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NOTES ON EXPLICIT AND INVERSION FORMULAS FOR THE CHEBYSHEV POLYNOMIALS OF THE FIRST TWO KINDS

FENG QI, DA-WEI NIU, AND DONGKYU LIM

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Abstract. In the paper, starting from the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds, by virtue of the Faà di Bruno formula, with the help of two identities for the Bell polynomials of the second kind, and making use of a new inversion theorem for combinatorial coefficients, the authors derive two nice explicit formulas and their corresponding inversion formulas for the Chebyshev polynomials of the first and second kinds.

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1. INTRODUCTION

It is well known [5–7, 31] that the Chebyshev polynomials of the first and second kinds T_n and $U_n(x)$ are very important in mathematical sciences and that, in the study of ordinary differential equations [5, pp. xxxv and 1004], they arise as solutions to the Chebyshev differential equations

$$(1-x^2)y''-xy'+n^2y=0 \quad \text{and} \quad (1-x^2)y''-3xy'+n(n+2)y=0$$

for the Chebyshev polynomials of the first and second kinds T_n and U_n respectively.

In [6, Eqs. (4.30) and (4.31)], the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds T_n and U_n read that

$$T_n(x)=(-1)^n\frac{2^n n!}{(2n)!}\bigl(1-x^2\bigr)^{1/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n-1/2}\bigr] \tag{1.1}$$

and

$$U_n(x)=(-1)^n\frac{2^n (n+1)!}{(2n+1)!}\bigl(1-x^2\bigr)^{-1/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n+1/2}\bigr]. \tag{1.2}$$

For variants of the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds T_n and U_n, please refer to, for example, [5, pp. 1003–1004], [7, p. 442], [19, Section 4], and [31, pp. 432–433].

In [5, p. 1003], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are written in the forms

$$T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{\mathrm{d}^n}{\mathrm{d} x^n}\left[\left(1-x^2\right)^{n-1/2}\right] \tag{1.3}$$

and

$$U_n(x) = \frac{(-1)^n (n+1)}{\sqrt{1-x^2}\,(2n+1)!!} \frac{\mathrm{d}^n}{\mathrm{d} x^n}\left[\left(1-x^2\right)^{n+1/2}\right]. \tag{1.4}$$

In [5, p. 1004] and [31, pp. 432–433], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are formulated as

$$T_n(x) = \frac{(-1)^n \sqrt{\pi}}{2^n \Gamma(n+1/2)} \left(1-x^2\right)^{1/2} \frac{\mathrm{d}^n}{\mathrm{d} x^n}\left[\left(1-x^2\right)^{n-1/2}\right] \tag{1.5}$$

and

$$U_n(x) = \frac{(-1)^n \sqrt{\pi}\,(n+1)}{2^{n+1} \Gamma(n+3/2)} \left(1-x^2\right)^{-1/2} \frac{\mathrm{d}^n}{\mathrm{d} x^n}\left[\left(1-x^2\right)^{n+1/2}\right], \tag{1.6}$$

where $\Gamma(z)$ stands for the classical gamma function which can be defined [8, 16] by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$$

or by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d} t, \quad \Re(z) > 0.$$

In [7, p. 442], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are arranged as

$$T_n(x) = \frac{\left(1-x^2\right)^{1/2}}{(-2)^n (1/2)_n} \frac{\mathrm{d}^n}{\mathrm{d} x^n}\left[\left(1-x^2\right)^{n-1/2}\right] \tag{1.7}$$

and

$$U_n(x) = \frac{(n+1)\left(1-x^2\right)^{-1/2}}{(-2)^n (3/2)_n} \frac{\mathrm{d}^n}{\mathrm{d} x^n}\left[\left(1-x^2\right)^{n+1/2}\right], \tag{1.8}$$

where $(x)_n$ for $n \ge 0$ and $x \in \mathbb{R}$ denotes the rising factorial which can be defined [22] by

$$(x)_n = \prod_{\ell=0}^{n-1}(x+\ell) = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)\cdots(x+n-1), & n \ge 1; \\ 1, & n = 0. \end{cases}$$

