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LARGEST DIGRAPHS CONTAINED IN ALL n-TOURNAMENTS

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Let $f(n)$ (resp. $g(n)$) be the largest m such that there is a digraph (resp. a spanning weakly connected digraph) on n-vertices and m edges which is a subgraph of every tournament on n-vertices. We prove that

$$n \log_2 n - c_1 n \geqq f(n) \geqq g(n) \geqq n \log_2 n - c_2 n \log\log n.$$

A directed graph G is *an unavoidable subgraph of all n-tournaments* or, simply *n-unavoidable,* if every tournament on n vertices contain san isomorphic copy of G, i.e., for each n-tournament T there exists an edge preserving injection of the vertices of G into the vertices of T. The problem of showing certain types of graphs to be n-unavoidable has been the subject of several papers, for example, it is known that every n-tournament contains a Hamiltonian path ([7]), an anti-directed Hamiltonian path ([4]) and a transitive subtournament on $[\log_2 n]$ vertices ([6]). Results of this type are also found in [1], [3], and [8]. In this paper we answer the following question: what is the maximum number of edges that an n-unavoidable subgraph can have?

Our graph theoretic terminology is standard. For a vertex v of a digraph $G=(V, E)$ we let $G^+(v)=\{w|\langle v, w\rangle \in E\}$. All logarithms are base 2.

Let $f(n)$ (resp. $g(n)$) be the largest m such that there exists a digraph resp. spanning, weakly connected diagraph) with m edges that is n-unavoidable subgraph. Trivially $f(n) \geqq g(n)$. Our main resullt is

Theorem. *There exists positive constants c_1 and c_2 such that for all positive integers n,*

$$n \lg n - c_1 n \geqq f(n) \geqq g(n) \geqq n \lg n - c_2 n \lg\lg n.$$

Proof. We start with the left inequality. Let $V=\{1, \ldots, n\}$ and let H be an n-unavoidable digraph on V with m edges. There are $2^{\binom{n}{2}}$ labeled n-tournaments on V, each of which contains $\theta(H)$ where θ is a permutation on V. For fixed θ,

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AMS subject classification (1980): 05 C 20, 05 C 35, 05 C 55 exactly $2^{\binom{n}{2}-m}$ labeled tournaments on V contain $\theta(H)$. Hence,

$$2^{\binom{n}{2}} \leqq n! \cdot 2^{\binom{n}{2}-m}$$

so

$$m \leqq \lg n! \leqq n \lg n - c_1 n,$$

for an appropriate c_1.

To prove the inequality on the right we proceed by a sequence of propositions. They provide an inductive construction for a spanning weakly connected n-unavoidable digraph with $n \lg n - c_2 n \lg \lg n$ edges.

For positive integers k and r we define $D(k, r)$ to be the complete bipartitite graph between vertex sets V_1 and V_2 of sizes k and r respectively, with every edge directed from v_1 to v_2.

Proposition 1. *If* $r \leqq (n+1)/2^k - 1$ *then* $D(k, r)$ *is n-unavoidable digraph.*

Proof. Let T be any n-tournament; we show by induction on k that the specified graphs are subgraphs of T. If $k=1$ then $r \leqq \frac{n-1}{2}$, so let V_1 consist of some vertex v of out degree at least $\frac{n-1}{2}$ and V_2 be a subset of $G^+(v)$ of size r. For $k>1$ and $r \leqq (n+1)/2^k - 1$ the numbers $k-1, 2r-1$ meet the conditions of the induction hypothesis so T contains the specified bipartite graph on vertex sets V_1' and V_2', with size $k-1$ and $2r+1$. Choose a vertex $w \in V_2'$ having out degree at least r in the subtournament spanned by V_2' and let $V_1 = V_1' \cup \{w\}$ and V_2 be any r vertices in $V_2' \cap G^+(w)$. All edges in T point from V_1 to V_2 so the required subgraph can be constructed. ∎

Proposition 2. *There exists a constant* $c_3>0$ *such that for all positive integers* n.

$$f(n) \geqq n \lg n - c_3 n \lg \lg n$$

Proof. Let $h(n) = n \lg n - c_3 n \lg \lg n$ (leaving c_3 unspecified) and let k and r be integers satisfying the hypothesis of Proposition 1. Every n-tournament contains $D(r)$, which has kr edges and, disjoint from this, a maximum $(n-k-r)$-unavoidable subgraph since the remaining vertices span an $(n-k-r)$-tournament. Thus

$$f(n) \geqq kr + f(n-k-r).$$

It suffices to show that k, r and c_3 can be chosen so that, for n sufficiently large

$$h(n) - h(n-k-r) \leqq kr.$$

Using $\lg (n-k-r) \geqq \lg n - \frac{k+r}{n-k-r} \lg e$ for $n \geqq k+r$ we have by routine computation:

$$h(n) - h(n-k-r) \leqq (k+r)(\lg n + \lg e - c_3 \lg \lg n)$$

Choose $k = \lfloor \lg n - 2 \lg \lg n \rfloor$ and $r = \lfloor \lg^2 n \rfloor - 1$. It is easily checked that k and r satisfy the condition on Proposition 1. Further

$$\begin{aligned} kr &\geqq (\lg n - 2 \lg \lg n - 1)(\lg^2 n - 2) \\ &\geqq \lg^3 n - 2 \lg \lg n \lg^2 n - \lg^2 n - 2 \lg n \\ &\geqq (k+r)(\lg n + \lg e - c_3 \lg \lg n), \end{aligned}$$

if c_3 is chosen bigger than 2 and n is large enough, Combining this with the previous inequality completes the proof. ∎

