Different levels of randomness in Random Ramsey theorems

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Abstract

Extremal graph theory, Ramsey theory and the theory of Random graphs are strongly connected to each other.

Starting from these fields, we formulate some problems and results which are related to different levels of randomness. The first one is completely non-random, being the ordinary Ramsey–Turán problem and in the subsequent three problems we formulate some randomized variations of the problem.

Speaking of graph properties, we shall consider them as sets of graphs and occasionally write $G \in \mathbb{P}$ instead of writing that G has property $\mathbb{P}$. A graph property $\mathbb{P}$ is called **monotone** if adding an edge to a $H_n \in \mathbb{P}$, we get an $H_n^* \in \mathbb{P}$.

We shall use three models of random graphs: the binomial, the hypergeometric and the stopping-rule model.

This abstract contains the most important definitions.

Four levels of Deterministic and Random Ramsey problems

Here we are interested primarily in the edge-phase-transitions: For fixed n we consider graphs G_n on n vertices and gradually increase $e(G_n)$ from 0 to $\binom{n}{2}$. Concerning a fixed property $\mathbb{P}$, for some number of edges, $f(n)$, we have a phase-transition and we are interested in finding this $f(n)$.

Phase transition means in our context that the typical structure of the graphs in consideration suddenly changes as we increase the number of edges. (Physicists prefer to have a stricter definition of phase transition.)

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In fact, there are two types of problems: the **vertex-problems**, where we increase the number of vertices and suddenly some "phase-transition" occurs, and the **edge-problems** where for a given n we consider graphs on n vertices and increase the number of edges.

Below we shall speak of random r-colorings of random or deterministically given graphs G_n. There are several ways to define random colorings. We agree here that we shall always use the "uniform coloring":

Definition (*Random coloring.*) Given a graph G_n, a random edge-coloring is a coloring when the edges are colored by $1, \dots, r$ and for each edge we choose each color uniformly and independently. The graph defined by the edges of color i is G_n^i.

Roughly speaking, we could say that we shall consider six problem-families. Out of these, four will be emphasized more than the remaining two.

The four general problems can be formulated as follows: Put

$$\mathcal{G}_{n,\ell} = \{G_n : e(G_n) = \ell\}.$$

Definition *(Deterministic-Deterministic.)* $f^r_{\mathbf{DD}}(n, \mathbb{P})$ is the minimum ℓ for which for every r-coloring of every G_n of $\ell := f^r_{\mathbf{DD}}(n, \mathbb{P})$ edges, one of the graphs G_n^i has property $\mathbb{P}$.

This minimum does not always exist. Observe that no randomness is in this definition and $f^r_{\mathbf{DD}}(n, \mathbb{P})$ is uniquely defined for any fixed n. Below, speaking of *"almost surely"* we mean that the probability of some event tends to 1 as $n \to \infty$.

Definition *(Deterministic-Random.)* We call $f^r_{\mathbf{DR}}(n, \mathbb{P})$ a weak **DR**-threshold function if

$$\frac{f(n)}{f^r_{\mathbf{DR}}(n, \mathbb{P})} \to \infty, \text{ as } n \to \infty,$$

for almost every r-coloring of every graph G_n of $f(n)$ edges at least one of the color-graphs G_n^i has property $\mathbb{P}$; on the other hand, if

$$\frac{f(n)}{f^r_{\mathbf{DR}}(n, \mathbb{P})} \to 0,$$

then almost every r-coloring of every G_n of $f(n)$ edges has no $G_n^i \in \mathbb{P}$.

In some sense, this threshold function is often uniquely determined, or determined up to a very small additive error term. The two threshold functions below are determined only up to a multiplicative constant.

Definition *(Random-Deterministic.)* $f^r_{\mathbf{RD}}(n,\mathbb{P})$ is a weak **RD**-threshold function assuming that
(a) if

$$\frac{f(n)}{f^r_{\mathbf{RD}}(n,\mathbb{P})} \to \infty, \text{ as } n \to \infty$$

then for every r-coloring of almost every G_n of $f(n)$ edges, (at least) one of the color graphs G^i_n $(i = 1, \ldots, r)$ has property $\mathbb{P}$; while
(b) if

$$\frac{f(n)}{f^r_{\mathbf{RD}}(n,\mathbb{P})} \to 0, \text{ as } n \to \infty,$$

then almost every G_n of $f(n)$ edges has an r-coloring where none of the color graphs G^i_n $(i = 1, \ldots, r)$ have property $\mathbb{P}$.

Definition *(Random-Random.)* We shall call $f^r_{\mathbf{RR}}(n,\mathbb{P})$ a weak **RR**-threshold function if for almost all r-colorings of almost all graphs G_n of $f(n)$ edges almost surely one of the color-graphs G^i_n has property $\mathbb{P}$ if

$$\frac{f(n)}{f^r_{\mathbf{RR}}(n,\mathbb{P})} \to \infty,$$

while for

$$\frac{f(n)}{f^r_{\mathbf{RR}}(n,\mathbb{P})} \to 0,$$

for almost all r-colorings of almost all G_n of $f(n)$ edges we have no G^i_n of property $\mathbb{P}$.

We can define the **strong threshold functions** $F^r_{\mathbf{RD}}$, $F^r_{\mathbf{DR}}$, $F^r_{\mathbf{RR}}$ similarly to the threshold functions $f^r_{\mathbf{RD}}$,$f^r_{\mathbf{DR}}$, $f^r_{\mathbf{RR}}$ above:

Definition *Strong threshold (hypergeometric).* We call $F^r_{UV}(n,\mathbb{P})$ a strong threshold function for "UV" if there exist two functions F^-_{UV} and F^+_{UV} such that $F^-_{UV}(n,\mathbb{P}) = (1+o(1))F^r_{UV}(n,\mathbb{P})$ and $F^+_{UV}(n,\mathbb{P}) = (1+o(1))F^r_{UV}(n,\mathbb{P})$ and $e(G_n) = F^+_{UV}$ implies "YES" while $e(G_n) = F^-_{UV}$ implies "NO" in the corresponding question, for r colors and $\mathbb{P}$.

All the definitions for ordinary graphs can be extended for hypergraphs or for digraphs.

Basic Questions

Having this hierarchy, we are interested in the following problems:

(1) Which are the basic relations between our threshold functions when the property $\mathbb{P}$ and r are fixed?
(2) If we have some inequalities relating the threshold functions, when do we have *strict* inequalities?
(3) When do we know that some of the functions are *much larger* than others?
(4) How are the threshold functions related to other, more well known graph theoretical functions?
(5) Which graph-theoretical properties of $\mathbb{P}$ influence the threshold functions, and how?