



CERTAIN RESULTS ON KENMOTSU PSEUDO-METRIC MANIFOLDS

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Abstract. In this paper, a systematic study of Kenmotsu pseudo-metric manifolds are introduced. After studying the properties of this manifolds, we provide necessary and sufficient condition for Kenmotsu pseudo-metric manifold to have constant φ -sectional curvature, and prove the structure theorem for ξ -conformally flat and φ -conformally flat Kenmotsu pseudo-metric manifolds. Next, we consider Ricci solitons on this manifolds. In particular, we prove that an η -Einstein Kenmotsu pseudo-metric manifold of dimension higher than 3 admitting a Ricci soliton is Einstein, and a Kenmotsu pseudo-metric 3-manifold admitting a Ricci soliton is of constant curvature $-\varepsilon$.

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1. INTRODUCTION

The study of contact metric manifolds with associated pseudo-Riemannian metrics were first started by Takahashi [10] in 1969. Since then, such structures were studied by several authors mainly focusing on the special case of Sasakian manifolds. The case of contact Lorentzian structures (η, g) , where η is a contact one-form and g is a Lorentzian metric associated to it, has a particular relevance for physics and was considered in [4] and [1]. A systematic study of almost contact semi-Riemannian manifolds was undertaken by Calvaruso and Perrone [3] in 2010, introducing all the technical apparatus which is needed for further investigations.

On the other hand, in 1972 Kenmotsu [9] investigated a class of contact metric manifolds satisfying some special conditions, and after onwards such manifolds are came to known as Kenmotsu manifolds. Recently, Wang and Liu [11] investigated almost Kenmotsu manifolds with associated pseudo-Riemannian metric. These are called almost Kenmotsu pseudo-metric manifolds. In this paper, we undertake the systematic study of Kenmotsu pseudo-metric manifolds.

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The present paper is organized as follows: In Section 2, we give the basics of Kenmotsu pseudo-metric manifold (M, g) . Certain properties of Kenmotsu pseudo-metric manifolds are provided in Section 3. We devote Section 4 to the study of curvature properties of Kenmotsu pseudo-metric manifold (M, g) and gave necessary and sufficient condition for (M, g) to have constant φ -sectional curvature. In Section 5, we prove necessary and sufficient condition for Kenmotsu pseudo-metric manifold to be ξ -conformally flat (and φ -conformally flat). The last section is focused on the study of Kenmotsu pseudo-metric manifolds whose metric is a Ricci soliton. We show that if (M, g) is a Kenmotsu pseudo-metric manifold admitting a Ricci soliton, then the soliton constant $\lambda = 2n\varepsilon$, where $\varepsilon = \pm 1$. Moreover, we show that if M is an η -Einstein manifold of dimension higher than 3 admitting Ricci soliton, then M is Einstein. Further we show that, a Kenmotsu pseudo-metric manifold (M, g) of dimension 3 admitting Ricci soliton is of constant curvature $-\varepsilon$, where $\varepsilon = \pm 1$. Finally, an illustrative example is constructed which verifies our results.

2. PRELIMINARIES

Let M be a $(2n + 1)$ dimensional smooth manifold. We say that M has an *almost contact structure* if there is a tensor field φ of type $(1, 1)$, a vector field ξ (called the *characteristic vector field* or *Reeb vector field*), and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \quad (2.1)$$

If M with (φ, ξ, η) -structure is endowed with a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \quad (2.2)$$

where $\varepsilon = \pm 1$, for all $X, Y \in TM$, then M is called an *almost contact pseudo-metric manifold*. The relation (2.2) is equivalent to

$$\eta(X) = \varepsilon g(X, \xi) \text{ along with } g(\varphi X, Y) = -g(X, \varphi Y). \quad (2.3)$$

In particular, in an almost contact pseudo-metric manifold, it follows that $g(\xi, \xi) = \varepsilon$ and so, the characteristic vector field ξ is a unit vector field, which is either space-like or time-like, but cannot be light-like.

The *fundamental 2-form* of an almost contact pseudo-metric manifold is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

which satisfies $\eta \wedge \Phi^n \neq 0$. An almost contact pseudo-metric manifold is said to be a *contact pseudo-metric manifold* if $d\eta = \Phi$. The Riemannian curvature tensor R is given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, which is opposite to the one used in [3]. The Ricci operator Q is determined by

$$Ric(X, Y) = g(QX, Y),$$

where Ric denotes the Ricci tensor. In an almost contact pseudo-metric manifold there always exists a special kind of local pseudo-orthonormal basis $\{e_i, \varphi e_i, \xi\}_{i=1}^n$, called a local pseudo φ -basis.

Consider the manifold $M \times \mathbb{R}$, where M is an almost contact pseudo-metric manifold. Denoting the vector field on $M \times \mathbb{R}$ by $(X, f \frac{d}{dt})$, where $X \in TM, t \in \mathbb{R}$, and f is a smooth function $M \times \mathbb{R}$, we define the structure J on $M \times \mathbb{R}$ by

$$J \left(X, f \frac{d}{dt} \right) = \left(\varphi X - f \xi, \eta(X) \frac{d}{dt} \right),$$

which defines an almost complex structure. If J is integrable, we say that the almost contact pseudo-metric structure (φ, ξ, η) is normal. Necessary and sufficient condition for integrability of J is [3]

$$[\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0.$$

The following can be easily obtained.

Proposition 1. *An almost contact pseudo-metric manifold is normal if and only if*

$$(\nabla_{\varphi X} \varphi)Y - \varphi(\nabla_X \varphi)Y + (\nabla_X \eta)(Y)\xi = 0, \tag{2.4}$$

where ∇ is the Levi-Civita connection.

