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PROBLEMS AND RESULTS ON ADDITIVE PROPERTIES OF GENERAL SEQUENCES, III

P. ERDŐS, A. SÁRKÖZY and V. T. SÓS

To Professor L. Fejes Tóth on his seventieth birthday

1. Let $\mathcal{A}=\{a_1, a_2, \dots\}$ $(a_1<a_2<\dots)$ be an infinite sequence of positive integers, put

$$A(n) = \sum_{\substack{a\in\mathcal{A}\\ a\le n}} 1,$$

and for $n=0$, 1, 2, ..., let $R(\mathcal{A}, n)$ or briefly R(n) denote the number of solutions of

$$a_x+a_y = n, \quad a_x\in\mathcal{A},\ a_y\in\mathcal{A}.$$

In Part I of this paper [3], P. Erdős and A. Sárközy proved that if F(n) is an arithmetic function satisfying $F(n)\to+\infty$, $F(n+1)\ge F(n)$ for $n\ge n_0$ and $F(n)= =o\big(n(\log n)^{-2}\big)$, then

$$R(n) - F(n) = o\left((F(n))^{1/2}\right)$$

cannot hold. In Part II [4], they showed that this theorem is nearly best possible. (See [1], [2] and [5] for further related results and problems.) In this paper, we continue the study of the regularity properties of the function R(n), In fact, here our goal is to show that under certain possibly simple assumptions on $\mathcal{A}$, $|R(n+1)-R(n)|$ cannot be bounded.

If $\mathcal{A}$ is "very thin" $\big(\mathcal{A}(n)=o(\sqrt[3]{n})\big)$ then R(n) can be bounded and then also $|R(n+1)-R(n)|$ is bounded. On the other hand, if $\mathcal{A}$ is "very dense" (e.g. $\mathcal{A}=\{1, 2, \dots, n, \dots\}$) then clearly, $|R(n+1)-R(n)|$ can be bounded again. One may guess that if $\mathcal{A}$ is not "very thin" and not "very dense" then $|R(n+1)-R(n)|$ cannot be bounded. This is not so as the following theorem shows:

THEOREM 1. Let $S_1<S_2<\dots$ and $t_1<t_2<\dots$ *be positive integers satisfying*

$$S_k < t_k \le \frac{S_{k+1}}{2}+1 \qquad (\text{for } k = 1, 2, \dots). \tag{1}$$

and put

$$\mathcal{A} = \{a_1, a_2, \dots\} = \bigcup_{k=1}^{+\infty} \{S_k, S_k+1, \dots, t_k\}.$$

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Then we have

$$|R(n+1)-R(n)| \leqq 3$$

for all n.

This theorem shows that if $\mathscr{A}$ consists of a "few" blocks of consecutive integers then $|R(n+1)-R(n)|$ can be bounded (independently of the counting function $A(n)$). On the other hand, one may guess that if the number of these blocks up to n, i.e.,

$$B(\mathscr{A}, n) = \sum_{\substack{a \leqq n \\ a-1 \notin \mathscr{A}, a \in \mathscr{A}}} 1$$

is "large" (in terms of n), then $|R(n+1)-R(n)|$ cannot be bounded. In fact, we will prove the following theorem:

THEOREM 2. *If $\mathscr{A} \neq \emptyset$ then*

$$S(N) \overset{\text{def}}{=} \sum_{n=1}^{N} (R(n+1)-R(n))^2 = o\big((B(\mathscr{A}, n))^2\big) \tag{2}$$

cannot hold.

The following corollaries are trivial consequences of Theorem 2.

COROLLARY 1. *If $\mathscr{A} \neq \emptyset$, then*

$$\max_{n \leqq N} |R(n+1)-R(n)| = o\big((B(\mathscr{A}, N)/N^{1/2})^{1/2}\big) \tag{3}$$

cannot hold.

In fact, Theorem 2 says that (3) is impossible in square mean.

COROLLARY 2. *If*

$$\lim_{N \to +\infty} \frac{B(\mathscr{A}, N)}{N^{1/2}} = +\infty$$

then $|R(n+1)-R(n)|$ cannot be bounded.