By virtue of the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we have

$$\Gamma\left(n+\frac{1}{2}\right) = \prod_{\ell=0}^{n-1}\left(n-\ell-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)_n \sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

and

$$\Gamma\left(n+\frac{3}{2}\right)=\prod_{\ell=0}^{n}\left(n-\ell+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\left(\frac{3}{2}\right)_n\frac{\sqrt{\pi}}{2}=\frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}\,.$$

Substituting these into (1.5) and (1.6) respectively leads to (1.3), (1.4), (1.7), and (1.8) which are equivalent to (1.1) and (1.2) respectively.

In [31, pp. 432–433], it was listed that

$$T_n(x)=\frac{n}{2}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}(2x)^{n-2m} \tag{1.9}$$

and

$$U_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{(n-m)!}{m!(n-2m)!}(2x)^{n-2m}, \tag{1.10}$$

where $n\in\mathbb{N}$ and $\lfloor t\rfloor$ denotes the floor function whose value equals the largest integer less than or equal to t.

In this paper, starting from the four formulas (1.1), (1.2), (1.9), and (1.10), by virtue of the Faà di Bruno formula, with the help of two identities for the Bell polynomials of the second kind, and making use of a new inversion theorem [28, Theorem 4.3] for combinatorial coefficients, we will derive the following two nice explicit formulas and their corresponding inversion formulas for the Chebyshev polynomials T_n and U_n.

2. Four lemmas

For proving our main results, Theorems 1 and 2 below, we need the following four lemmas.

Lemma 1 ([4, pp. 134 and 139])**.** *For $n\ge k\ge 0$, the Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big). \tag{2.1}$$

Lemma 2 ([4, p. 135])**.** *For $n\ge k\ge 0$, we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}), \tag{2.2}$$

where a and b are any complex numbers.

Lemma 3 ([13, Theorem 4.1] and [27, Section 3]). *For $n \ge k \ge 0$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy*

$$\mathrm{B}_{n,k}(x,1,0,\dots,0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n}, \tag{2.3}$$

where $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \ge 0$.

Lemma 4 ([28, Theorem 4.3]). *For $n \ge k \ge 1$, let s_k and S_k be two sequences independent of n. Then*

$$\frac{s_n}{n!} = \sum_{k=1}^{n} (-1)^k \binom{k}{n-k} S_k$$

if and only if

$$n S_n = \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} \binom{2n-k-1}{n-1} s_k.$$

3. Main results and their proofs

Now we begin to state and prove our main results, Theorems 1 and 2 below.

Theorem 1. *For $n \ge 0$, the Chebyshev polynomials T_n and U_n can be explicitly computed by*

$$T_n(x) = x^n \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{n}{2\ell} \left(1 - \frac{1}{x^2}\right)^{\ell} \tag{3.1}$$

and

$$U_n(x) = x^n \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2\ell+1} \left(1 - \frac{1}{x^2}\right)^{\ell}. \tag{3.2}$$

Proof. By virtue of the formulas (2.1), (2.2), and (2.3), we have

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(1-x^2\right)^{n-1/2}\right] &= \sum_{k=1}^{n} \frac{\mathrm{d}^k u^{n-1/2}}{\mathrm{d}u^k} \mathrm{B}_{n,k}(-2x,-2,0\dots,0) \\
&= \sum_{k=1}^{n} \prod_{\ell=0}^{k-1} \left(n-\ell-\frac{1}{2}\right) u^{n-k-1/2} (-2)^k \mathrm{B}_{n,k}(x,1,0\dots,0) \\
&= \sum_{k=1}^{n} \frac{1}{2^k} \prod_{\ell=0}^{k-1} (2n-2\ell-1) \left(1-x^2\right)^{n-k-1/2} (-2)^k \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n} \\
&= \frac{n!}{(2x)^n} \left(1-x^2\right)^{n-1/2} \sum_{k=1}^{n} (-1)^k \binom{k}{n-k} \frac{(2n-1)!!}{[2(n-k)-1]!!} \frac{2^k}{k!} \left(\frac{x^2}{1-x^2}\right)^k
\end{aligned}$$