An almost Kenmotsu pseudo-metric manifold is an almost contact pseudo-metric manifold with $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. A normal almost Kenmotsu pseudo-metric manifold is called a Kenmotsu pseudo-metric manifold [11]. Equivalently, from (2.4) we have the following:

Definition 1. Almost contact pseudo-metric manifold is said to be *Kenmotsu pseudo-metric manifold* if

$$(\nabla_X \varphi)Y = -\eta(Y)\varphi X - \varepsilon g(X, \varphi Y)\xi. \tag{2.5}$$

From (2.5), we see

$$\nabla \xi = I - \eta \otimes \xi. \tag{2.6}$$

A straight forward calculation gives the following:

Proposition 2. *On Kenmotsu pseudo-metric manifold (M, g) , we have*

$$(\nabla_X \eta)Y = \varepsilon g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$\mathcal{L}_\xi g = 2g - \varepsilon \eta \otimes \eta, \tag{2.8}$$

$$\mathcal{L}_\xi \varphi = 0, \tag{2.9}$$

$$\mathcal{L}_\xi \eta = 0, \tag{2.10}$$

where \mathcal{L} denotes the Lie derivative.

3. SOME PROPERTIES OF KENMOTSU PSEUDO METRIC MANIFOLDS

For $X \in \text{Ker } \eta$, either space-like or time-like, the ξ -sectional curvature $K(\xi, X)$ and φ -sectional curvature $K(X, \varphi X)$ are defined respectively as

$$K(\xi, X) = \frac{g(R(\xi, X)X, \xi)}{\varepsilon g(X, X)},$$

$$K(X, \varphi X) = \frac{g(R(\varphi X, X)X, \varphi X)}{g(X, X)^2}.$$

Now we prove:

Proposition 3. *If (M, g) is a Kenmotsu pseudo-metric manifold, then we have*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (3.1)$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \quad (3.2)$$

$$R(X, \xi)Y = \varepsilon g(X, Y)\xi - \eta(Y)X, \quad (3.3)$$

$$\text{Ric}(X, \xi) = -2n\eta(X) \quad (\Rightarrow Q\xi = -2n\varepsilon\xi), \quad (3.4)$$

$$K(\xi, \cdot) = -\varepsilon, \quad (3.5)$$

$$(\nabla_Z R)(X, Y, \xi) = \varepsilon\{g(X, Z)Y - g(Y, Z)X\} - R(X, Y)Z. \quad (3.6)$$

Proof. Equations (2.6) and (2.7) give (3.1). Equations (3.2), (3.3), (3.4) and (3.5) are consequences of (3.1). Equation (3.6) follows from (2.6), (2.7) and (3.1). \square

Definition 2. An almost contact pseudo-metric manifold for which

$$\varphi^2(\nabla_W R)(X, Y, Z) = 0,$$

for all $X, Y, Z, W \in TM$ is said to be *globally φ -symmetric*.

Using (3.2) and (3.6), we have the following:

Corollary 1. *A globally φ -symmetric Kenmotsu pseudo-metric manifold is of constant curvature $-\varepsilon$.*

A Kenmotsu pseudo-metric manifold M is said to be η -Einstein if the Ricci tensor satisfies

$$\text{Ric}(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \quad (3.7)$$

where a and b are certain smooth functions on M . If $b = 0$, then M is called an *Einstein* manifold.

From (3.4), we have

$$\varepsilon a + b = -2n. \quad (3.8)$$

Contracting (3.7) and using (3.8), we get

$$a = \left(\frac{r}{2n} + \varepsilon\right), \quad b = -\left(\frac{\varepsilon r}{2n} + 2n + 1\right), \quad (3.9)$$

where r is the scalar curvature. Thus, we have:

Proposition 4. *A Kenmotsu pseudo-metric manifold (M, g) is η -Einstein if and only if*

$$Ric(X, Y) = \left(\frac{r}{2n} + \varepsilon\right)g(X, Y) - \left(\frac{\varepsilon r}{2n} + 2n + 1\right)\eta(X)\eta(Y). \quad (3.10)$$

In particular, we have the following:

Corollary 2. *A Kenmotsu pseudo-metric manifold (M, g) is Einstein if and only if*

$$Ric(X, Y) = -2n\varepsilon g(X, Y). \quad (3.11)$$

Proposition 5. *If the Kenmotsu pseudo-metric manifold (M, g) is η -Einstein, then*

$$X(b) + 2b\eta(X) = 0, \quad (3.12)$$

for $n > 1$, and for any vector field $X \in TM$.

Proof. Equation (3.10) is equivalent to

$$QY = aY + b\varepsilon\eta(Y)\xi, \quad (3.13)$$

where a and b are as in (3.9). It is well known that

$$\operatorname{div} Q = \frac{1}{2}Dr, \quad (3.14)$$

where D denotes the gradient. Equations (3.13) and (3.14) yields to

$$(n-1)Y(a) = \varepsilon\{\xi(b)\eta(Y) + 2nb\eta(Y)\}.$$

For $Y = \xi$, it gives $\xi(b) = -2b$, and so we get (3.12) for $n > 1$. \square

Corollary 3. *If b (or a) is constant in an η -Einstein Kenmotsu pseudo-metric manifold, then it is Einstein.*

4. CURVATURE PROPERTIES OF KENMOTSU PSEUDO METRIC MANIFOLDS

First we prove the following Lemma which is very useful in subsequent sections.

