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Extremals of Functions on Graphs with Applications to Graphs and Hypergraphs*

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The method used in an article by T. S. Matzkin and E. G. Straus [*Canad. J. Math.* **17** (1965), 533–540] is generalized by attaching nonnegative weights to t-tuples of vertices in a hypergraph subject to a suitable normalization condition. The edges of the hypergraph are given weights which are functions of the weights of its t-tuples and the graph is given the sum of the weights of its edges. The extremal values and the extremal points of these functions are determined. The results can be applied to various extremal problems on graphs and hypergraphs which are analogous to P. Turán's Theorem [*Colloq. Math.* **3** (1954), 19–30: (*Hungarian*) *Mat. Fiz. Lapok* **48** (1941), 436–452].

1. INTRODUCTION

In [2] Motzkin and Straus gave nonnegative weights $x_1,..., x_n$ with $x_1 + \cdots + x_n = 1$ to the n vertices of a graph G and showed that the function

$$f_G(x) = \sum_{(ij) \in E} x_i x_j$$

summed over the edges of G, has maximal value $\frac{1}{2}(1 - 1/k)$, where k is the order of a maximal clique in G. This maximum is attained by assigning values $x_i = 1/k$ to the vertices of the clique and 0 to all other vertices.

In this article we derive a family of direct generalizations of this result by attaching weights to complete subgraphs of graphs, or more generally, to subsets of edges of homogeneous hypergraphs and then define functions on graphs of a fixed order in terms of the weights of their complete subgraphs. We then maximize the sum of these functions over all the subgraphs of a certain type of given order in the graph.

The chief analytic tool is a simple lemma.

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LEMMA 1.1. *Let*

$$f(x_1,\dots,x_m; y_1,\dots,y_n) = g(x_1,\dots,x_m) + h(y_1,\dots,y_n) + C,$$

where g, h *are continuous homogeneous functions of degree* l *for* $x_i \geqslant 0$, $y_j \geqslant 0$ *and* C *is a constant. Then* f *attains a maximum in the domain* D *given* $x_i \geqslant 0$ $(i = 1,\dots,m)$, $y_j \geqslant 0$ $(j = 1,\dots,n)$, $\sum_{i=1}^m x_i^l + \sum_{j=1}^n y_j^l = c$, *where* c *is a fixed nonnegative constant, at a point* $(\bar{x}_1,\dots,\bar{x}_m; \bar{y}_1,\dots,\bar{y}_n)$, *where either* $\bar{x}_1 = \cdots = \bar{x}_m = 0$ *or* $\bar{y}_1 = \cdots = \bar{y}_n = 0$.

Proof. We want to show that a maximum point in D with a minimal number of nonzero coordinates has the desired property. For this purpose we assume that there exists a maximum point $(\bar{\bar{x}}_1,\dots,\bar{\bar{x}}_m; \bar{\bar{y}}_1,\dots,\bar{\bar{y}}_n)$ in D with $a^l = \sum_{i=1}^m \bar{\bar{x}}_i^l > 0$, $b^l = \sum_{j=1}^n \bar{\bar{y}}_j^l > 0$. Then set

$$x_i = a\xi_i,\ i = 1,\dots,m, \qquad \text{and} \qquad y_j = b\eta_j,\ j = 1,\dots,n,$$

so that

$$f(\bar{\bar{x}}; \bar{\bar{y}}) = g(\xi_1,\dots,\xi_m) \sum_{i=1}^m \bar{\bar{x}}_i^l + h(\eta_1,\dots,\eta_n) \sum_{j=1}^n \bar{\bar{y}}_j^l + C.$$

Here $g(\xi_1,\dots,\xi_m)$ is a constant

$$A = g(\xi_1,\dots,\xi_m) = \max g(x_1,\dots,x_m)$$

with the maximum taken over all x with $x_i \geqslant 0$, $\sum_{i=1}^m x_i^l = 1$. Similarly $h(\eta_1,\dots,\eta_n)$ is a constant

$$B = h(\eta_1,\dots,\eta_n) = \max h(y_1,\dots,y_n)$$

with the maximum taken over all y with $y_j \geqslant 0$, $\sum_{j=1}^n y_j^l = 1$. Now

$$\begin{aligned} f(\bar{\bar{x}}, \bar{\bar{y}}) &= A \sum_{i=1}^m \bar{\bar{x}}_i^l + B \sum_{j=1}^n \bar{\bar{y}}_j^l + C \\ &= (A - B) \sum_{i=1}^m \bar{\bar{x}}_i^l + Bc + C \\ &= (B - A) \sum_{j=1}^n \bar{\bar{y}}_j^l + Ac + C \end{aligned}$$

