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EXTENSION OF SOME CHENEY-SHARMA TYPE OPERATORS TO A TRIANGLE WITH ONE CURVED SIDE

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Abstract. We extend the definition of some Cheney-Sharma type operators to a triangle with one curved side. We construct their product and Boolean sum, we study their interpolation properties, the orders of accuracy and we give different expressions of the corresponding remainders. We also give some illustrative examples.

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1. INTRODUCTION

In order to match all the boundary information on a curved domain (as Dirichlet, Neumann or Robin boundary conditions for differential equation problems), there were considered interpolation operators on domains with curved sides (see, e.g., [3], [4], [5], [6], [8], [9], [10]).

The aim of this paper is to construct some Cheney-Sharma type operators defined on a triangle with one curved side and to study the interpolation properties, the orders of accuracy and the remainders of the corresponding approximation formulas.

Using the interpolation properties of such operators, blending function interpolants can be constructed, that exactly match the function on some sides of the given region. Important applications of these blending functions are in computer aided geometric design, in finite element method for differential equations problems and for construction of surfaces which satisfy some given conditions.

2. UNIVARIATE OPERATORS

Let $m \in \mathbb{N}$ and β be a nonnegative parameter. The Cheney-Sharma operator of second kind $Q_m : C[0,1] \to C[0,1]$, introduced in [7], is given by

$$(Q_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f(\tfrac{k}{m}), \tag{2.1}$$

with

$$q_{m,i}(x) = \binom{m}{i} \frac{x(x+i\beta)^{i-1}(1-x)[1-x+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}.$$

We recall some results regarding these Cheney-Sharma type operators, studied, for example, in [1], [14].

Remark 1. 1) Notice that for $\beta = 0$, the operator Q_m becomes the Bernstein operator.

2) In [14], it has been proved that the Cheney-Sharma operator Q_m interpolates a given function at the endpoints of the interval.

3) In [7] and [14], it has been proved that the Cheney-Sharma operator Q_m reproduces the constant and the linear functions, so its degree of exactness is 1 (denoted $\text{dex}(Q_m) = 1$).

4) In [7] it is given the following result

$$\begin{aligned}(Q_m e_2)(x) = x(1+m\beta)^{1-m}[&S(2,m-2,x+2\beta,1-x) \\ &-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)],\end{aligned} \tag{2.2}$$

where $e_i(x) = x^i$, $i \in \mathbb{N}$, and

$$S(j,m,x,y) = \sum_{k=0}^{m} \binom{m}{k} (x+k\beta)^{k+j-1}[y+(m-k)\beta]^{m-k}, \tag{2.3}$$

$j = \overline{0,m}$, $m \in \mathbb{N}$, $x,y \in [0,1]$, $\beta > 0$.

We consider the standard triangle $\tilde{T}_h$ (see Figure 1), with vertices $V_1 = (0,h)$, $V_2 = (h,0)$ and $V_3 = (0,0)$, with two straight sides Γ_1, Γ_2, along the coordinate axes, and with the third side Γ_3 (opposite to the vertex V_3) defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$, for $h > 0$. Also, we have $f(x) \le h$ and $g(y) \le h$, for $x,y \in [0,h]$.

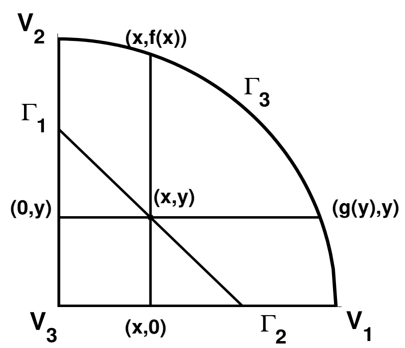

FIGURE 1. Triangle $\tilde{T}_h$.