Lemma 1. *On Kenmotsu pseudo-metric manifold (M, g) , we have the following identities:*

$$R(X, Y)\varphi Z - \varphi R(X, Y)Z = \varepsilon\{g(Y, Z)\varphi X - g(X, Z)\varphi Y + g(X, \varphi Z)Y - g(Y, \varphi Z)X\}, \quad (4.1)$$

$$R(\varphi X, \varphi Y)Z = R(X, Y)Z + \varepsilon\{g(Y, Z)X - g(X, Z)Y + g(Y, \varphi Z)\varphi X - g(X, \varphi Z)\varphi Y\}. \quad (4.2)$$

Proof. The Ricci identity shows that

$$\nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X, Y]} \varphi = R(X, Y)\varphi - \varphi R(X, Y).$$

Computing the left-hand side using (2.5) yields (4.1). The equation (4.2) is a result of (4.1). \square

Note that the necessary and sufficient condition for a Sasakian pseudo-metric manifold to have constant φ -sectional curvature c is [10]

$$\begin{aligned} 4R(X, Y)Z = & (c + 3\varepsilon)\{g(Y, Z)X - g(X, Z)Y\} \\ & + (\varepsilon c - 1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\ & + (c - \varepsilon)\{\eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(X, \varphi Z)\varphi Y \\ & - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\}. \end{aligned}$$

Here we prove:

Theorem 1. *The necessary and sufficient condition for a Kenmotsu pseudo-metric manifold M to have constant φ -sectional curvature c is*

$$\begin{aligned} 4R(X, Y)Z = & (c - 3\varepsilon)\{g(Y, Z)X - g(X, Z)Y\} \\ & + (c + \varepsilon)\{\varepsilon\eta(X)\eta(Z)Y - \varepsilon\eta(Y)\eta(Z)X \\ & + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(X, \varphi Z)\varphi Y \\ & - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\}. \end{aligned} \quad (4.3)$$

Proof. Suppose that M has constant φ -sectional curvature c . Then for all vector fields $U, V \in \text{Ker } \eta$, we have

$$R(U, \varphi U, U, \varphi U) = -cg(U, U)^2. \quad (4.4)$$

Using (4.1), we get

$$\begin{aligned} R(U, \varphi V, U, \varphi V) = & R(U, \varphi V, V, \varphi U) + \varepsilon\{g(U, U)g(V, V) \\ & - g(U, V)^2 - g(U, \varphi V)^2\}, \end{aligned} \quad (4.5)$$

$$R(U, \varphi U, V, \varphi U) = R(U, \varphi U, U, \varphi V), \quad (4.6)$$

for all $U, V \in \text{Ker } \eta$. Putting $U + V$ in (4.4), and using (4.2), (4.5), (4.6) and the first Bianchi identity, we obtain

$$\begin{aligned} & 2R(U, \varphi U, U, \varphi V) + 2R(V, \varphi V, V, \varphi U) + 3R(U, \varphi V, V, \varphi U) + R(U, V, U, V) \\ & = -c\{2g(U, V)^2 + 2g(U, U)g(U, V) + 2g(U, V)g(V, V) + g(U, U)g(V, V)\}. \end{aligned}$$

Replacing V by $-V$ and then summing the resulting equation to the above equation gives

$$3R(U, \varphi V, V, \varphi U) + R(U, V, U, V) = -c\{2g(U, V)^2 + g(U, U)g(V, V)\}. \quad (4.7)$$

Replacing V by φV in (4.7) and then using the identities (4.2) and (4.5), we get

$$4R(U, V, U, V) = (c - 3\varepsilon)\{g(U, V)^2 - g(U, U)g(V, V)\} - 3(c + \varepsilon)g(U, \varphi V)^2. \quad (4.8)$$

For $U, V, Z, W \in \text{Ker } \eta$, we determine $R(U + Z, V + W, U + Z, V + W)$ and then using (4.8) we obtain

$$4R(U, V, Z, W) + 4R(U, W, Z, V) = (c - 3\varepsilon)\{g(U, V)g(Z, W)$$

$$\begin{aligned}
 &+g(U, W)g(V, Z) - 2g(U, Z)g(V, W)\} - 3(c + \varepsilon)\{g(U, \varphi V)g(Z, \varphi W) \\
 &\quad +g(U, \varphi W)g(Z, \varphi V)\}. \tag{4.9}
 \end{aligned}$$

Interchanging V and Z in (4.9), and then subtracting the resulting equation with (4.9) and by virtue of the first Bianchi identity we obtain

$$\begin{aligned}
 4R(U, W, Z, V) &= (c - 3\varepsilon)\{g(U, V)g(Z, W) - g(U, Z)g(V, W)\} \\
 &- (c + \varepsilon)\{g(U, \varphi V)g(Z, \varphi W) - g(U, \varphi Z)g(V, \varphi W) + 2g(U, \varphi W)g(Z, \varphi V)\}. \tag{4.10}
 \end{aligned}$$

Now if $X, Y, Z, W \in TM$, then replacing U, V, Z, W by $\varphi X, \varphi Y, \varphi Z, \varphi W$ in (4.10), and using (4.1), (4.2), and $\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z)$ we get (4.3). The converse is trivial. \square

Theorem 2. *If a Kenmotsu pseudo-metric manifold has constant φ -sectional curvature c , then it is a space of constant curvature and $c = -\varepsilon$.*

Proof. From (4.3), it is easy to obtain (3.7), where $a = \frac{1}{2}(n(c - 3\varepsilon) + (c + \varepsilon))$ and $b = \frac{-1}{2}\varepsilon(n + 1)(c + \varepsilon)$. Since a and b are constants, from Corollary 3 it follows that $c = -\varepsilon$. \square

5. SOME STRUCTURE THEOREMS

The tangent space $T_p M$ of an almost contact pseudo-metric manifold M can be decomposed as $T_p M = \varphi(T_p M) \oplus L(\xi_p)$, where $L(\xi_p)$ is the linear subspace of $T_p M$ generated by ξ_p . Thus the conformal curvature tensor C is defined as a map

$$C : T_p M \times T_p M \times T_p M \rightarrow \varphi(T_p M) \oplus L(\xi_p), \quad p \in M,$$

such that

$$\begin{aligned}
 C(X, Y)Z &= R(X, Y)Z - \frac{1}{2n - 1}\{Ric(Y, Z)X + g(Y, Z)QX - Ric(X, Z)Y \\
 &\quad - g(X, Z)QY\} + \frac{r}{2n(2n - 1)}\{g(Y, Z)X - g(X, Z)Y\}. \tag{5.1}
 \end{aligned}$$

Then there arise three cases:

- The projection of the image of C in $\varphi(T_p M)$ is zero, that is,

$$C(X, Y, Z, \varphi W) = 0, \quad \text{for any } X, Y, Z, W \in T_p M. \tag{5.2}$$

- Projection of the image of C in $L(\xi_p)$ is zero, that is,

$$C(X, Y)\xi = 0, \quad \text{for all } X, Y \in T_p M. \tag{5.3}$$

- Projection of the image of $C \mid_{\varphi(T_p M) \times \varphi(T_p M) \times \varphi(T_p M)}$ in $\varphi(T_p M)$ is zero, that is,

$$\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0, \quad \text{for all } X, Y, Z \in T_p M. \tag{5.4}$$

An almost contact pseudo-metric manifold satisfying the cases (5.2), (5.3) and (5.4) are said to be conformally symmetric [14], ξ -conformally flat [13] and φ -conformally flat [2], respectively.