can be a maximum only if $A = B$. But in that case the value of f remains unchanged if we replace $\bar{\bar{x}}_i$ by $s\bar{\bar{x}}_i$ $(i = 1,\dots,m)$ and $\bar{\bar{y}}_j$ by $t\bar{\bar{y}}_j$ $(j = 1,\dots,n)$, where $s \geqslant 0$, $t \geqslant 0$ and

$$s^l \sum_{i=1}^m \bar{\bar{x}}_i^l + t^l \sum_{j=1}^n \bar{\bar{y}}_j^l = c.$$

In particular, by choosing $s=0$ or $t=0$, we get the desired maximum point.

Since the simple idea used in Lemma 1.1 is the transfer of the values of one set of variables to another set of variables, we shall refer to this lemma as the Transfer Lemma.

In order to illustrate the method without getting lost in generality we state the following important special case. Attach a nonnegative weight $x(H_l)$ to every complete l-subgraph H_l of a graph G, normalized by the condition

$$\sum_{H_l \subseteq G} x(H_l)^l = 1.$$

To every complete $(l+1)$-subgraph H_{l+1} of G attach the weight

$$x(H_{l+1}) = \prod_{H_l \subset H_{l+1}} x(H_l).$$

Then define

$$f_G(x) = \sum_{H_{l+1} \subseteq G} x(H_{l+1}).$$

Then we get the following.

THEOREM 1.2. $\max_x f_G(x) = \binom{k}{l+1} / \binom{k}{l}^{(l+1)/l}$, *where k is the order of a maximal clique K of G. This maximum is attained by attaching weights $\binom{k}{l}^{-1/l}$ to the l-subgraphs of K and weight* 0 *to all other complete l-subgraphs.*

The case $l=1$ is the Motzkin–Straus Theorem. Just as that theorem gave a new and simple proof of Turán's Theorem [3] concerning the maximum number of edges in a graph on n vertices without k-cliques, so the present theorem establishes a generalization, a special case of which was conjectured by Erdös [1], concerning the maximum number of k-cliques in a graph with a given number of edges (but unspecified number of vertices) which contains no complete $(k+1)$-subgraph.

Proof of Theorem 1.2. Let u, v be two independent vertices of G and let $x(H_l)$ be a weight function giving a maximal value of $f_G(x)$. Then write $f(x) = g_u(x) + g_v(x) + g_0(x)$, where

$$g_u(x) = \sum_{u \in H_{l+1}} x(H_{l+1}),$$

$$g_v(x) = \sum_{v \in H_{l+1}} x(H_{l+1}),$$

$$g_0(x) = \sum_{u, v \notin H_{l+1}} x(H_{l+1}).$$

We now divide the variables $x(H_l)$ into three classes

$$X_u = \{x(H_l) \mid u \in H_l\},$$
$$X_v = \{x(H_l) \mid v \in H_l\},$$
$$X_0 = \{x(H_l) \mid u, v \notin H_l\}$$

and hold X_0 fixed. Then g_u is homogeneous of degree l in the variables in X_u and g_v is homogeneous of degree l in the variables in X_v. Also

$$\sum_{u \in H_l} x(H_l)^l + \sum_{v \in H_l} x(H_l)^l = 1 - \sum_{u,v \notin H_l} x(H_l)^l = c.$$

So the Transfer Lemma applies and we get a maximal value where either all variables in X_u or all variables in X_v are 0. Thus the maximal weight function $x(H_l)$ with a minimal number of nonzero weights assigns all its positive weights to the subgraphs of a clique H_k of order k.

It is now easy to see that f_G is maximal if all $H_l \subset H_k$ are given equal weight, $\binom{k}{l}^{-1/l}$, which yields

$$f_G(x) = \binom{k}{l+1} \Big/ \binom{k}{l}^{(l+1)/l}.$$

The right side is an increasing function of k and this proves the theorem.

