\alpha-1}dt.$$

Since

$$\int_0^1 t^{\alpha-1}dt = \frac{1}{\alpha},$$

and

$$\int_a^b f(x)dx = \int_a^b f\big(a+b-x\big)dx,$$

then we obtain

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3.5}$$

On the other hand since f is convex, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx,$$

which along with (3.5), implies the inequality (3.1).
To obtain inequality (3.2), we consider that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\Big[f\big(ta+(1-t)b\big) + f\big(tb+(1-t)a\big)\Big].$$

Multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$ we get

$$\frac{2}{\alpha}f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]$$
$$\leq \int_0^1 f\big(ta+(1-t)b\big)dt + \int_0^1 f\big(tb+(1-t)a\big)dt$$

which implies the inequality (3.2). Finally inequality (3.3) comes from (3.1) and (3.2). □

Remark 3. If f and g are nonnegative decreasing functions defined on $[0,1]$ and B is an upper bound for them, then $B-f$ and $B-g$ are nonnegative increasing functions and so

$$\int_0^1 (B-f(x))dx \int_0^1 (B-g(x))dx \leq \int_0^1 (B-f(x))(B-g(x))dx,$$

which gives again

$$\int_0^1 f(x)dx \int_0^1 g(x)dx \le \int_0^1 f(x)g(x)dx.$$

This implies that inequalities (3.1) and (3.3) of Theorem 10 can be obtained if $f : [a,b] \to \mathbb{R}$ be a decreasing nonnegative convex function and $0 < \alpha \le 1$.

4. Application to special means

The following means for real numbers $a,b \in \mathbb{R}$ are known:

$$A(a,b) = \frac{a+b}{2} \qquad arithmetic\ mean,$$

$$L_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} \qquad generalized\ log-mean,\ n \in \mathbb{N},\ a<b.$$

Consider $f(x) = x^n$ for $x \ge 0$, $n \in \mathbb{N}$. If $x \in [a,b]$,

$$l = na^{n-1} \le f'(x) = nx^{n-1} \le nb^{n-1} = L.$$

So from Theorem 6, we obtain

$$\left|\frac{a^n+b^n}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \le \frac{n(b-a)(b^{n-1}-a^{n-1})}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right),$$

where

$$J_{a+}^\alpha f(b) = \int_a^b (b-t)^{\alpha-1} t^n dt = \sum_{k=0}^n \frac{a^{n-k}(b-a)^{\alpha+k} P(n,k)}{\prod_{i=0}^k (\alpha+i)},$$

$$J_{b-}^\alpha f(a) = \int_a^b (t-a)^{\alpha-1} t^n dt = \sum_{k=0}^n \frac{(-1)^k b^{n-k}(b-a)^{\alpha+k} P(n,k)}{\prod_{i=0}^k (\alpha+i)},$$

and

$$P(n,k) = \frac{n!}{(n-k)!},$$

which is the number of possible permutations of k objects from a set of n.
In special case if we consider $\alpha = 1$, then we have

$$J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) = \frac{2(b^{n+1}-a^{n+1})}{n+1}.$$

So

$$\left|\frac{a^n+b^n}{2}-\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right| \le \frac{n(b-a)(b^{n-1}-a^{n-1})}{8},$$

or equivalently

$$\left|A(a^n,b^n)-L_n^n(a,b)\right| \le \frac{n(b-a)(b^{n-1}-a^{n-1})}{8} \tag{4.1}$$

$$\le \frac{n(b-a)(b^{n-1}+a^{n-1})}{8} = \frac{n(b-a)A(a^{n-1},b^{n-1})}{4}.$$

Inequality (4.1) gives a refinement for inequality (1.5) (in the case that $\alpha = 1$ and $f(x) = x^n$) that turns to the inequality obtained in Proposition 3.1 in [3], where $0 \le a < b$, $n \in \mathbb{N}$ and $n \ge 2$.
It follows that $f'(x) = nx^{n-1}$, satisfies a Lipschitz condition for

$$K = \sup_{x\in[a,b]} n(n-1)x^{n-2} = n(n-1)b^{n-2}.$$

So from Theorem 7 we have

$$\left|A(a^n,b^n)-L_n^n(a,b)\right| \le \frac{n(n-1)b^{n-2}(b-a)^2}{12}.$$

At last, Theorem 8 and 9 imply the following inequalities.

$$\left|f\big(A(a,b)\big)-L_n^n(a,b)\right| \le \frac{nb^{n-1}(b-a)}{4},$$

and

$$\left|f\big(A(a,b)\big)-L_n^n(a,b)\right| \le \frac{n(n-1)b^{n-1}(b-a)^2}{12}+\frac{n}{4}A^{n-1}(a,b).$$

REFERENCES

[1] R. Ahlswede and D. E. Daykin, "Integrals inequalities for increasing functions," vol. 86, no. 3, pp. 391–394, 1979, doi: https://doi.org/10.1017/S030500410005622X.

[2] S. S. Dragomir, "The Ostrowski's Integral Inequality for Lipschitzian Mappings and Applications," *Comput. Math. Appl.*, vol. 38, pp. 33–37, 1999, doi: https://doi.org/10.1016/S0898-1221(99)00282-5.

[3] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Appl. Math. Lett.*, vol. 11, pp. 91–95, 1998, doi: https://doi.org/10.1016/S0893-9659(98)00086-X.

[4] M. Iqbal, M. I. Bhatti, and K. Nazeer, "Generalization of inequalities analogous to hermite-hadamard inequality via fractional integrals," *Bull. Korean Math. Soc.*, vol. 25, no. 3, pp. 707–716, 2015, doi: http://dx.doi.org/10.4134/BKMS.2015.52.3.707.

[5] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Appl. Math. Comp.*, vol. 147, no. 1, pp. 137–146, 2004, doi: https://doi.org/10.1016/S0096-3003(02)00657-4.

[6] D. S. Mitrinović and I. B. Lacković, "Hermite and convexity," *Aequationes .Math.*, vol. 28, pp. 229–232, 1985, doi: https://doi.org/10.1007/BF02189414.
[7] A. W. Robert and D. E. Varbeg, *Convex Functions*. New York and London: Academic Press, 1973.
[8] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-hadamard's inequalities for fractional integrals and related fractional inequalities," *Math. Comput. Model.*, vol. 57, pp. 2403–2407, 2013, doi: https://doi.org/10.1016/j.mcm.2011.12.048.

Author's address

M. Rostamian Delavar
Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P. O. Box 1339, Bojnord 94531, Iran
E-mail address: m.rostamian@ub.ac.ir