This is a consequence of Corollary 1.

Finally, we will show that Theorem 2 is nearly best possible:

THEOREM 3. *For all $\varepsilon > 0$, there exists an infinite sequence $\mathscr{A}$ such that*

$$B(\mathscr{A}, n) \gg N^{1/2-\varepsilon} \tag{i}$$

and

(ii) *$R(n)$ is bounded (in fact, $R(n) < 3+\varepsilon^{-1}$ for large n) so that also $|R(n+1)-R(n)|$ is bounded.*

Furthermore, by using a construction of Erdős, we can show that there exists an infinite sequence $\mathscr{A}$ such that

$$\lim_{N \to +\infty} \sup \frac{B(\mathscr{A}, N)}{N^{1/2}} > 0$$

and $|R(n+1)-R(n)|$ is bounded.

There is some gap between the lower bounds and upper bounds given above. In fact, we conjecture that Corollary 2 can be sharpened in the following way: If

$$\lim_{N\to+\infty} \sup \frac{B(\mathscr{A}, N)}{N^{1/2}} = +\infty$$

or

$$\lim_{N\to+\infty} \inf \frac{B(\mathscr{A}, N)}{N^{1/2}} > 0$$

(perhaps, it suffices to assume that

$$\lim_{N\to+\infty} \inf \frac{B(\mathscr{A}, N) \log N}{N^{1/2}} = +\infty),$$

then $|R(n+1)-R(n)|$ cannot be bounded. Unfortunately, we have not been able to prove this.

2. In this section, we prove Theorem 1.

For an arbitrary positive integer n, let us define the positive integer k by

$$t_{k-1} < n/2 \leqq t_k \tag{4}$$

(if $n/2 \leqq t_1$, then we put $k=1$). Then (1) and (4) yield that

$$2t_{k-1} < 2(n/2) = n \tag{5}$$

and

$$S_{k+1} \geqq 2(t_k+k) \geqq 2\left(\frac{n}{2}+1\right) = n+2. \tag{6}$$

Let m denote one of the numbers n, $n+1$. Then in view of (5) and (6)

$$a_x + a_y = m, \quad a_x \in \mathscr{A}, a_y \in \mathscr{A} \tag{7}$$

implies that

$$t_{k-1} < \max(a_x, a_y) < S_{k+1}. \tag{8}$$

By the construction of the sequence $\mathscr{A}$, we have

$$\mathscr{A} \cap \{t_{k-1}, S_{k+1}\} = \{S_k, S_k+1, \dots, t_k\}. \tag{9}$$

By (8) and (9),

$$S_k \leqq \max(a_x, a_y) \leqq t_k. \tag{10}$$

In view of (9) and (10), (a_x, a_y) is a solution of (7) if and only if it satisfies one and only one of the following equations:

$$a_x + a_y = m, \quad S_k \leqq a_x \leqq t_k, \quad a_y \leqq t_{k-1}, \quad a_y \in \mathscr{A}, \tag{11}$$

$$a_x + a_y = m, \quad a_x = t_{k-1}, \quad a_x \in \mathscr{A}, \quad S_k \leqq a_y \leqq t_k, \tag{12}$$

$$a_x + a_y = m, \quad S_k \leqq a_x \leqq t_k, \quad S_k \leqq a_y \leqq t_k. \tag{13}$$

Denoting the number of solutions of (11), (12) and (13) by $R_1(m)$, $R_2(m)$ and $R_3(m)$, respectively, clearly we have

$$R_2(m) = R_1(m)$$

and

(14) $$R(m) = R_1(m) + R_2(m) + R_3(m) = 2R_1(m) + R_3(m).$$

If a_x, a_y is a solution of (11) or (13) with n in place of m, i.e.,

(15) $$a_x + a_y = n, \quad S_k \leqq a_x \leqq t_k, \quad a_y \leqq t_{k-1}, \quad a_y \in \mathcal{A}$$

or

(16) $$a_x + a_y = n, \quad S_k \leqq a_x \leqq t_k, \quad S_k \leqq a_y \leqq t_k,$$