We begin with the following:

Theorem 3. *Let M be a ξ -conformally flat Kenmotsu pseudo-metric manifold of dimension higher than 3. Then the scalar curvature r of M satisfies*

$$Dr = \varepsilon \xi(r)\xi, \quad (5.5)$$

where D denotes gradient.

Proof. Since M is ξ -conformally flat, from (5.3) the equation (5.1) becomes

$$\begin{aligned} R(U, V)\xi &= \frac{1}{2n-1} \{Ric(V, \xi)U + \varepsilon\eta(V)QU - Ric(U, \xi)V - \varepsilon\eta(U)QV\} \\ &\quad - \frac{\varepsilon r}{2n(2n-1)} \{\eta(V)U - \eta(U)V\}, \end{aligned} \quad (5.6)$$

for any $U, V \in TM$, and this further gives

$$\begin{aligned} R(U, \xi)V &= \frac{1}{2n-1} \{g(V, Q\xi)U + \varepsilon\eta(V)QU - g(QU, V)\xi - g(U, V)Q\xi\} \\ &\quad - \frac{r}{2n(2n-1)} \{\varepsilon\eta(V)U - g(U, V)\xi\}. \end{aligned} \quad (5.7)$$

Putting $V = \xi$ in (5.6), then differentiating it covariantly along W and using (5.7), we get:

$$\begin{aligned} (\nabla_W R)(U, \xi)\xi &= \frac{1}{2n-1} \{g((\nabla_W Q)\xi, \xi)U + \varepsilon(\nabla_W Q)U - g((\nabla_W Q)U, \xi)\xi \\ &\quad - \varepsilon\eta(U)(\nabla_W Q)\xi\} - \frac{Wr}{2n(2n-1)} \{\varepsilon U - \varepsilon\eta(U)\xi\}. \end{aligned}$$

Taking the inner product of the above equation with Y and contracting with respect to U and W yield

$$\begin{aligned} \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(e_i, \xi)\xi, Y) &= \frac{1}{2n-1} \{g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y, \xi)\} \\ &\quad + \frac{\varepsilon(2n-2)}{4n(2n-1)} \{Yr - \eta(Y)\xi(r)\}, \end{aligned} \quad (5.8)$$

where $\{e_i\}$ is a pseudo-orthonormal basis in M and $\varepsilon_i = g(e_i, e_i)$. From the second Bianchi identity we easily obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(Y, \xi)\xi, e_i) = g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y, \xi). \quad (5.9)$$

Then from (5.8) and (5.9), noting that $n > 1$ we get

$$g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y, \xi) = \frac{\varepsilon}{4n} \{Yr - \eta(Y)\xi(r)\}.$$

Since ∇Q is symmetric, the above equation becomes

$$g((\nabla_Y Q)\xi, \xi) - g((\nabla_\xi Q)\xi, Y) = \frac{\varepsilon}{4n} \{Yr - \eta(Y)\xi(r)\}. \tag{5.10}$$

From (3.4), the left hand side of above equation vanishes. Then (5.10) leads to $Yr = \eta(Y)\xi(r)$ which gives (5.5). \square

Theorem 4. *A Kenmotsu pseudo-metric manifold M is ξ -conformally flat if and only if it is an η -Einstein manifold.*

Proof. If M is ξ -conformally flat, then

$$\begin{aligned} R(X, \xi)\xi &= \frac{1}{2n-1} \{Ric(\xi, \xi)X + \varepsilon QX - Ric(X, \xi)\xi - \varepsilon\eta(X)Q\xi\} \\ &\quad - \frac{\varepsilon r}{2n(2n-1)} \{X - \eta(X)\xi\}. \end{aligned}$$

Making use of equations (3.1) and (3.4) in above gives

$$Q = \left(\frac{r}{2n} + \varepsilon\right)I - \left(\frac{\varepsilon r}{2n} + 2n + 1\right)\varepsilon\eta \otimes \xi,$$

which is equivalent to (3.10).

Conversely, suppose that M is η -Einstein. Formula (5.1) gives

$$\begin{aligned} C(X, Y)\xi &= R(X, Y)\xi - \frac{1}{2n-1} \{Ric(Y, \xi)X + \varepsilon\eta(Y)QX - Ric(X, \xi)Y \\ &\quad - \varepsilon\eta(X)QY\} + \frac{\varepsilon r}{2n(2n-1)} \{\eta(Y)X - \eta(X)Y\}. \end{aligned}$$

Now using identities (3.1), (3.4) and (3.13) results in

$$\begin{aligned} C(X, Y)\xi &= R(X, Y)\xi - \frac{1}{2n-1} \left\{ (2n - \varepsilon a) + \frac{\varepsilon r}{2n} \right\} (\eta(X)Y - \eta(Y)X) \\ &= R(X, Y)\xi - (\eta(X)Y - \eta(Y)X) = 0, \end{aligned}$$

and this concludes the proof. \square

Theorem 5. *A Kenmotsu pseudo-metric manifold of dimension higher than 3 is φ -conformally flat if and only if it is a space of constant curvature $-\varepsilon$.*