$$= \frac{n!(2n-1)!!}{(2x)^n}\left(1-x^2\right)^{n-1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{2^k}{k![2(n-k)-1]!!}\left(\frac{x^2}{1-x^2}\right)^k,$$

where $n \in \mathbb{N}$, $u = u(x) = 1-x^2$, and the double factorial of negative odd integers $-2n-1$ is defined by

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = (-1)^n\frac{2^n n!}{(2n)!}, \quad n \geq 0.$$

Substituting the above established equality into (1.1) and simplifying lead to

$$T_n(x) = \sum_{k=1}^{n}\frac{(-1)^{n-k}}{4^{n-k}}\binom{k}{n-k}\binom{n}{k}\frac{[2(n-k)]!!}{[2(n-k)-1]!!}x^{2k-n}\left(1-x^2\right)^{n-k}$$

which can be rearranged, by replacing $n-k$ by ℓ, as

$$T_n(x) = x^n\sum_{\ell=0}^{n-1}\frac{(-1)^\ell}{4^\ell}\binom{n}{n-\ell}\binom{n-\ell}{\ell}\frac{(2\ell)!!}{(2\ell-1)!!}\left(\frac{1}{x^2}-1\right)^\ell.$$

Since

$$\frac{1}{4^\ell}\binom{n}{n-\ell}\binom{n-\ell}{\ell}\frac{(2\ell)!!}{(2\ell-1)!!} = \binom{n}{2\ell},$$

we arrives at the identity (3.1).

Repeating the above process, we can obtain

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(1-x^2\right)^{n+1/2}\right] &= \sum_{k=1}^{n}\frac{\mathrm{d}^k u^{n+1/2}}{\mathrm{d}u^k}\mathrm{B}_{n,k}(-2x,-2,0\dots,0)\\
&= \sum_{k=1}^{n}\prod_{\ell=0}^{k-1}\left(n-\ell+\frac{1}{2}\right)u^{n-k+1/2}(-2)^k\,\mathrm{B}_{n,k}(x,1,0\dots,0)\\
&= \sum_{k=1}^{n}\frac{1}{2^k}\prod_{\ell=0}^{k-1}(2n-2\ell+1)\left(1-x^2\right)^{n-k+1/2}(-2)^k\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}\\
&= \frac{n!}{(2x)^n}\left(1-x^2\right)^{n+1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{(2n+1)!!}{[2(n-k)+1]!!}\frac{2^k}{k!}\left(\frac{x^2}{1-x^2}\right)^k\\
&= \frac{n!(2n+1)!!}{(2x)^n}\left(1-x^2\right)^{n+1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{2^k}{k![2(n-k)+1]!!}\left(\frac{x^2}{1-x^2}\right)^k.
\end{aligned}$$

Substituting this into (1.2) and simplifying lead to

$$U_n(x) = \sum_{k=1}^{n}\frac{(-1)^{n-k}}{2^{2n-2k+1}}\binom{k}{n-k}\binom{n+1}{k}\frac{[2(n-k+1)]!!}{[2(n-k)+1]!!}x^{2k-n}\left(1-x^2\right)^{n-k}.$$

Replacing $n-k$ by ℓ reveals that

$$U_n(x)=\sum_{\ell=0}^{n-1}\frac{(-1)^\ell}{2^{2\ell+1}}\binom{n+1}{n-\ell}\binom{n-\ell}{\ell}\frac{[2(\ell+1)]!!}{(2\ell+1)!!}x^{n-2\ell}\big(1-x^2\big)^\ell.$$