then $a_u = a_x + 1$, $a_v = a_y$ is a solution of

(17) $$a_u + a_v = n+1, \quad S_k \leqq a_u \leqq t_k, \quad a_v \leqq t_{k-1}, \quad a_v \in \mathcal{A},$$

or

(18) $$a_u + a_v = n+1, \quad S_k \leqq a_u \leqq t_k, \quad S_k \leqq a_v \leqq t_k,$$

respectively, except at most the solution $a_x = t_k$, $a_y = n - t_k$ of (17) or (18). On the other hand, in this way we get all the solutions of (17) and (18), except at most the solution $a_u = S_k$, $a_v = n + 1 - S_k$. Thus we have

(19) $$R_i(n) - 1 \leqq R_i(n+1) \leqq R_i(n) + 1 \quad \text{for } i = 1, 3.$$

(14) and (19) yield that

$$|R(n+1) - R(n)| = \left|2\big(R_1(n+1) - R_1(n)\big) + \big(R_3(n+1) - R_3(n)\big)\right| \leqq$$
$$\leqq 2|R_1(n+1) - R_1(n)| + |R_3(n+1) - R_3(n)| \leqq 2 \cdot 1 + 1 = 3$$

which completes the proof of Theorem 1.

3. Sections 3-6 will be devoted to the proof of Theorem 2. We start out from the indirect assumption that $\mathcal{A} \neq 0$ is a sequence satisfying (2) in Theorem 2.

First we are going to show that there exist infinitely many integers N such that

(20) $$\frac{B(\mathcal{A}, N+j)}{B(\mathcal{A}, N)} < \left(\frac{N+j}{N}\right)^2 \quad \text{for } j = 1, 2, \ldots.$$

In fact, if this inequality holds only for finitely many integers N, then there exists an integer N_0 such that

$$B(\mathcal{A}, N_0) \geqq 1$$

and for $N \geqq N_0$ there exists an integer $N' = N'(N)$ satisfying $N' > N$ and

$$\frac{B(\mathcal{A}, N')}{B(\mathcal{A}, N)} \geqq \left(\frac{N'}{N}\right)^2.$$

Then we get by induction that there exist integers $N_0 < N_1 < N_2 < \dots < N_j < \dots$ such that

$$\frac{B(\mathcal{A}, N_{j+1})}{B(\mathcal{A}, N_j)} \geqq \left(\frac{N_{j+1}}{N_j}\right)^2 \qquad (\text{for } j = 1, 2, \dots).$$

(In fact, N_{j+1} can be defined by $N_{j+1} = N'(N_j)$.) Hence

$$\frac{B(\mathcal{A}, N_{k+1})}{B(\mathcal{A}, N_0)} = \prod_{j=0}^{k} \frac{B(\mathcal{A}, N_{j+1})}{B(\mathcal{A}, N_j)} \geqq \prod_{j=0}^{k} \left(\frac{N_{j+1}}{X_j}\right)^2 = \left(\frac{N_{k+1}}{N_0}\right)^2$$

so that

$$(21) \qquad B(\mathcal{A}, N_{k+1}) \geqq \left(\frac{N_{k+1}}{N_0}\right)^2 B(\mathcal{A}, N_0) \geqq \frac{1}{N_0^2} N_{k+1}^2 > N_{k+1}^{3/2}$$

for large enough k. On the other hand, clearly we have

$$(22) \qquad B(\mathcal{A}, N_{k+1}) = \sum_{\substack{a \leqq N_{k+1} \\ a-1 \notin \mathcal{A}, a \in \mathcal{A}}} 1 \leqq \sum_{a \leqq N_{k+1}} 1 = N_{k+1}.$$

(21) and (22) cannot hold simultaneously and this contradiction proves the existence of infinitely many integers N satisfying (20).

4. Throughout the remaining part of the proof of Theorem 2, we use the following notation :

N denotes a large integer satisfying (20). We write $e^{2\pi i\alpha} = e(\alpha)$, and we put $r = e^{-1/N}$, $z = re(\alpha)$ where α is a real variable (so that a function of form p(z) is a function of the real variable α: $p(z) = p(re(\alpha)) = P(\alpha)$). We write

$$f(z) = \sum_{j=1}^{+\infty} z^{a_j}.$$

(By $r < 1$, this infinite series and all the other infinite series in the remaining part of the proof are absolutely convergent.)