Proof. Note that the φ -conformally flat condition $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$ is equivalent to $C(\varphi X, \varphi Y, \varphi Z, \varphi W) = 0$, and so from (5.1) we get

$$\begin{aligned} &R(\varphi X, \varphi Y, \varphi Z, \varphi W) \\ &= \frac{1}{2n-1} \{ Ric(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + g(\varphi Y, \varphi Z) Ric(\varphi X, \varphi W) \\ &\quad - Ric(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - g(\varphi X, \varphi Z) Ric(\varphi Y, \varphi W) \} \end{aligned}$$

$$-\frac{r}{2n(2n-1)}\{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\}. \quad (5.11)$$

Let $\{E_i = e_i, E_{n+i} = \varphi e_i, E_{2n+1} = \xi\}_{i=1}^n$ be a local pseudo-orthonormal φ -basis. Taking $X = W = E_i$ in (5.11) and summing, we get

$$\begin{aligned} & \sum_{i=1}^{2n} \varepsilon_i R(\varphi E_i, \varphi Y, \varphi Z, \varphi E_i) \\ &= \sum_{i=1}^{2n} \varepsilon_i \left[\frac{1}{2n-1} \{Ric(\varphi Y, \varphi Z)g(\varphi E_i, \varphi E_i) + g(\varphi Y, \varphi Z)Ric(\varphi E_i, \varphi E_i) \right. \\ & \quad \left. - Ric(\varphi E_i, \varphi Z)g(\varphi Y, \varphi E_i) - g(\varphi E_i, \varphi Z)Ric(\varphi Y, \varphi E_i)\} \right. \\ & \quad \left. - \frac{r}{2n(2n-1)} \{g(\varphi Y, \varphi Z)g(\varphi E_i, \varphi E_i) - g(\varphi E_i, \varphi Z)g(\varphi Y, \varphi E_i)\} \right] \\ &= \left(\frac{2n-2}{2n-1} \right) Ric(\varphi Y, \varphi Z) + \frac{1}{2n-1} \left(\frac{r}{2n} + \varepsilon 2n \right) g(\varphi Y, \varphi Z), \end{aligned} \quad (5.12)$$

where $\varepsilon_i = g(E_i, E_i)$. It can be easily verified that

$$\begin{aligned} \sum_{i=1}^{2n} \varepsilon_i R(\varphi E_i, \varphi Y, \varphi Z, \varphi E_i) &= Ric(\varphi Y, \varphi Z) - \varepsilon R(\xi, \varphi Y, \varphi Z, \xi) \\ &= Ric(\varphi Y, \varphi Z) + \varepsilon g(\varphi Y, \varphi Z). \end{aligned}$$

So that equation (5.12) becomes

$$Ric(\varphi Y, \varphi Z) = \left(\varepsilon + \frac{r}{2n} \right) g(\varphi Y, \varphi Z).$$

Substituting this in (5.11), one obtains

$$\begin{aligned} & R(\varphi X, \varphi Y, \varphi Z, \varphi W) \\ &= \frac{r+4n\varepsilon}{2n(2n-1)} \{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\}. \end{aligned} \quad (5.13)$$

From (4.2), (4.1), (3.2) and (2.2), we get

$$\begin{aligned} R(\varphi X, \varphi Y, \varphi Z, \varphi W) &= R(X, Y, Z, W) + \eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(Z)g(Y, W) \\ & \quad - \eta(Y)\eta(W)g(X, Z) + \eta(X)\eta(W)g(Y, Z). \end{aligned} \quad (5.14)$$

Now (5.13) and (5.14) imply

$$\begin{aligned} R(X, Y, Z, W) &= \frac{r+4n\varepsilon}{2n(2n-1)} \{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\} \\ & \quad - \eta(Y)\eta(Z)g(X, W) + \eta(X)\eta(Z)g(Y, W) \\ & \quad + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z). \end{aligned} \quad (5.15)$$

Now taking the scalar product of (4.1) with W and by virtue of (5.15) we get an equation and then contracting the resulting equation with respect to X and W gives

$$(2n - 2) \left(\frac{r + 4n\varepsilon}{2n(2n - 1)} + \varepsilon \right) g(Y, \varphi Z) = 0.$$

Since $n > 1$, it follows that

$$r = -\varepsilon 2n(2n + 1). \tag{5.16}$$

Using (5.16) and (2.2) in (5.15), we get

$$R(X, Y, Z, W) = -\varepsilon \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\},$$

and so that the manifold is of constant curvature $-\varepsilon$.

The converse is trivial. □

Corollary 4. *A conformally flat Kenmotsu pseudo-metric manifold of dimension higher than 3 is a space of constant curvature $-\varepsilon$.*

The above corollary for Riemannian case has been proved in [9].

Now contracting (5.15), we obtain (3.10). Thus we can state the following:

Corollary 5. *A φ -conformally flat Kenmotsu pseudo-metric manifold is an η -Einstein manifold.*

In view of Theorem 4 and Corollary 5, we have the following:

Corollary 6. *A φ -conformally flat Kenmotsu pseudo-metric manifold is always ξ -conformally flat.*

6. RICCI SOLITON ON KENMOTSU PSEUDO-METRIC MANIFOLDS

A Ricci soliton on a pseudo-Riemannian manifold (M, g) is defined by

$$(\mathcal{L}_V g)(X, Y) + 2Ric(X, Y) + 2\lambda g(X, Y) = 0, \tag{6.1}$$

where λ is a constant. Ricci soliton is a natural generalization of the Einstein metric (that is, $Ric(X, Y) = ag(X, Y)$, for some constant a), and is a special self similar solution of Hamilton's Ricci flow (see [8]) $\frac{\partial}{\partial t} g(t) = -2Ric(t)$ with initial condition $g(0) = g$. We say that the Ricci soliton is *steady* when $\lambda = 0$, *expanding* when $\lambda > 0$ and *shrinking* when $\lambda < 0$.