COROLLARY. *Let L_l denote the number of complete l-subgraphs of a graph G. Then, if $L_{k+1} = 0$, we have*

$$L_{l+1} \leqslant \binom{k}{l+1} \binom{k}{l}^{-(l+1)/l} L_l^{(l+1)/l}.$$

Proof. If we set $x(H_l) = L_l^{-1/l}$ for all $H_l \subseteq G$, we get, by Theorem 1.2,

$$f_G(x) = L_{l+1}/L_l^{(l+1)/l} \leqslant \binom{k}{l+1} \Big/ \binom{k}{l}^{(l+1)/l}.$$

The case $l = 2$ shows that a graph with E edges and no complete $(k+1)$-graphs contains T triangles where

$$T \leqslant \binom{k}{3} \binom{k}{2}^{-3/2} E^{3/2}$$

which is the value obtained for Turán graphs, as conjectured by Erdös [1].

In Section 2 we give various generalizations of Theorem 1.2 which follow from the method of proof. In Section 3 we characterize the weight distributions which give maximal values. In Section 4 we give applications to graphs and hypergraphs.

2. The Main Theorem

We first state a general combinatorial inequality and then give a few more concrete versions.

THEOREM 2.1. *Given a graph G containing (not necessarily disjoint) sets of vertices S, T and U. To each $t \in T$ we assign a nonnegative weight $x(t)$ so that $\sum_{t\in T} x(t)^l = 1$ and to each $u \in U$ we assign a weight $y(u)$ which is a continuous function of the $x(t)$ where (tu) is an edge of G so that $y(u)$ is homogeneous of degree l in each set of variables*

$$X_{s,u} = \{x(t) \mid (st) \in G,\ (su) \in G,\ (tu) \in G\}.$$

Finally, let $f_G(x) = \sum_{u\in U} y(u)$. Then $f_G(x)$ attains a maximum for a distribution $x(t)$ in which any two elements $s_1, s_2 \in S$ for which there exist $u_1, u_2 \in U$ so that both X_{s_1,u_1} and X_{s_2,u_2} contain positive elements have the property of being connected to a common element u of U. In other words, if we construct the graph G_S with vertex set S where two vertices s_1, s_2 are connected if and only if there exists a $u \in U$ so that $(s_1u), (s_2u) \in G$, then all s for which some $X_{s,u}$ has positive elements are vertices of a complete subgraph of G_S.

Proof. Define

$$X_s = \bigcup_{u\in U} X_{s,u}.$$

If $X_{s_1} \cap X_{s_2} = \varnothing$ then we can divide the set of variables $X = \{x(t) \mid t \in T\}$ into three sets $X_{s_1} \cup X_{s_2} \cup X_0$ and write

$$f_G(x) = g_{s_1}(x) + g_{s_2}(x) + g_0(x),$$

where

$$g_{s_1}(x) = \sum_{(s_1u)\in G} y(u),$$

$$g_{s_2}(x) = \sum_{(s_2u)\in G} y(u),$$

$$g_0(x) = \sum_{\substack{(s_1u)\notin G\\(s_2u)\notin G}} y(u).$$

If X_0 is held fixed then $g_{s_1}(x)$ is a homogeneous function of degree l in the variables in X_{s_1} and $g_{s_2}(x)$ is a homogeneous function of degree l in the variables in X_{s_2} where

$$\sum_{x(t)\in X_{s_1}} x(t)^l + \sum_{x(t)\in X_{s_2}} x(t)^l = 1 - \sum_{x(t)\in X_0} x(t)^l = c$$

so that, by the Transfer Lemma, a maximum is attained with all elements of X_{s_1} or X_{s_2} equal to 0. Thus the maximum $x(t)$ with a minimal number of nonzero values has the desired property.

Now consider a hypergraph G of u-tuples where $1 \leqslant s \leqslant t \leqslant u$ and attach nonnegative weights $x(T)$ to every t-tuple $T \subset G$ subject to the condition

$$\sum_{T\subset G} x(T)^{\binom{u-s}{t-s}} = 1.$$

To each edge U of G attach weight

$$y(U) = A \sum_{T\subset U} x(T)^{\binom{u-s}{t-s}} + B \prod_{T\subset U} x(T), \qquad A, B \text{ real, not both } 0,$$

and define $f_G(x) = \sum_{U\in G} y(U)$, summed over the edges of G. We are interested in the maximum of $f_G(x)$. This maximum may be trivial, for example, because G contains no edge or because it is the value 0, attained by attaching weight 1 to a single t-tuple which is not contained in any edge. To the hypergraph G we associate the ordinary graph G_s whose vertices are the s-tuples of G and wherein two s-tuples S_1, S_2 of G are connected if $S_1 \cup S_2$ belongs to an edge U of G. We define $w(S) = \sum_{S\subseteq T} x(T)$.

