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Strong characterizing sequences in simultaneous diophantine approximation ☆

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Abstract

Answering a question of Liardet, we prove that if $1, \alpha_1, \alpha_2, \ldots, \alpha_t$ are real numbers linearly independent over the rationals, then there is an infinite subset A of the positive integers such that for real β, we have ($||\ ||$ denotes the distance to the nearest integer)

$$\sum_{n \in A} ||n\beta|| < \infty$$

if and only if β is a linear combination with integer coefficients of $1, \alpha_1, \alpha_2, \ldots, \alpha_t$. The proof combines elementary ideas with a deep theorem of Freiman on set addition. Using Freiman's theorem, we prove a lemma on the structure of Bohr sets, which may have independent interest.

Keywords: Characterizing sequences; Bohr sets; Freiman's theorem

1. Introduction

In [1], together with Jean-Marc Deshouillers, we proved the following theorem ($||\ ||$ denotes the distance to the nearest integer).

Theorem. *Assume that $1, \alpha_1, \alpha_2, \ldots, \alpha_t$ are real numbers linearly independent over the rationals. Then there is an infinite subset A of the positive integers such that for real β,*

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we have

$$\lim_{n\in A, n\to\infty} ||n\beta|| = 0$$

if and only if $\beta\in G$, *where* G *is the group generated by* $1, \alpha_1, \alpha_2, \dots, \alpha_t$.

We call A a characterizing sequence of G.

Actually, we proved there a stronger theorem: the same statement is true for any countable subgroup of the reals with $1\in G$, but to extend the theorem for that case is a technical matter. For the sake of simplicity, in the present paper we consider only the special case. Liardet [2] asked the following problem: can one replace the condition

$$\lim_{n\in A, n\to\infty} ||n\beta|| = 0$$

in the above theorem by

$$\sum_{n\in A} ||n\beta|| < \infty ?$$

Our answer is affirmative.

Theorem. *Assume that* $1, \alpha_1, \alpha_2, \dots, \alpha_t$ *are real numbers linearly independent over the rationals. Then there is an infinite subset* A *of the positive integers such that for real* β, *we have*

$$\sum_{n\in A} ||n\beta|| < \infty,$$

if and only if $\beta\in G$, *where* G *is the group generated by* $1, \alpha_1, \alpha_2, \dots, \alpha_t$. *Furthermore, for* $\beta\notin G$ *we even have*

$$\lim_{n\in A, n\to\infty} \inf ||n\beta|| > 0.$$

This is a strengthening of the quoted theorem of [1], so we may call such an A a strong characterizing sequence of G.

Our proof combines the ideas of the proof in [1] with a deep theorem of Freiman on set addition. Using Freiman's theorem, we prove a lemma on the structure of Bohr sets. Since this lemma (Lemma 1 below) may have independent interest, we state it here, in the Introduction.

Bohr sets are defined in the following way: if $\alpha_1, \alpha_2, \dots, \alpha_t$ are arbitrary (but fixed) real numbers (so independence is not assumed here), N is a positive integer and $\varepsilon > 0$, let

$$H_{N,\varepsilon} = \{1 \leqslant n \leqslant N\colon\ ||n\alpha_1|| \leqslant \varepsilon, ||n\alpha_2|| \leqslant \varepsilon, \ldots, ||n\alpha_t|| \leqslant \varepsilon\}.$$

The implied constants in $\ll$ depend only on t in the following lemma.

Lemma 1. *Let* $\varepsilon > 0$ *be small enough (depending on* t*). Then*

$$H_{N,\varepsilon} \subseteq \left\{ \sum_{i=1}^{R} k_i n_i\colon\ 1 \leqslant k_i \leqslant K_i \text{ for } 1 \leqslant i \leqslant R \right\} \tag{1}$$

with some $R \geqslant 1$ *and suitable nonzero integers* n_i *and positive integers* K_i *satisfying* $R \ll 1$,

$$||n_i\alpha_j|| \ll \frac{\varepsilon}{K_i} \quad (1 \leqslant i \leqslant R, 1 \leqslant j \leqslant t)$$

and

$$|n_i| \ll \frac{N}{K_i} \quad (1 \leqslant i \leqslant R).$$

Consequently, for any element n *of the right-hand side of* (1) *we have*

$$|n| \ll N \quad \textit{and} \quad ||n\alpha_j|| \ll \varepsilon \quad (1 \leqslant j \leqslant t).$$

Remark 1. It would be interesting to analyze the dependence of R on the dimension t of the Bohr set.