Before producing the main results, we prove the following:

Lemma 2. *A Kenmotsu pseudo-metric manifold (M, g) satisfies*

$$(\nabla_X Q)\xi = -QX - 2n\varepsilon X, \tag{6.2}$$

$$(\nabla_\xi Q)X = -2QX - 4n\varepsilon X. \tag{6.3}$$

Proof. Differentiating $Q\xi = -2n\varepsilon\xi$, and recalling (2.6) provide (6.2). Now differentiating (3.1) along W leads to

$$(\nabla_W R)(X, Y)\xi = -R(X, Y)W + \varepsilon g(X, W)Y - \varepsilon g(Y, W)X.$$

Contracting this with respect to X and W gives us

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(e_i, Y)\xi, Z) = Ric(Y, Z) + 2ng(Y, Z). \quad (6.4)$$

From the second Bianchi identity, one can easily obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(Z, \xi)Y, e_i) = g((\nabla_Z Q)\xi, Y) - g((\nabla_\xi Q)Z, Y). \quad (6.5)$$

Fetching (6.5) into (6.4) and with the aid of (6.2), we infer that

$$g((\nabla_\xi Q)Z, Y) = -2Ric(Y, Z) - 4ng(Y, Z),$$

which proves (6.3). \square

Theorem 6. *Let (M, g) be a Kenmotsu pseudo-metric manifold. If (g, V) is a Ricci soliton, then the soliton constant $\lambda = 2n\varepsilon$, and so the soliton is either expanding or shrinking depending on the casual character of the Reeb vector field ξ .*

Proof. Differentiating (6.1) covariantly along Z gives

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = -2(\nabla_Z Ric)(X, Y). \quad (6.6)$$

From Yano [12], we know the following well known commutation formula:

$$\begin{aligned} (\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V, X]} g)(Y, Z) \\ = -g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y), \end{aligned}$$

for all $X, Y, Z \in TM$. Since $\nabla g = 0$, the previous equation gives

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y), \quad (6.7)$$

for all $X, Y, Z \in TM$. As $\mathcal{L}_V \nabla$ is a symmetric, it follows from (6.7) that

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X, Y), Z) \\ = \frac{1}{2}(\nabla_X \mathcal{L}_V g)(Y, Z) + \frac{1}{2}(\nabla_Y \mathcal{L}_V g)(Z, X) - \frac{1}{2}(\nabla_Z \mathcal{L}_V g)(X, Y). \end{aligned} \quad (6.8)$$

Making use of (6.6) in (6.8) we have

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z Ric)(X, Y) - (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(Z, X). \quad (6.9)$$

Putting $Y = \xi$ in (6.9) and using (6.2) and (6.3), we obtain

$$(\mathcal{L}_V \nabla)(X, \xi) = 2QX + 4n\varepsilon X.$$

Differentiating the preceding equation along Y and using (2.6), we obtain

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = -(\mathcal{L}_V \nabla)(X, Y) + 2\eta(Y)\{QX + 2n\varepsilon X\} + 2(\nabla_Y Q)X.$$

Feeding the above obtained expression into the following relation (see [12])

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z), \tag{6.10}$$

and using the symmetry of $\mathcal{L}_V \nabla$, we immediately obtain

$$\begin{aligned} (\mathcal{L}_V R)(X, Y)\xi = & 2\eta(X)\{QY + 2n\varepsilon Y\} - 2\eta(Y)\{QX + 2n\varepsilon X\} \\ & + 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\}. \end{aligned} \tag{6.11}$$

Setting $Y = \xi$ in the foregoing equation, we get

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{6.12}$$

Now taking the Lie-derivative of $R(X, \xi)\xi = -X + \eta(X)\xi$ along V gives

$$(\mathcal{L}_V R)(X, \xi)\xi - 2\eta(\mathcal{L}_V \xi)X + \varepsilon g(X, \mathcal{L}_V \xi)\xi = (\mathcal{L}_V \eta)(X)\xi,$$

which by virtue of (6.12) becomes

$$(\mathcal{L}_V \eta)(X)\xi = -2\eta(\mathcal{L}_V \xi)X + \varepsilon g(X, \mathcal{L}_V \xi)\xi. \tag{6.13}$$

With the help of (3.4), the equation (6.1) takes the form

$$(\mathcal{L}_V g)(X, \xi) = -2\lambda\varepsilon\eta(X) + 4n\eta(X). \tag{6.14}$$

Changing X to ξ in the aforementioned equation gives

$$\eta(\mathcal{L}_V \xi) = \lambda - 2n\varepsilon. \tag{6.15}$$

Now Lie-differentiating $\eta(X) = \varepsilon g(X, \xi)$ yields $(\mathcal{L}_V \eta)(X) = \varepsilon(\mathcal{L}_V g)(X, \xi) + \varepsilon g(X, \mathcal{L}_V \xi)$.

Using this and (6.15) in (6.13) provides $(\lambda - 2n\varepsilon)(X - \eta(X)\xi) = 0$. Tracing the previous equation yield $\lambda = 2n\varepsilon$. □

Corollary 7. *A Kenmotsu manifold admitting the Ricci soliton is always expanding with $\lambda = 2n$.*

Lemma 3. *Let (M, g) be a Kenmotsu pseudo-metric manifold. If (g, V) is a Ricci soliton, then the Ricci tensor satisfies*

$$(\mathcal{L}_V Ric)(X, \xi) = -X(r) + \xi(r)\eta(X). \tag{6.16}$$

Proof. Contracting equation (6.11) with respect to X and recalling the well-known formulas

$$\operatorname{div} Q = \frac{1}{2}Dr \quad \text{and} \quad \operatorname{trace} \nabla Q = Dr,$$

we easily obtain

$$(\mathcal{L}_V Ric)(Y, \xi) = -Y(r) - 2\eta(Y)\{r + \varepsilon 2n(2n + 1)\}. \tag{6.17}$$