THEOREM 2.2. *If $f_G(x)$ attains a nontrivial maximum then it attains a maximum for a weight distribution $x(T)$ where the set*

$$\{S \mid |S| = s, w(S) > 0\}$$

induces a complete subgraph of G_s.

Proof. We apply Theorem 2.1 where we construct an auxiliary graph G^* whose vertices are the set S of s-tuples of vertices of G, the set T of t-tuples of vertices of G and the set U of edges of G. Two vertices of G^* are connected if one of the sets includes the other.

As an example of both the range and the limitations of Theorem 2.1 we mention the following.

EXAMPLE 2.3. Let H be a triangle-free graph and let G be the hypergraph of 4-cycles of H. Since H is triangle-free each 4-cycle is determined by its set of vertices and thus G is a homogeneous 4-hypergraph. In Theorem 2.1 choose S the vertices, T the edges and U the 4-cycles of G with connection by inclusion. Attach weights $x(T)$ to the edges T of H and the weight $y(U) = \prod_{T \subset U} x(T)$ to the 4-cycles. With the normalization $\sum_{T \in H} x(T)^2 = 1$ we attain a maximum for $f_G(x) = \sum_{U \in G} y(U)$ by limiting attention to a subgraph H' of H in which every two points belong to a common 4-cycle. In particular H' has diameter $\leqslant 2$ and every two vertices at distance 2 are joined by at least two 2-chains. It remains to maximize $f_G(x)$ over the possible choices of H'. Once this maximum M_G is found we have $C_4 \leqslant M_G E^2$, where C_4 is the number of 4-cycles of H.

3. WEIGHT DISTRIBUTIONS WHICH GIVE MAXIMAL VALUES

DEFINITION 3.1. A graph is maximally k-chromatic (complete k-partite) if it is k-chromatic and if the addition of an edge makes it $(k+1)$-chromatic.

Now we again restrict attention to hypergraphs which are complete-subgraph graphs of an ordinary graph G and to the weight function $y(U)$ which attaches the product $\prod_{T \subset U} x(T)$ of the complete t-subgraphs of U to a complete u-subgraph U. That is, we set $A = 0$, $B = 1$ in the definition in Section 2.

Using the notation of Section 2 we assign to each complete subgraph S of G_2 the weight $w(S)$. We then can formulate a partial converse of Theorem 2.1.

THEOREM 3.2. *Let G be the hypergraph whose edges are the complete u-subgraphs of an ordinary graph G_2 and let k be the order of a maximal clique of G_2. Then with the above definition of $y(U)$ the function $f_G(x)$ attains its maximum only when the subgraph of G_s induced by those S for which $w(S) > 0$ is maximally $\binom{k}{s}$-chromatic.*

Proof. By induction on the number n of vertices in the set $N = \bigcup_{w(S)>0} S$.

We first show that we must have $n \geqslant k$ and that for $n = k$ we get a maximum only if N is the set of vertices of a k-clique K of G_2. If this were not so then, by Theorem 2.1, the function $f_G(x)$ would also attain its maximum where N is a set with $|N| < k$ elements. The value of $f_G(x)$ we get by setting

$$x(T) = \binom{k}{t}^{-1/\binom{u-s}{t-s}}, \qquad T \subset K,$$

is

$$f_G = \binom{k}{u}\binom{k}{t}^{-\binom{u}{t}/\binom{u-s}{t-s}}$$

which is an increasing function of k and cannot be attained when $|N| < k$.

Now assume $n > k$ and the theorem is true for $n-1$. Since not all the vertices of N are connected in G_2, there are two vertices v_1 and v_2 in N which are not connected. By the Transfer Lemma we can transfer the weights $w(S)$ with $v_1 \in S$ to some $w(S^*)$ with $v_2 \in S^*$ and get the same maximal value. Thus the s-tuples S with $w(S) > 0$ and $v_1 \in S$ form a maximally $\binom{k}{s}$-chromatic subgraph H of G_s. Now if for some S_1 with $w(S_1) > 0$, $v_1 \in S_1$ were connected to all the color classes of H then this would violate the maximality of k. Thus we may assume that S_1 is not connected to any s-tuple in the color class of S^*, but then S_1 must be connected to all s-tuples in the other color classes or else the transfer of its weight to $w(S^*)$ would have caused a strict increase in f_G. Thus we adjoin all the s-tuples with $v_1 \in S$ and $w(S) > 0$ to the class of S^* and the graph remains maximally $\binom{k}{s}$-chromatic.