We start out from the integral

$$\mathcal{J} = \int_0^1 |f(z)(1-z)|^2 \, d\alpha.$$

We will give lower and upper bounds for $\mathcal{J}$. The lower bound for $\mathcal{J}$ will be greater than the upper bound, and this contradiction will prove that the indirect assumption (2) cannot hold which will complete the proof of Theorem 2.

5. In this section, we give a lower bound for $\mathcal{J}$. We write

$$f(z)(1-z) = \sum_{n=1}^{+\infty} b_n z^n.$$

Then for $n-1 \notin \mathcal{A}$, $n \in \mathcal{A}$ we have $b_n = 1$, thus by the Parseval formula, we have

$$\mathcal{J} = \int_0^1 |f(z)(1-z)|^2 \, d\alpha = \int_0^1 \Big|\sum_{n=1}^{+\infty} b_n z^n\Big|^2 d\alpha = \sum_{n=1}^{+\infty} b_n^2 r^{2n} \geqq \tag{23}$$

$$\geqq r^{2N} \sum_{\substack{n \leqq N \\ n-1 \notin \mathcal{A},\, n \in \mathcal{A}}} b_n^2 = e^{-2} \sum_{\substack{n \leqq N \\ n-1 \notin \mathcal{A},\, n \in \mathcal{A}}} 1 = e^{-2} B(\mathcal{A}, N).$$

6. In this section, we give an upper bound for $\mathcal{J}$. By the Cauchy inequality and the Parseval formula, and in view of (2) and (20), for all $\varepsilon > 0$ and for $N > N_0(\varepsilon)$ we have

$$\mathcal{J} = \Big|\int_0^1 |f(z)(1-z)^2| \, d\alpha\Big| = \int_0^1 |f^2(z)(1-z)|\,|1-z| \, d\alpha \leqq$$

$$\leqq \int_0^1 |f^2(z)(1-z)|\,(1+|z|) \, d\alpha \leqq 2 \int_0^1 |f^2(z)(1-z)| \, d\alpha =$$

$$= 2 \int_0^1 \Big|\Big(\sum_{j=1}^{+\infty} z^{a_j}\Big)^2 (1-z)\Big| \, d\alpha = 2 \int_0^1 \Big|\Big(\sum_{n=1}^{+\infty} R(n) z^n\Big)(1-z)\Big| \, d\alpha =$$

$$= 2 \int_0^1 \Big|\sum_{n=1}^{+\infty} \big(R(n)-R(n-1)\big) z^n\Big| \, d\alpha \leqq 2 \Big(\int_0^1 \Big|\sum_{n=1}^{+\infty} \big(R(n)-R(n-1)\big) z^n\Big|^2 d\alpha\Big)^{1/2} =$$

$$= 2\Big(\sum_{n=1}^{+\infty} \big(R(n)-R(n-1)\big)^2 r^{2n}\Big)^{1/2} = 2\left((1-r^2)\frac{1}{1-r^2} \sum_{n=1}^{+\infty} \big(R(n)-R(n-1)\big)^2 r^{2n}\right)^{1/2} =$$

$$= 2\Big((1-r^2) \sum_{n=1}^{+\infty} S(n-1)\, r^{2n}\Big)^{1/2} < 2\Big((1-r^2) \sum_{n=1}^{+\infty} S(n)\, r^{2n}\Big)^{1/2} = \tag{24}$$

$$= 2\Big((1-e^{-2/N})\Big(\sum_{n=1}^{N} S(n) r^{2n} + \sum_{n=N+1}^{+\infty} S(n) r^{2n}\Big)\Big)^{1/2} <$$

$$< 2\left(\frac{2}{N}\Big(\sum_{n=1}^{N} S(N) + \sum_{n=N+1}^{+\infty} S(n)\, r^{2n}\Big)\right)^{1/2} < 3\Big(S(N) + N^{-1} \sum_{n=N+1}^{+\infty} S(n)\, r^{2n}\Big)^{1/2} <$$