Remark 2. Our work is related to the papers [3,4] (see [1] for more details in this connection).

2. Lemmas on Bohr sets

In this section $\alpha_1, \alpha_2, \ldots, \alpha_t$ are arbitrary real numbers, and the implied constants in $\ll$ depend only on t.

To prove Lemma 1 stated in the Introduction we need Lemma 2. If A and B are two subsets of the integers, then we write

$$A + B = \{a + b\colon\ a \in A, b \in B\}.$$

Lemma 2. *We have*

$$|H_{N,\varepsilon}+H_{N,\varepsilon}|\leqslant C|H_{N,\varepsilon}|,$$

where C *is a constant depending only on* t (*the dimension of the Bohr set*).

Proof. It is clear that $H_{N,\varepsilon}+H_{N,\varepsilon}\subseteq H_{2N,2\varepsilon}$. We divide the interval $[1,2N]$ into two parts, the interval $[-2\varepsilon,2\varepsilon]$ into four parts, so the cube $[-2\varepsilon,2\varepsilon]^t$ into 4^t parts, and the lemma follows easily by the pigeon-hole principle. $\square$

Proof of Lemma 1. By Ruzsa's version of Freiman's theorem (see [5]; Freiman's original work is [6]) and Lemma 2 we have

$$H_{N,\varepsilon}\subseteq\left\{a+\sum_{i=1}^{r}l_id_i\colon\ 1\leqslant l_i\leqslant L_i\ \text{ for }\ 1\leqslant i\leqslant r\right\}$$

with some $r\geqslant 1$ and suitable integers a and d_i and positive integers L_i, where

$$|H_{N,\varepsilon}|\geqslant DL_1L_2\ldots L_r$$

with some $0<D<1$. Here the numbers r and D depend only on C of Lemma 2 (so depend only on t).

Assume that $L_1\geqslant\frac{2}{D}$. Then it is clear that we can fix $l_2,l_3,\ldots,l_r$ such that

$$\left|\left\{1\leqslant l_1\leqslant L_1\colon\ a+\sum_{i=1}^{r}l_id_i\in H_{N,\varepsilon}\right\}\right|\geqslant DL_1\geqslant 2.$$

Then there are two different numbers in this set, say l_1 and λ_1, with the property

$$0<|l_1-\lambda_1|<\tfrac{2}{D},$$

and since l_1 and λ_1 are elements of the above set, by the definition of $H_{N,\varepsilon}$ we have

$$||(l_1-\lambda_1)d_1\alpha_j||\leqslant 2\varepsilon\quad\text{for }\ 1\leqslant j\leqslant t$$

and

$$|(l_1-\lambda_1)d_1|\leqslant N.$$

Applying this argument several times and taking least common multiple, we find a positive integer T such that

$$T\ll 1,\quad ||Td_i\alpha_j||\ll\varepsilon,\quad |Td_i|\ll N \tag{2}$$

for $1\leqslant j\leqslant t$ and for every $1\leqslant i\leqslant r$ satisfying $L_i\geqslant\frac{2}{D}$. We want to improve the last two inequalities in (2).

To this end we assume again that $L_1 \geqslant \frac{2}{D}$. If we fix suitably $l_2, l_3, \ldots, l_r$, then we can find a residue class $\tau \pmod{T}$ such that

$$\left| \left\{ 1 \leqslant l_1 \leqslant L_1 \colon \; l_1 \equiv \tau \pmod{T}, a + \sum_{i=1}^{r} l_i d_i \in H_{N,\varepsilon} \right\} \right| \gg L_1.$$