Substituting $Y = \xi$, we have $(\mathcal{L}_V Ric)(\xi, \xi) = -\xi(r) - 2\{r + \varepsilon 2n(2n + 1)\}$. On the other hand, contracting (6.12) gives $(\mathcal{L}_V Ric)(\xi, \xi) = 0$. Using this in the previous equation leads to

$$\xi(r) = -2(r + \varepsilon 2n(2n + 1)). \quad (6.18)$$

Hence (6.18) and (6.17) give (6.16). \square

Combining Theorem 3 and 4, we state the following:

Lemma 4. *An η -Einstein Kenmotsu pseudo-metric manifold M of dimension higher than 3 satisfies*

$$Dr = \varepsilon \xi(r)\xi. \quad (6.19)$$

Now we prove:

Theorem 7. *Let (M, g) be an η -Einstein Kenmotsu pseudo-metric manifold of dimension higher than 3. If (g, V) is a Ricci soliton, then M is Einstein.*

Proof. Making use of (6.19) in (6.16), we have $(\mathcal{L}_V Ric)(X, \xi) = 0$. Now, Lie-differentiating the first relation of (3.4) along V , using (3.10), (6.14), $\lambda = 2n\varepsilon$ and $\eta(\mathcal{L}_V \xi) = 0$, we obtain

$$(r + \varepsilon 2n(2n + 1))\mathcal{L}_V \xi = 0.$$

If $r = -\varepsilon 2n(2n + 1)$, then (3.10) shows that M is Einstein.

So we assume $r \neq -\varepsilon 2n(2n + 1)$ in some open set \mathcal{O} of M . Hence $\mathcal{L}_V \xi = 0$ on \mathcal{O} , and so it follows from (2.6) that

$$\nabla_\xi V = V - \eta(V)\xi. \quad (6.20)$$

Clearly, (6.14) shows that $(\mathcal{L}_V g)(X, \xi) = 0$. This together with (6.20), we have

$$g(\nabla_X V, \xi) = -g(\nabla_\xi V, X) = -g(X, V) + \eta(X)\eta(V). \quad (6.21)$$

From Duggal and Sharma [5], we know that

$$(\mathcal{L}_V \nabla)(X, Y) = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} V + R(V, X)Y.$$

Setting $Y = \xi$ in the above equation and by virtue of (2.6), (3.1), (6.20) and (6.21), we have $r = -\varepsilon 2n(2n + 1)$. This leads to a contradiction as $r \neq -\varepsilon 2n(2n + 1)$ on \mathcal{O} and completes the proof. \square

Now we consider Kenmotsu pseudo-metric 3-manifolds which admits Ricci solitons.

Theorem 8. *Let (M, g) be a Kenmotsu pseudo-metric 3-manifold. If (g, V) is a Ricci soliton, then M is of constant curvature $-\varepsilon$.*

Proof. The Riemannian curvature tensor of pseudo-Riemannian 3-manifold is given by

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y \\ &\quad - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \quad (6.22)$$

Taking $Y = Z = \xi$ in (6.22) and using (3.1) and (3.4) gives

$$Q = \left(\frac{r}{2} + 1\right)I - \left(\frac{r}{2} + 3\right)\eta \otimes \xi. \tag{6.23}$$

Making use of this in (6.11) gives

$$\begin{aligned} (\mathcal{L}_V R)(X, Y)\xi &= X(r)\{Y - \eta(Y)\xi\} + Y(r)\{-X + \eta(X)\xi\} \\ &\quad - (r + 6\varepsilon)\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \tag{6.24}$$

Replacing Y by ξ in the above equation and comparing it with (6.12), we obtain

$$\{\xi(r) + (r + 6\varepsilon)\}\{-X + \eta(X)\xi\} = 0.$$

Contracting the above equation gives $\xi(r) + (r + 6\varepsilon) = 0$, and consequently it follows from (6.18) that $r = -6\varepsilon$. Then from (6.23) we have $QX = -2\varepsilon X$, and substituting this into (6.22) shows that M is of constant curvature $-\varepsilon$. \square

Corollary 8. *There does not exist a Kenmotsu pseudo-metric manifold (M, g) admitting the Ricci soliton $(g, V = \xi)$.*

Proof. If $V = \xi$, then from (2.8) the Ricci soliton equation (6.1) would become

$$Ric = -(1 + \lambda)g + \varepsilon\eta \otimes \eta, \tag{6.25}$$

which means M is η -Einstein. Then due to Theorem 7 and 8, M must be Einstein, and this will be a contradiction to equation (6.25) as $\varepsilon \neq 0$. \square

Remark 1. Clearly, Theorem 7 and 8 are generalizations of the results of Ghosh proved in [6] and [7]. Note that our approach and technique to obtain the result is different to that of Ghosh.

Now we provide an example of a Kenmotsu pseudo-metric 3-manifold which admits a Ricci soliton and verify our results.

Example 1. Let $M = N \times I$, where N is an open connected subset of \mathbb{R}^2 and I is an open interval in \mathbb{R} . Let (x, y, z) be the Cartesian coordinates in M . Define the structure (φ, ξ, η, g) on M as follows:

$$\begin{aligned} \varphi\left(\frac{\partial}{\partial x}\right) &= \frac{\partial}{\partial y}, & \varphi\left(\frac{\partial}{\partial y}\right) &= -\frac{\partial}{\partial x}, & \varphi\left(\frac{\partial}{\partial z}\right) &= 0, \\ \xi &= \frac{\partial}{\partial z}, & \eta &= dz, \\ (g_{ij}) &= \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{2z} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}. \end{aligned}$$

Now from Koszul's formula, the Levi-Civita connection ∇ is given by

$$\begin{aligned}\nabla_{\partial_1}\partial_1 &= -\varepsilon e^{2z}\partial_3, & \nabla_{\partial_1}\partial_2 &= 0, & \nabla_{\partial_1}\partial_3 &= \partial_1, \\ \nabla_{\partial_2}\partial_1 &= 0, & \nabla_{\partial_2}\partial_2 &= -\varepsilon e^{2z}\partial_3, & \nabla_{\partial_2}\partial_3 &= \partial_2, \\ \nabla_{\partial_3}\partial_1 &= \partial_1, & \nabla_{\partial_3}\partial_2 &= \partial_2, & \nabla_{\partial_3}\partial_3 &= 0,\end{aligned}\tag{6.26}$$

where $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$ and $\partial_3 = \frac{\partial}{\partial z}$. From (6.26), one can easily verify

$$(\nabla_{\partial_i}\varphi)\partial_j = -\eta(\partial_j)\varphi\partial_i - \varepsilon g(\partial_i, \varphi\partial_j)\xi,\tag{6.27}$$

for all $i, j = 1, 2, 3$, and so M is a Kenmotsu pseudo-metric manifold with the above (φ, ξ, η, g) structure.