$$< 3\Big(\varepsilon(B(\mathcal{A}, N))^2 + N^{-1} \sum_{n=N=1}^{+\infty} \varepsilon (B(\mathcal{A}, n))^2 r^{2n}\Big)^{1/2} <$$

$$< 3\left(\varepsilon\,(B(\mathcal{A}, N))^2 + \varepsilon N^{-1} \sum_{n=N+1}^{+\infty} \left(B(\mathcal{A}, N)\left(\frac{n}{N}\right)^2\right)^2 r^{2n}\right)^{1/2} =$$

$$= 3\varepsilon^{1/2} B(\mathcal{A}, N)\Big(1 + N^{-5} \sum_{n=N+1}^{+\infty} n^4 r^{2n}\Big)^{1/2}$$

since we have

$$1 - e^{-x} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots < x \quad \text{for} \quad 0 < x < 1.$$

For $0<x<1$ we have

$$(1-x)^{-5} = 1+\sum_{n=1}^{+\infty}\binom{n+4}{4}x^n > \frac{1}{24}\sum_{n=1}^{+\infty} n^4 x^n$$

thus we obtain from (24) that for large N,

$$\mathscr{I} < 3\varepsilon^{1/2}B(\mathscr{A}, N)\left(1+N^{-5}\sum_{n=N+1}^{+\infty} n^4 r^{2n}\right)^{1/2} <$$

$$< 3\varepsilon^{1/2}B(\mathscr{A}, N)\left(1+N^{-5}\sum_{n=1}^{+\infty} n^4 r^{2n}\right)^{1/2} <$$

(25) $$< 3\varepsilon^{1/2}B(\mathscr{A}, N)\left(1+N^{-5}\cdot 24(1-r^2)^{-5}\right)^{1/2} =$$

$$= 3\varepsilon^{1/2}B(\mathscr{A}, N)\left(1+24N^{-5}(1-e^{-2/N})^{-5}\right)^{1/2} <$$

$$< 3\varepsilon^{1/2}B(\mathscr{A}, N)\left(1+24N^{-5}\left(\frac{1}{N}\right)^{-5}\right)^{1/2} = 15\varepsilon^{1/2}B(\mathscr{A}, N)$$

since we have

$$1-e^{-x} = x-\frac{x^2}{2!}+\frac{x^3}{3!}-\ldots > x-\frac{x^2}{2!} = x\left(1-\frac{x}{2}\right) > \frac{x}{2} \quad \text{for} \quad 0<x<1.$$

7. In this section, we complete the proof of Theorem 2. By (23) and (25), for all ε and $N>N_0(\varepsilon)$ we have

$$e^{-2}B(\mathscr{A}, N) \leqq \mathscr{I} < 15\varepsilon^{1/2}B(\mathscr{A}, N)$$

hence

$$e^{-2}B(\mathscr{A}, N) < 15\varepsilon^{1/2}B(\mathscr{A}, N)$$

$$\frac{1}{15e^2} < \varepsilon^{1/2}.$$

But for sufficiently small ε (e.g., for $\varepsilon=3\cdot 10^{-5}$), this inequality cannot hold. Thus in fact, the indirect assumption (2) leads to a contradiction which completes the proof of Theorem 2.

8. Sections 8, 9 and 10 will be devoted to the proof of Theorem 3. The proof is based on the probabilistic method of Erdős and Rényi [1], [2]. The Halberstam—Roth book [5] contains an excellent exposition of this method thus we use the terminology and notation of this book. In this section, we give a survey of those notations, facts and results connected with this probabilistic method which will be needed in the proof of Theorem 3.

Let Ω denote the set of the strictly increasing sequences of positive integers.

LEMMA 1. *Let*

(26) $$\alpha_1, \alpha_2, \alpha_3, \ldots$$

be real numbers satisfying

(27) $$0 \leqq \alpha_n \leqq 1 \qquad (n = 1, 2, \ldots).$$

Then there exists a probability space (Ω, S, μ) *with the following two properties:*

(i) *For every natural number* n, *the event* $E^{(n)}=\{\mathcal{A}: \mathcal{A}\in\Omega, n\in\mathcal{A}\}$ *is measurable, and* $\mu(E^{(n)})=\alpha_n$,

(ii) *The events* $E^{(1)}, E^{(2)}, \dots$ *are independent,*

This is Theorem 13 in [5], p. 142.