Hence there is an integer $M_1 \gg L_1$ and a number $E > 0$ depending only on t with the property that for every $1 \leqslant j \leqslant t$, there is a real x_j and there is an integer n such that with the notations

$$S_{1,j} = \{1 \leqslant m \leqslant M_1 \colon \; ||x_j + m(Td_1\alpha_j)|| \leqslant \varepsilon\} \tag{3}$$

and

$$S_2 = \{1 \leqslant m \leqslant M_1 \colon \; |n + m(Td_1)| \leqslant N\}, \tag{4}$$

we have

$$|S_{1,j}| \geqslant EM_1, \quad |S_2| \geqslant EM_1. \tag{5}$$

Recall from (2) that $||Td_1\alpha_j|| \ll \varepsilon$. Then it follows by (3) (dividing the interval $[1, M_1]$ into intervals of length smaller than $\frac{1}{||Td_1\alpha_j||}$) that

$$|S_{1,j}| \ll (1 + M_1||Td_1\alpha_j||)\frac{\varepsilon}{||Td_1\alpha_j||}.$$

If ε is small enough (depending on t), then using (5) and $M_1 \gg L_1$ we get

$$||Td_1\alpha_j|| \ll \frac{\varepsilon}{L_1}. \tag{6}$$

On the other hand, by (2) and (4) we have

$$|S_2| \ll \frac{N}{|Td_1|},$$

and so (5) gives

$$|d_1| \ll \frac{N}{L_1}. \tag{7}$$

We see that (6) and (7) indeed improve (2).

Summing up: if ε is small enough, we can divide $\{1, 2, \ldots, r\}$ into a disjoint union

$$\{1, 2, \ldots, r\} = I_1 \cup I_2$$

such that

$$L_i < \frac{2}{D} \quad \text{for } i \in I_1,$$

$$||Td_i\alpha_j|| \ll \frac{\varepsilon}{L_i} \quad \text{and} \quad |d_i| \ll \frac{N}{L_i} \text{ for } i \in I_2 \text{ and } 1 \leqslant j \leqslant t. \tag{8}$$

Now, it is clear that there is a set H_1 of integers satisfying $|H_1| \ll 1$ and $H_{N,\varepsilon} \subseteq H_1 + H_2$, where

$$H_2 = \left\{ \sum_{i \in I_2} (Td_i)l_i \colon 1 \leqslant l_i \leqslant \left[\frac{L_i}{T} \right] \right\}.$$

Of course, we can assume that $H_{N,\varepsilon} \cap (h + H_2) \neq \emptyset$ for every $h \in H_1$, and so we know

$$||h\alpha_j|| \ll \varepsilon \quad \text{for } 1 \leqslant j \leqslant t \text{ and } |h| \ll N \tag{9}$$

for $h \in H_1$, if we know (9) for $h \in H_2$ and $h \in H_{N,\varepsilon}$. But for $h \in H_2$ (9) follows from (8); for $h \in H_{N,\varepsilon}$ (9) is true by definition. The lemma follows from the above observations (as n_i we can take $Td_i (i \in I_2)$ and each element of H_1). □

Lemma 3. *If ω is a real number, $k \geqslant 1$ is an integer, and*

$$||\omega||, ||2\omega||, ||4\omega||, \ldots, ||2^k\omega|| \leqslant \delta < \frac{1}{10},$$

then $||\omega|| \leqslant \frac{\delta}{2^k}$.

Proof. We use induction on k. The case $k = 1$ is clear since

$$\frac{\delta}{2} < ||\omega|| \leqslant \delta < \frac{1}{10}$$

implies $\delta < ||2\omega||$. If $k > 1$, then by the $k = 1$ case we have

$$||2^j\omega|| \leqslant \frac{\delta}{2} \quad \text{for } 1 \leqslant j \leqslant k - 1$$

and then the assertion for $k - 1$ implies the assertion for k. □

Lemma 4. *If $H_{N,\varepsilon}$ is a Bohr set, and $\varepsilon > 0$ is small enough (depending on t), then there is a set S consisting of positive integers with the following three properties:*

(i) $\max_{n \in S} n \ll N$,
(ii) $\sum_{n \in S} ||n\alpha_j|| \ll \varepsilon$ for $1 \leqslant j \leqslant t$,
(iii) $\max_{n \in H_{N,\varepsilon}} ||n\beta|| \ll \max_{n \in S} ||n\beta||$ *for every real* β.