With the help of (6.26), we find that:

$$\begin{aligned}R(\partial_1, \partial_2)\partial_3 &= R(\partial_2, \partial_3)\partial_1 = R(\partial_1, \partial_3)\partial_2 = 0, \\ R(\partial_1, \partial_3)\partial_1 &= R(\partial_2, \partial_3)\partial_2 = \varepsilon e^{2z}\partial_3, \\ R(\partial_1, \partial_2)\partial_1 &= \varepsilon e^{2z}\partial_2, & R(\partial_2, \partial_3)\partial_3 &= -\partial_2, \\ R(\partial_1, \partial_3)\partial_3 &= -\partial_1, & R(\partial_1, \partial_2)\partial_2 &= -\varepsilon e^{2z}\partial_1.\end{aligned}\tag{6.28}$$

Let $e_1 = e^{-z}\partial_1$, $e_2 = e^{-z}\partial_2$ and $e_3 = \xi = \partial_3$. Clearly, $\{e_1, e_2, e_3\}$ forms an orthonormal φ -basis of vector fields on M . Making use of (6.28) one can easily show that M is Einstein, that is, $Ric(Y, Z) = -2\varepsilon g(Y, Z)$, for any $Y, Z \in TM$.

Let us consider the vector field

$$V = \alpha \frac{\partial}{\partial y},\tag{6.29}$$

where α is a non-zero constant. Making use of (6.26) one can easily show that V is Killing with respect to g , that is, we have

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0,$$

for any $X, Y \in TM$. Hence g is a Ricci soliton, that is, (6.1) holds true with V as in (6.29) and $\lambda = 2\varepsilon$. Further (6.28) shows that

$$R(X, Y)Z = -\varepsilon\{g(Y, Z)X - g(X, Z)Y\},$$

for any $X, Y \in TM$, which means M is of constant curvature $-\varepsilon$ and so Theorem 8 is verified.

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REFERENCES

- [1] A. Bejancu and K. L. Duggal, “Real hypersurfaces of indefinite Kaehler manifolds.” *Int. J. Math. Math. Sci.*, vol. 16, no. 3, pp. 545–556, 1993, doi: [10.1155/S0161171293000675](https://doi.org/10.1155/S0161171293000675).
- [2] J. Cabrerizo, L. Fernandez, M. Fernandez, and G. Zhen, “The structure of a class of K -contact manifolds.” *Acta Math. Hungar.*, vol. 82, no. 4, pp. 331–340, 1990, doi: [10.1023/A:1006696410826](https://doi.org/10.1023/A:1006696410826).
- [3] G. Calvaruso and D. Perrone, “Contact pseudo-metric manifolds.” *Differ. Geom. Appl.*, vol. 28, no. 2, pp. 615–634, 2010, doi: [10.1016/j.difgeo.2010.05.006](https://doi.org/10.1016/j.difgeo.2010.05.006).
- [4] K. L. Duggal, “Space time manifolds and contact structures.” *Int. J. Math. Math. Sci.*, vol. 13, no. 3, pp. 545–554, 1990, doi: [10.1155/S0161171290000783](https://doi.org/10.1155/S0161171290000783).
- [5] K. L. Duggal and R. Sharma, *Symmetries of Spacetimes and Riemannian Manifolds*. Kluwer: Springer, 1999. doi: [10.1007/978-1-4615-5315-1](https://doi.org/10.1007/978-1-4615-5315-1).
- [6] A. Ghosh, “Kenmotsu 3-metric as a Ricci soliton.” *Chaos Solitons Fractals*, vol. 44, pp. 647–650, 2011.
- [7] A. Ghosh, “An η -Einstein Kenmotsu metric as a Ricci soliton,” *Publ. Math. Debrecen*, vol. 82, pp. 591–598, 2013.
- [8] R. Hamilton, “The Ricci flow on surfaces.” *Contemp. Math.*, vol. 71, pp. 237–261, 1988.
- [9] K. Kenmotsu, “A class of almost contact Riemannian manifolds.” *Tohoku Math. J.*, vol. 24, no. 1, pp. 93–103, 1972, doi: [10.2748/tmj/1178241594](https://doi.org/10.2748/tmj/1178241594).
- [10] T. Takahashi, “Sasakian manifold with pseudo-Riemannian metrics.” *Tohoku Math. J.*, vol. 21, no. 2, pp. 271–290, 1969, doi: [10.2748/tmj/1178242996](https://doi.org/10.2748/tmj/1178242996).
- [11] Y. Wang and X. Liu, “Almost Kenmotsu pseudo-metric manifolds.” *An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.)*, vol. LXII, no. 1, pp. 241–256, 2016.
- [12] K. Yano, *Integral formulas in Riemannian geometry*. New York: Marcel Dekker, 1970.
- [13] G. Zhen, J. Cabrerizo, L. Fernandez, and M. Fernandez, “On ξ -conformally flat contact metric manifolds.” *Indian J. Pure Appl. Math.*, vol. 28, pp. 725–734, 1997.
- [14] G. Zhen, “Conformally symmetric K -contact manifolds.” *Chinese Quart. J. Math.*, vol. 7, no. 1, pp. 5–10, 1992.

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