For $m,n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$, we consider the following extensions of the Cheney-Sharma operator given in (2.1):

$$(Q_m^x F)(x,y) = \sum_{i=0}^{m} q_{m,i}(x,y) F\left(i\frac{g(y)}{m}, y\right), \tag{2.4}$$
$$(Q_n^y F)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, j\frac{f(x)}{n}\right),$$

with

$$q_{m,i}(x,y) = \binom{m}{i} \frac{\frac{x}{g(y)}(\frac{x}{g(y)}+i\beta)^{i-1}(1-\frac{x}{g(y)})[1-\frac{x}{g(y)}+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}},$$
$$q_{n,j}(x,y) = \binom{n}{j} \frac{\frac{y}{f(x)}(\frac{y}{f(x)}+jb)^{j-1}(1-\frac{y}{f(x)})[1-\frac{y}{f(x)}+(n-j)b]^{n-j-1}}{(1+nb)^{n-1}},$$

where

$$\Delta_m^x = \left\{ i\frac{g(y)}{m} \,\middle|\, i=\overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ j\frac{f(x)}{n} \,\middle|\, j=\overline{0,n} \right\}$$

are uniform partitions of the intervals $[0,g(y)]$ and $[0,f(x)]$.

Remark 2. As the Cheney-Sharma operator of second kind interpolates a given function at the endpoints of the interval, we may use the operators Q_m^x and Q_n^y as interpolation operators.

Theorem 1. *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $Q_m^x F = F$ on $\Gamma_1 \cup \Gamma_3$,
(ii) $Q_n^y F = F$ on $\Gamma_2 \cup \Gamma_3$.

Proof. (i) We may write

$$\begin{aligned}(Q_m^x F)(x,y) = \tfrac{1}{(1+m\beta)^{m-1}} \Big\{ &(1-\tfrac{x}{g(y)})[1-\tfrac{x}{g(y)}+m\beta]^{m-1} F(0,y) \\ &+ \tfrac{x}{g(y)}(1-\tfrac{x}{g(y)}) \sum_{i=1}^{m-1} \tbinom{m}{i} (\tfrac{x}{g(y)}+i\beta)^{i-1} \\ &\cdot [1-\tfrac{x}{g(y)}+(m-i)\beta]^{m-i-1} F\left(i\tfrac{g(y)}{m}, y\right) \\ &+ \tfrac{x}{g(y)}(\tfrac{x}{g(y)}+m\beta)^{m-1} F(g(y),y) \Big\}.\end{aligned} \tag{2.5}$$

Considering (2.5), we may easily prove that

$$(Q_m^x F)(0,y) = F(0,y),$$
$$(Q_m^x F)(g(y),y) = F(g(y),y).$$

(ii) Similarly, writing

$$(Q_n^y F)(x,y) = \tfrac{1}{(1+nb)^{n-1}} \Big\{ (1-\tfrac{y}{f(x)})[1-\tfrac{y}{f(x)}+nb]^{n-1} F(x,0)$$

$$+\tfrac{y}{f(x)}(1-\tfrac{y}{f(x)})\sum_{j=1}^{n-1}\tbinom{n}{j}(\tfrac{y}{f(x)}+jb)^{j-1}$$
$$\cdot[1-\tfrac{y}{f(x)}+(n-j)b]^{n-j-1}F\left(x,j\tfrac{f(x)}{n}\right)$$
$$+\tfrac{y}{f(x)}(\tfrac{y}{f(x)}+nb)^{n-1}F(x,f(x))\Big\},$$

we get that

$$(Q_n^yF)(x,0)=F(x,0),$$
$$(Q_n^yF)(x,f(x))=F(x,f(x)).$$

□

Theorem 2. *The operators Q_m^x and Q_n^y have the following orders of accuracy:*

(i) $(Q_m^x e_{ij})(x,y)=x^iy^j,\ \ i=0,1;\ j\in\mathbb{N};$
(ii) $\left(Q_n^y e_{ij}\right)(x,y)=x^iy^j,\ \ i\in\mathbb{N};\ j=0,1,$ where $e_{ij}(x,y)=x^iy^j,\ \ i,j\in\mathbb{N}.$

Proof. (i) We have

$$(Q_m^x e_{ij})(x,y)=y^j\sum_{i=0}^{m}q_{m,i}(x,y)[i\tfrac{g(y)}{m}]^i,$$

and since the degree of exactness of the univariate Cheney-Sharma operator is equal to 1 (see Remark 1), the result follows.