Proof. We use the notations of Lemma 1. We define

$$S = \{2^{l_i}|n_i|:\ 1 \leqslant 2^{l_i} \leqslant K_i, 1 \leqslant i \leqslant R\}.$$

The first two required properties of S are then trivial from Lemma 1. We prove the third one. We may assume that

$$\max_{n \in S} ||n\beta|| < \frac{1}{10}.$$

Then by Lemma 3, we have

$$||n_i\beta|| \leqslant \frac{1}{K_i} \max_{n \in S} ||n\beta||$$

for $1 \leqslant i \leqslant R$, and using Lemma 1, this proves the present lemma. □

3. Proof of the Theorem

It is not needed for the general proof, but we think that it is interesting to give first a construction of a suitable set in the one-dimensional case: if $t = 1, \alpha = \alpha_1$,

$$\alpha = [a_0; a_1, a_2, \ldots]$$

is its continued fraction expansion, and p_m/q_m is the sequence of its convergents, then

$$A = \{2^l q_m:\ 1 \leqslant 2^l \leqslant a_{m+1}, m = 1, 2, \ldots\}$$

is a set satisfying the conditions listed in the Theorem. This can be easily proved using Theorem 1* of [1] and our present Lemma 3, but instead of analyzing it further, we turn to the proof of the Theorem for any $t \geqslant 1$.

In the sequel, $1, \alpha_1, \alpha_2, \ldots, \alpha_t$ are linearly independent over the rationals. The following lemma is a simple consequence of Lemma 2.2 in [1]. For the sake of completeness, we sketch its proof here.

Lemma 5. *Let $\varepsilon > 0, T \geqslant 1$ and $\delta > 0$, and assume that $\varepsilon T \leqslant \frac{1}{4}$. Then there is a positive integer N such that if*

$$\max_{n \in H_{N,\varepsilon}} ||n\beta|| \leqslant T\varepsilon \qquad (*)$$

for a real β, then

$$||\beta - (K_1\alpha_1 + \cdots + K_t\alpha_t)|| < \delta$$

with some integers $K_1, \ldots, K_t$ *satisfying*

$$|K_1| + \cdots + |K_t| \leqslant T. \qquad (**)$$

Proof. By a compactness argument, it is enough to prove the following:

Statement. *Let* $\varepsilon > 0, T \geqslant 1$ *and assume that* $\varepsilon T \leqslant \frac{1}{4}$. *Then, if* $(*)$ *is true for every positive integer* N, *then*

$$\beta \equiv K_1\alpha_1 + \cdots + K_t\alpha_t \pmod 1$$

with some integers $K_1, \ldots, K_t$ *satisfying* $(**)$.

To prove it, we note that by the conditions, the set

$$\{(n\alpha_1, n\alpha_2, \ldots, n\alpha_t, n\beta): \ n \in Z\}$$

is not dense in $(R/Z)^{t+1}$, so, by Kronecker's theorem, the numbers $\alpha_1, \alpha_2, \ldots, \alpha_t, \beta$ and 1 cannot be linearly independent over the rationals. Hence, there are integers $K_1, K_2, \ldots, K_{t+1}$ and a positive integer K such that

$$\beta \equiv \frac{K_1}{K}\alpha_1 + \cdots + \frac{K_t}{K}\alpha_t + \frac{K_{t+1}}{K} \pmod 1.$$

We first prove that K_1/K is an integer. If this is not the case, then there is an integer $1 \leqslant R < K$ such that $||RK_1/K|| \geqslant 1/3$. For that R and any $\delta > 0$, we can choose a large enough r such that

$$||(R/K) - r\alpha_1|| < \delta, \quad ||r\alpha_2||, \ldots, ||r\alpha_t|| < \delta,$$

and then, taking $n = rK$, this gives us (if δ is small enough) that $||n\alpha_1||, \ldots, ||n\alpha_t|| < \varepsilon$, but $||n\beta|| > 1/4$. This contradiction shows that K divides K_1, and similarly, K divides $K_2, \ldots, K_t$.