We denote by $\varrho(\mathcal{A}, n)$ the characteristic function of the event $E^{(n)}$:

$$\varrho(\mathcal{A}, n)=\begin{cases}1 & \text{if } n\in\mathcal{A}\\ 0 & \text{if } n\notin\mathcal{A}\end{cases}$$

so that

$$A(n)=\sum_{j=1}^{n}\varrho(\mathcal{A}, j).$$

Furthermore, we denote the number of solutions of

$$a_x+a_y=n, \quad a_x\in\mathcal{A}, \quad a_y\in\mathcal{A}, \quad a_x<a_y \tag{28}$$

by $r(n)=r(\mathcal{A}, n)$ so that

$$|R(\mathcal{A}, n)-2r(\mathcal{A}, n)|\leqq 1 \tag{29}$$

(where $R(\mathcal{A}, n)$ is the number of solutions of (28) without the restriction $a_x<a_y$).

LEMMA 2. *If the sequence (26) satisfies (27) and*

$$\alpha_j=\alpha j^{-c} \quad \textit{for } j\geqq j_0$$

where α, c *are constants such that* $0<\alpha$, $0<c<1$, *then, with probability* 1, we *have*

$$A(n)\sim\frac{\alpha}{1-c}n^{1-c}.$$

This lemma is a consequence of Lemmas 10 and 11 in [5], pp. 144-145.

The crucial point of the proof is the use of the following result of Erdős and Rényi [2]:

LEMMA 3. *If* $\varepsilon>0$ *and the sequence (26) is defined by*

$$\alpha_j=\frac{1}{2}j^{(2+\varepsilon)^{-1}-1} \quad \textit{for} \quad j=1, 2, \dots \tag{30}$$

then, with probability 1,

$$R(\mathcal{A}, n)\,(\leqq 2r(\mathcal{A}, n)+1)<4(1+\varepsilon^{-1})+1 \quad \textit{for} \quad n>n_1(\varepsilon, \mathcal{A}).$$

See Theorem 2 and its proof in [5], pp. 111 and 151-152; see also (29).

We shall need also the Borel-Cantelli lemma:

LEMMA 4. *Let* (X, S, μ) *be a probability space and let* $F_1, F_2, \dots$ *be a sequence of measurable events. If*

$$\sum_{j=1}^{+\infty}\mu(F_j)<+\infty,$$

then, with probability 1, *at most a finite number of the events* F_j *can occur.*

See [5], p. 135.

9. For $\mathscr{A} \in \Omega$, we write

$$T(\mathscr{A}, n) = \sum_{\substack{a \leq n \\ a-1 \in \mathscr{A}, a \in \mathscr{A}}} 1$$

so that

$$B(\mathscr{A}, n) + T(\mathscr{A}, n) = \sum_{\substack{a \leq n \\ a-1 \notin \mathscr{A}, a \in \mathscr{A}}} 1 + \sum_{\substack{a \leq n \\ a-1 \in \mathscr{A}, a \in \mathscr{A}}} 1 = \sum_{\substack{a \leq n \\ a \in \mathscr{A}}} 1 = A(n). \tag{31}$$

LEMMA **5.** *If the sequence (26) satisfies (27) and*

$$\sum_{j=1}^{+\infty} \alpha_j \alpha_{j+1} < +\infty, \tag{32}$$

then, with probability 1,

$$T(\mathscr{A}, n) < 4 \log n \text{ for } n > n_2(\mathscr{A})$$

(where n_2 may depend on both the sequence (26) and $\mathscr{A}$).