Due to

$$\frac{1}{2^{2\ell+1}}\binom{n+1}{n-\ell}\binom{n-\ell}{\ell}\frac{[2(\ell+1)]!!}{(2\ell+1)!!}=\binom{n+1}{2\ell+1},$$

we derive (3.2). The proof of Theorem 1 is complete. □

Theorem 2. *For $n\in\mathbb{N}$, we have*

$$\sum_{k=1}^{n}\binom{2n-k-1}{n-1}(2x)^kT_k(x)=\frac{1}{2}(2x)^{2n} \tag{3.3}$$

and

$$\sum_{k=1}^{n}k\binom{2n-k-1}{n-1}(2x)^kU_k(x)=n(2x)^{2n}. \tag{3.4}$$

Proof. We notice that the formulas (1.9) and (1.10) can be rearranged as

$$T_n(x)=\frac{n}{2}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\binom{n-m}{m}\frac{(2x)^{n-2m}}{n-m} \tag{3.5}$$

and

$$U_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\binom{n-m}{m}(2x)^{n-2m}. \tag{3.6}$$

The inversion theorem in Lemma 4 can be restated as

$$(-1)^n\frac{s_n}{n!}=\sum_{\ell=0}^{n-1}(-1)^\ell\binom{n-\ell}{\ell}S_{n-\ell}=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^\ell\binom{n-\ell}{\ell}S_{n-\ell}$$

if and only if

$$nS_n=\sum_{\ell=1}^{n}\frac{(-1)^\ell}{(\ell-1)!}\binom{2n-\ell-1}{n-1}s_\ell.$$

The formulas (3.5) and (3.6) can be rearranged as

$$\frac{2}{n}(2x)^nT_n(x)=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^\ell\binom{n-\ell}{\ell}\frac{(2x)^{2(n-\ell)}}{n-\ell}$$

and

$$(2x)^nU_n(x)=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^\ell\binom{n-\ell}{\ell}(2x)^{2(n-\ell)}.$$

Consequently, we obtain

$$n\frac{(2x)^{2n}}{n} = \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}(-1)^k 2(k-1)!(2x)^k T_k(x)$$

and

$$n(2x)^{2n} = \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}(-1)^k k!(2x)^k U_k(x)$$

which can be simplified as (3.3) and (3.4). The proof of Theorem 2 is complete. □

4. Remarks

In this section, we will list several remarks to explain more about the formula (2.3), Lemma 4, our main results, and other things.

Remark 1. To the best of our knowledge, the nice formula (2.3) was first concluded in [13] and has been extensively applied in the papers [9–15, 17, 19, 21, 23–25, 27, 29, 30] and closely related references therein. The formula (2.3) has been generalized in the papers [15, 17, 20] and closely related references therein.

Remark 2. To the best of our knowledge, Lemma 4 is a new inversion theorem and has been applied in the paper [10, 18, 19, 26].

Remark 3. Because both the formula (2.3) and Lemma 4 are new, our main results stated in Theorems 1 and 2, or at least their proofs, are also new.

Remark 4. The Chebyshev polynomials are classical, but their study is still very active. As examples, we recommend three newly-published papers [1–3] to readers. Considering the length of this paper, we would not like to detail main results in these three papers and the closely-related references therein.

Remark 5. This paper is a slightly revised version of the preprint [19].

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Authors’ addresses

Feng Qi
Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China; School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China
E-mail address: `qifeng618@gmail.com`
URL: `https://qifeng618.wordpress.com`

Da-Wei Niu
Department of Science, Henan University of Animal Husbandry and Economy, Zhengzhou 450046, Henan, China
E-mail address: `nnddww@gmail.com`
URL: `https://orcid.org/0000-0003-4033-7911`

Dongkyu Lim
Department of Mathematics Education, Andong National University, Andong 36729, Republic of Korea
E-mail address: `dgrim84@gmail.com`
URL: `https://orcid.org/0000-0002-0928-8480`