We now prove that K_{t+1}/K is also an integer. If not, then for a $1 \leqslant R < K$ we have $||RK_{t+1}/K|| \geqslant 1/3$. For any $\delta > 0$ we can choose a large enough r such that with $n = R + rK$ we have $||n\alpha_1||, \ldots, ||n\alpha_t|| < \delta$. Then, similarly as above, for small enough δ we will have $||n\alpha_1||, \ldots, ||n\alpha_t|| < \varepsilon$, but $||n\beta|| > 1/4$. Hence K divides K_{t+1}. So we can assume that $K = 1$, i.e.,

$$\beta \equiv K_1\alpha_1 + \cdots + K_t\alpha_t \pmod 1$$

and it is easy to see that our condition can be satisfied only if $(**)$ is true. Lemma 5 is proved. □

We now prove the theorem. Let δ_k be a strictly decreasing sequence (to be determined later) tending to 0. Then, by Lemma 5, we can choose a sequence N_k of positive integers such that $H_{N_k,2^{-k-2}} \neq \emptyset$, and if

$$\max_{n \in H_{N_k,2^{-k-2}}} ||n\beta|| \leqslant \tfrac{1}{4} \tag{10}$$

for a real β, then

$$||\beta - (K_1\alpha_1 + \cdots + K_t\alpha_t)|| < \delta_k \tag{11}$$

with some integers $K_1, \ldots, K_t$ satisfying

$$|K_1| + \cdots + |K_t| \leqslant 2^k. \tag{12}$$

By Lemma 4, for large enough k, say for $k \geqslant K_0$ we can choose a set S_k for $H_{N_k,2^{-k-2}}$ satisfying the properties listed in that lemma. Observe that by (ii) of Lemma 4, we have

$$\lim_{k \to \infty} \left(\min_{n \in S_k} n \right) = \infty. \tag{13}$$

Define

$$A = \bigcup_{k \geqslant K_0} S_k. \tag{14}$$

Assume that for a real β we have

$$\lim_{n \in A, n \to \infty} ||n\beta|| = 0. \tag{15}$$

Then, by (13) and (14), we must have

$$\lim_{k \to \infty} \left(\max_{n \in S_k} ||n\beta|| \right) = 0,$$

and so by (iii) of Lemma 4, (10) is valid for large enough k, if β satisfies (15). This implies (see (11) and (12)) that for such β and for every large enough k, one has

$$||\beta - (K_{1,k}\alpha_1 + \cdots + K_{t,k}\alpha_t)|| < \delta_k \tag{16}$$

for suitable integers satisfying

$$|K_{1,k}| + \cdots + |K_{t,k}| \leqslant 2^k. \tag{17}$$

Using (16) for k and $k+1$, and using also that δ_k is decreasing, we find that

$$||(K_{1,k} - K_{1,k+1})\alpha_1 + \cdots + (K_{t,k} - K_{t,k+1})\alpha_t|| < 2\delta_k. \tag{18}$$

If we define

$$\delta_k = \frac{1}{2}\left(\min_{0<|K_1|+\cdots+|K_t|\leqslant 2^{k+2}} ||K_1\alpha_1 + \cdots + K_t\alpha_t||\right),$$

then we obtain from (18) (using (17) for k and $k+1$) that

$$K_{j,k} = K_{j,k+1} \quad \text{for } 1\leqslant j\leqslant t.$$

This is true for every large enough k, so there are integers K_j for every j such that $K_{j,k} = K_j$ for large k. Since $\delta_k \to 0$, this easily implies $\beta \in G$ by (16). Hence we proved that if (15) is true for β, then $\beta \in G$.

On the other hand, for every $1\leqslant j\leqslant t$, by the definition of the sets S_k, by (ii) of Lemma 4 and by (14) we obtain

$$\sum_{n\in A} ||n\alpha_j|| \leqslant \sum_{k\geqslant K_0} \sum_{n\in S_k} ||n\alpha_j|| \ll \sum_{k\geqslant K_0} 2^{-k-2} \ll 1.$$

This proves the theorem. □

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