PROOF. We have to give an upper bound for $\mu(\{\mathscr{A}: T(\mathscr{A}, n) \geq 4 \log n\})$. We write

$$\lambda_n = 2 \log n.$$

Then

$$T(\mathscr{A}, n) = \sum_{\substack{a \leq n \\ a-1 \in \mathscr{A}, a \in \mathscr{A}}} 1 = \sum_{j=2}^{n} \varrho(\mathscr{A}, j-1)\varrho(\mathscr{A}, j) \geq 2\lambda_n$$

implies that either

$$T_1(\mathscr{A}, n) \stackrel{\text{def}}{=} \sum_{1 \leq i \leq n/2} \varrho(\mathscr{A}, 2i-1)\varrho(\mathscr{A}, 2i) \geq \lambda_n,$$

or

$$T_2(\mathscr{A}, n) \stackrel{\text{def}}{=} \sum_{1 \leq i < n/2} \varrho(\mathscr{A}, 2i)\varrho(\mathscr{A}, 2i+1) \geq \lambda_n$$

holds so that

$$\begin{aligned} &\mu(\{\mathscr{A}: T(\mathscr{A}, n) \geq 2\lambda_n\}) \leq \\ &\leq \mu(\{\mathscr{A}: T_1(\mathscr{A}, n) \geq \lambda_n\}) + \mu(\{\mathscr{A}: T_2(\mathscr{A}, n) \geq \lambda_n\}) = \\ &= \sum_{d \geq \lambda_n} \mu(\{\mathscr{A}: T_1(\mathscr{A}, n) = d\}) + \sum_{d \geq \lambda_n} \mu(\{\mathscr{A}: T_2(\mathscr{A}, n) = d\}) = \\ &= \sum_{d = \lambda_n} u_n(d) + \sum_{d \geq \lambda_n} v_n(d) \end{aligned} \tag{33}$$

where

$$u_n(d) = \mu(\{\mathscr{A}: T_1(\mathscr{A}, n) = d\}) = \mu(\{\mathscr{A}: \sum_{1 \leq i \leq n/2} \varrho(\mathscr{A}, 2i-1)\varrho(\mathscr{A}, 2i) = d\}) =$$

$$= \sum_{1 \leq i_1 < \ldots < i_d \leq n/2} \prod_{j=1}^{d} \alpha_{2i_j-1}\alpha_{2i_j}(1-\alpha_{2i_j-1}\alpha_{2i_j})^{-1} \prod_{1 \leq i \leq n/2} (1-\alpha_{2i-1}\alpha_{2i})$$

and similarly,

$$v_n(d) = \mu\big(\{\mathscr{A}: T_2(\mathscr{A}, n) = d\}\big) = \mu\big(\{\mathscr{A}: \sum_{1 \leqq i < n/2} \varrho(\mathscr{A}, 2i)\,\varrho(\mathscr{A}, 2i+1) = d\}\big) =$$

$$= \sum_{1 \leqq i_1 < \ldots < i_d < n/2} \prod_{j=1}^{d} \alpha_{2i_j}\alpha_{2i_j+1}(1-\alpha_{2i_j}\alpha_{2i_j+1})^{-1} \prod_{1 \leqq i < n/2} (1-\alpha_{2i}\alpha_{2i+1})$$

so that for any real number x,

$$U_n(x) \overset{\text{def}}{=} \sum_{0 \leqq d \leqq n/2} u_n(d)\,x^d = \sum_{1 \leqq i \leqq n/2} \big((1-\alpha_{2i-1}\alpha_{2i}) + \alpha_{2i-1}\alpha_{2i}x\big) \tag{34}$$

and

$$V_n(x) \overset{\text{def}}{=} \sum_{0 \leqq d < n/2} v_n(d)\,x^d = \prod_{1 \leqq i < n/2} \big((1-\alpha_{2i}\alpha_{2i+1}) + \alpha_{2i}\alpha_{2i+1}x\big). \tag{35}$$