4. A Conjecture of Erdös and Related Applications

We already stated the conjecture of Erdös [1] concerning the maximal number of complete l-subgraphs in a graph without $(k+1)$-cliques and with a given number of edges but an unspecified number of vertices. As a result of Theorems 2.2 and 3.2 we can now give a more complete result.

THEOREM 4.1. *Let $1 \leqslant l < m \leqslant k$ and L_j denote the number of complete j-subgraphs of the graph G. If $L_{k+1} = 0$ then*

$$L_m \leqslant \binom{k}{m}\binom{k}{l}^{-m/l} L_l^{m/l}.$$

Equality is attained for Turán graphs (complete k-partite graphs with equal parts) and only for Turán graphs.

Proof. We apply Theorem 2.2 with $u = m$, $t = l$, $s = 1$ to the hypergraph G_u of complete u-subgraphs of G. Then

$$\max f_{G_u}(x) \leqslant \binom{k}{m}\binom{k}{l}^{-\binom{m}{l}/\binom{m-1}{l-1}} = \binom{k}{m}\binom{k}{l}^{-m/l}.$$

Now, if we set $x(L) = L_l^{-1/\binom{m-1}{l-1}}$ for all complete l-subgraphs L of G, then

$$f_{G_u}(x) = L_m L_l^{-\binom{m}{l}/\binom{m-1}{l-1}} = L_m L_l^{-m/l} \leqslant \max f_{G_u}(x)$$
$$\leqslant \binom{k}{m}\binom{k}{l}^{-m/l}$$

which proves the inequality.

By Theorem 3.2 equality can only hold when the vertices of the complete m-subgraphs form a complete k-partite graph. It is now easy to see that the maximum is attained only when the parts are of equal magnitude.

Finally we consider an example of the general weight function described in Section 2. Here the character of the information changes considerably. For the simplest case let $s = t = 1$ and $u = 2$. Attach nonnegative weights x_i to the vertices $i = 1, 2,..., n$ of the graph G so that $\sum x_i = 1$ and give the weight $x(ij) = a(x_i + x_j) + x_i x_j$ to the edge (ij). By Theorem 2.2 we know that the function $f_G(x) = \sum_{(ij) \in G} x(ij)$ attains a maximum if all the weights are attached to the vertices of a complete subgraph K of G, though in this case we cannot be sure that K is a maximal clique. Thus

$$\max f_G(x) = \max \left(a \sum_{i \in K} v_i x_i + \frac{1}{2} \sum_{\substack{i,j \in K \\ i \neq j}}\sum x_i x_j \right),$$

where v_i is the valence (in G) of the vertex i. Now, since $\sum x_i = 1$, we have

$$\sum_{i \neq j} \sum x_i x_j = \sum_{i \in K} x_i(1 - x_i) = 1 - \sum_{i=1}^{k} x_i^2.$$

Thus

$$\max f_G(x) = \tfrac{1}{2} + \max \sum_{i \in K} (a x_i v_i - \tfrac{1}{2} x_i^2). \tag{4.2}$$

If $0 < x_i < 1$ then differentiation yields, with Lagrange multiplier λ of $x_1 + \cdots + x_n - 1$,

$$a v_i - x_i = \lambda, \qquad x_i = a v_i - \lambda,$$

$$1 = \sum x_i = a \sum v_i - k\lambda.$$

So, setting $\sum_{i \in K} v_i = v(K)$, we get

$$\lambda = \frac{1}{k}(a v(K) - 1),$$

$$x_i = \alpha(v_i - \bar{v}) + \frac{1}{k}, \qquad \bar{v} = \frac{v(K)}{k}.$$

These x_i satisfy our inequality $0 < x_i < 1$ only when

$$\bar{v} - v_{\min} < \frac{1}{k\alpha} \quad \text{when} \quad \alpha > 0 \tag{4.3}$$

and

$$v_{\max} - \bar{v} < \frac{1}{k|\alpha|} \quad \text{when} \quad \alpha < 0. \tag{4.4}$$

If (4.3) and (4.4) are satisfied then the maximum in (4.2) is

$$\begin{aligned} &\frac{1}{2} + \alpha\frac{1}{k}\sum v_i + \alpha^2 \sum v_i(v_i - \bar{v}) \\ &\quad - \frac{1}{2}\frac{1}{k} - \frac{\alpha}{k}\sum (v_i - \bar{v}) - \frac{\alpha^2}{2}\sum (v_i - \bar{v})^2 \\ &= \frac{1}{2}\left(1 - \frac{1}{k}\right) + \alpha\bar{v} + \frac{\alpha^2}{2}\sum (v_i - \bar{v})^2 \\ &= \frac{1}{2}\left(1 - \frac{1}{k}\right) + \alpha\bar{v} + \frac{\alpha^2}{2}\sigma^2, \end{aligned} \tag{4.5}$$

where $\sigma^2 = \sigma^2(K) = \sum_{i \in K} (v_i - \bar{v})^2$.