Proposition 3. *Every n-tournament has a vertex v and a partition V_1, V_2 of the remaining vertices with $|V_1|=\left\lceil\frac{n-1}{2}\right\rceil$ and $|V_2|=\left\lfloor\frac{n-1}{2}\right\rfloor$, so that $V_1\subseteq G^+(v)$ but $V_1\not\subseteq G^+(w)$ for any $w\in V_2$.*

Proof. Let T be an n-tournament and choose v, V_1, V_2 so that $|V_1|=\left\lceil\frac{n-1}{2}\right\rceil$, $|V_2|=\left\lfloor\frac{n-1}{2}\right\rfloor$, $V_1\subseteq G^+(v)$ and the size of $V_2'=\{w\in V_2\colon V_1\not\subseteq G^+(w)\}$ is maximum. We need to show that $V_2'=V_2$. Suppose not. Construct a set V_1' by selecting, for each $w\in V_2'$, a vertex x in V_1 so that $x\notin G^+(w)$. Since $|V_2'|<|V_2|$ we have $|V_1'|<|V_1|$. Choose $u\in V_2-V_2'$ and $x\in V_1-V_1'$. If (v, u) is an arc of T then exchanging u and x increases $|V_2'|$ and if (u, v) is in T then exchanging u with v and then v with x also increases $|V_2'|$, contradicting the maximality of V_2' so $V_2'=V_2$. ∎

Proposition 4.

$$g(n)\geq f\left(\left\lceil\frac{n-1}{2}\right\rceil-1\right)+g\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right)+\left\lceil\frac{n+1}{2}\right\rceil.$$

Proof. Let G' be a graph on vertex set $V_1'\cup V_2'\cup\{v', w'\}$ where $|V_1'|=\left\lceil\frac{n-1}{2}\right\rceil-1$, $|V_2'|=\left\lfloor\frac{n-1}{2}\right\rfloor$, with edges as follows. The subgraph spanned by V_1' is a maximum $\left(\left\lceil\frac{n-1}{2}\right\rceil-1\right)$-unavoidable digraph G_1', and that spanned by V_2' is a maximum spanning connected $\left\lfloor\frac{n-1}{2}\right\rfloor$-unavoidable digraph G_2'. In addition G' has edges $\langle v', w'\rangle$, $\langle v', y'\rangle$ for each $y\in V_1'$ and $\langle w', x'\rangle$ for exactly one $x'\in V_2'$ (see Figure 1).

G' is connected and spanning and has $f\left(\left\lceil\frac{n-1}{2}\right\rceil-1\right)+g\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right)+\left\lceil\frac{n+1}{2}\right\rceil$ edges; we now show that G' is n-unavoidable.

Let T be any n-tournament and choose v, V_1, V_2 according to Proposition 3. The subtournament of T spanned by V_2 contains a copy G_2 of G_2'; let $x\in V_2$ be the vertex corresponding to $x'\in V_2'$. Since $V_1\not\subseteq G^+(x')$, there is a vertex $w\in V_1$ with $\langle w, x\rangle$ in T. The subtournament on $V_1-\{w\}$ contains a copy G_1 of G_1'.

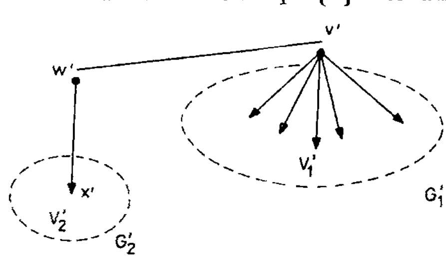

Fig. 1

Taking G_1 and G_2 together with the edges from v to V_1 and $\langle w, x\rangle$ yields a copy G of G'. ▮

Using induction and Proposition 4 we can now prove the last inequality of the theorem. By Propositions 4 and 2 and the induction hypothesis

$$
\begin{aligned}
g(n) &\geqq f\left(\frac{n-3}{2}\right)+g\left(\frac{n-3}{2}\right)\\
&\geqq \left(\frac{n-3}{2}\right)\lg\left(\frac{n-3}{2}\right)-c_3\left(\frac{n-3}{2}\right)\lg\lg\left(\frac{n-3}{2}\right)+\frac{n-3}{2}\lg\left(\frac{n-3}{2}\right)-c_2\lg\lg\left(\frac{n-3}{2}\right)\\
&\geqq (n-3)\lg(n-3)-(n-3)-(c_3+c_2)\left(\frac{n-3}{2}\right)\lg\lg\left(\frac{n-3}{2}\right)\\
&\geqq n\lg n-c_2 n\lg\lg n
\end{aligned}
$$

as long as c_2 is any number greater than c_3 and n is sufficiently large. This completes the proof of the theorem. ▮ ▮

Remark. In a forthcoming paper we investigate different problems concerning n-unavoidable graphs. Some classes of rooted directed trees that are or are not unavoidable are identified. In particular we consider the class of claws, rooted digraphs in which each branch is a path. We also produce, for each n, a spanning rooted digraph of small depth that is n-unavoidable.

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