By (32), (33), (34) and (35), we have

$$\mu\big(\{\mathscr{A}: T(\mathscr{A}, n) \geqq 2\lambda_n\}\big) \leqq \sum_{d \geqq \lambda_n} u_n(d) + \sum_{d \geqq \lambda_n} v_n(d) \leqq$$

$$\leqq \sum_{d \geqq \lambda_n} u_n(d)\,e^{d-\lambda_n} \sum_{d \geqq \lambda_n} v_n(d)\,e^{d-\lambda_n} = e^{-\lambda_n}\Big(\sum_{d \geqq \lambda_n} u_n(d)\,e^d + \sum_{d \geqq \lambda_n} v_n(d)\,e^d\Big) \leqq$$

$$\leqq e^{-\lambda_n}\Big(\sum_{0 \leqq d \leqq n/2} u_n(d)\,e^d + \sum_{0 \leqq d < n/2} v_n(d)\,e^d\Big) = e^{-2\log n}\big(U_n(e) + V_n(e)\big) =$$

$$= n^{-2}\Big(\prod_{1 \leqq i \leqq n/2} (1+\alpha_{2i-1}\alpha_{2i}(e-1)) + \prod_{1 \leqq i < n/2} (1+\alpha_{2i}\alpha_{2i+1}(e-1))\Big) \leqq$$

$$\leqq n^{-2} \prod_{1 \leqq j < n} \big(1+\alpha_j\alpha_{j+1}(e-1)\big) < n^{-2} \prod_{1 \leqq j < n} \exp\big(\alpha_j\alpha_{j+1}(e-1)\big) =$$

$$= n^{-2}\exp\Big((e-1)\sum_{1 \leqq j < n} \alpha_j\alpha_{j+1}\Big) < n^{-2}\exp\Big(2\sum_{j=1}^{+\infty} \alpha_j\alpha_{j+1}\Big) < cn^{-2}$$

(where c depends on the sequence (26)) since

$$1+x < e^x \quad \text{for} \quad x > 0.$$

Thus we have

$$\sum_{n=1}^{+\infty} \mu\big(\{\mathscr{A}: T(\mathscr{A}, n) \geqq 4\log n\}\big) < \sum_{n=1}^{+\infty} cn^{-2} < +\infty$$

so that by the Borel—Cantelli lemma (Lemma 4), with probability 1, at most a finite number of the events $T(\mathscr{A}, n) \geqq 4 \log n$ $(n = 1, 2, \ldots)$ can occur which completes the proof of the lemma.

10. In this section, we complete the proof of Theorem 3.

Let us define the sequence (26) by

$$\alpha_j = \frac{1}{2} j^{-(1/2)-\varepsilon}. \tag{36}$$

Then by Lemma 3 $\left(\text{with } \frac{4\varepsilon}{1-2\varepsilon} \text{ in place of } \varepsilon\right)$, with probability 1, $R(\mathscr{A}, n)$ is bounded. (In fact, for large n we have

$$R(\mathscr{A}, n) < 4\left(1+\left(\frac{4\varepsilon}{1-2\varepsilon}\right)^{-1}\right)+1 = 3+\varepsilon^{-1}.)$$

Furthermore, by Lemma 2, with probability 1, we have

$$A(n) \sim \frac{1}{2}\left(\frac{1}{2}-\varepsilon\right)^{-1} n^{(1/2)-\varepsilon}$$

so that, with probability 1,

$$A(n) > \frac{1}{2} 2n^{(1/2)-\varepsilon} = n^{(1/2)-\varepsilon} \tag{37}$$

for n large enough.

By Lemma 5 {note that, clearly, the sequence (36) satisfies (32)), with probability 1,

$$T(\mathscr{A}, n) < 4 \log n \tag{38}$$

for n large enough.

In view of (31), (37) and (38) yield that, with probability 1,

$$B(\mathscr{A}, n) = A(n) - T(\mathscr{A}, n) > n^{(1/2)-\varepsilon} - 4 \log n > \frac{1}{2} n^{(1/2)-\varepsilon} \quad \text{for} \quad n > n_3(\varepsilon, \mathscr{A}).$$

Thus, with probability 1, both (i) and (ii) in Theorem 3 hold, so that there exists infinitely many sequences satisfying both (i) and (ii), which completes the proof of Theorem 3.

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MTA MATEMATIKAI KUTATÓ INTÉZET
POSTAFIÓK 127
H—1364 BUDAPEST
HUNGARY

86-3637 — Szegedi Nyomda — Felelős vezető: Surányi Tibor igazgató