We now introduce the following notation.

$$\bar{V} = \frac{2E}{n} = \text{average valence of } G,$$

$$\Sigma^2 = \sum_{i \in G} (v_i - \bar{V})^2,$$

P_3 = number of 3-chains of G,

T = number of triangles of G.

THEOREM 4.6. *If G is triangle-free then*

$$P_3 \leqslant E^2 - 2E\bar{V} - \Sigma^2 + E.$$

Proof. We postpone the use of the hypothesis that G is triangle-free and make the choice $x_i = v_i/2E$ for the weights of the vertices of G. Then

$$(4.7)\quad f_G(v_1/2E,\dots,v_n/2E)=\frac{\alpha}{2E}\sum_{(i,j)\in G}(v_i+v_j)+\frac{1}{4E^2}\sum_{(i,j)\in G}v_iv_j$$

$$=\frac{\alpha}{2E}\sum_{i=1}^{n}v_i^2+\frac{1}{4E^2}\sum_{(i,j)\in G}((v_i-1)(v_j-1)+v_i+v_j-1)$$

$$=\left(\frac{\alpha}{2E}+\frac{1}{4E^2}\right)\sum v_i^2-\frac{1}{4E}+\frac{1}{4E^2}\sum_{(i,j)\in G}(v_i-1)(v_j-1).$$

Now

$$\Sigma^2=\sum(v_i-\bar{V})^2=\sum v_i^2-4E\bar{V}+n\bar{V}^2=\sum v_i^2-2E\bar{V}$$

and

$$\sum_{(ij)\in G}(v_i-1)(v_j-1)=P_3+3T.$$

Substituting in (4.7) we get

$$(4.8)\qquad 4E^2f_G\left(\frac{v_i}{2E},\dots,\frac{v_n}{2E}\right)=2E\alpha(2E\bar{V}+\Sigma^2)$$

$$+2E\bar{V}+\Sigma^2-E+P_3+3T.$$

Comparison with (4.5) yields

$$(4.9)\quad P_3+3T\leqslant 2E^2-2E\bar{V}-\Sigma^2+E$$

$$+\max_{K\subseteq G}\left[-2E^2/k+\min_{\alpha}\{(4E^2(v-\bar{V})-2E\Sigma^2)\,\alpha+2E^2\sigma^2\alpha^2\}\right],$$

where K ranges over the complete subgraphs of G and α ranges over the interval $(1/(k(\bar{v}-v_{\max})),\ 1/(k(\bar{v}-v_{\min})))$. Now assume that G is triangle-free, that is, $T=0$. Then $k=1$ or 2. In particular, if $\alpha\leqslant 0$ then $\max f_G(x)$ must be attained on an edge K. Here we have only two valences v_i, v_j with, $v_i\geqslant v_j$ and $\bar{v}=(v_i+v_j)/2$, $\sigma^2=(v_i-v_j)^2/4$ and

$$(4.10)\qquad P_3\leqslant E^2-2E\bar{V}-\Sigma^2+E$$

$$+\min_{\alpha}\max_{(ij)\in G}\left[(4E^2(\bar{v}-\bar{V})-2E\Sigma^2)\,\alpha+2E^2\sigma^2\alpha^2\right].$$

The choice $\alpha = 0$ in (4.9) yields

$$P_3 \leqslant E^2 - 2E\bar{V} - \Sigma^2 + E. \tag{4.11}$$

If there is an edge (ij) for which

$$2\bar{v} = v_i + v_j \leqslant 2\bar{V} + \Sigma^2/E, \tag{4.12}$$

then we cannot improve on (4.11) with negative values of α. However, if we average the left side of (4.12) over all edges of G we get the right side of (4.12). So (4.12) must hold for some edge. For an improvement on (4.11) with $\alpha > 0$ we would need the inequality in the sense opposite to (4.12) for all edges. This is impossible for the same reason as before. So the generality given by the choice of α is illusory in this case.

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