Property (ii) is proved in the same way. □

We consider the approximation formula

$$F=Q_m^xF+R_m^xF,$$

where R_m^xF denotes the approximation error.

Theorem 3. *If $F(\cdot,y)\in C[0,g(y)]$ then we have*

$$\left|(R_m^xF)(x,y)\right|\le(1+\tfrac{1}{\delta}\sqrt{A_m-x^2})\omega(F(\cdot,y);\delta),\quad\forall\delta>0,\tag{2.6}$$

where $\omega(F(\cdot,y);\delta)$ is the modulus of continuity and $A_m=x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]$, with S given in (2.3).

Proof. By Theorem 2 we have that $\mathrm{dex}(Q_m^x)=1$, thus we may apply the following property of linear operators (see, for example, [2], [13]):

$$\left|(Q_m^xF)(x,y)-F(x,y)\right|\le[1+\delta^{-1}\sqrt{(Q_m^xe_{20})(x,y)-x^2}]\omega(F(\cdot,y);\delta),\ \ \forall\delta>0,$$

and taking into account (2.2), we get (2.6). □

Theorem 4. *If $F(\cdot,y)\in C^2[0,g(y)]$ then*

$$\begin{aligned}(R_m^xF)(x,y)=\tfrac{1}{2}F^{(2,0)}(\xi,y)\{x^2-x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)\\ -(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\},\end{aligned}\tag{2.7}$$

for $\xi \in [0, g(y)]$ *and* $\beta > 0$.

Proof. Taking into account that $\mathrm{dex}(Q_m^x) = 1$, by Theorem 2, and applying the Peano's theorem (see, e.g., [12]), it follows

$$(R_m^x F)(x,y) = \int_0^{g(y)} K_{20}(x,y;s) F^{(2,0)}(s,y) ds,$$

where

$$K_{20}(x,y;s) = (x-s)_+ - \sum_{i=0}^{m} q_{m,i}(x,y) \left(i\tfrac{g(y)}{m} - s\right)_+.$$

For a given $\nu \in \{1, \ldots, m\}$ one denotes by $K_{20}^{\nu}(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{g(y)}{m}, \nu\frac{g(y)}{m}\right]$, i.e.,

$$K_{20}^{\nu}(x,y;\nu) = (x-s)_+ - \sum_{i=\nu}^{m} q_{m,i}(x,y) \left(i\frac{g(y)}{m} - s\right),$$

whence,

$$K_{20}^{\nu}(x,y;s) = \begin{cases} x - s - \sum\limits_{i=\nu}^{m} q_{m,i}(x,y) \left(i\frac{g(y)}{m} - s\right), & s < x \\ -\sum\limits_{i=\nu}^{m} q_{m,i}(x,y) \left(i\frac{g(y)}{m} - s\right), & s \geq x. \end{cases}$$

It follows that $K_{20}^{\nu}(x,y;s) \leq 0$, for $s \geq x$.

For $s < x$ we have

$$K_{20}^{\nu}(x,y;s) = x - s - \sum_{i=0}^{m} q_{m,i}(x,y) \left[i\frac{g(y)}{m} - s\right] + \sum_{i=0}^{\nu-1} q_{m,i}(x,y) \left[i\frac{g(y)}{m} - s\right].$$

Applying Theorem 2, we get

$$\sum_{i=0}^{m} q_{m,i}(x,y) \left[i\frac{g(y)}{m} - s\right] = (Q_m^x e_{10})(x,y) - s(Q_m^x e_{00})(x,y) = x - s,$$

whence it follows that

$$K_{20}^{\nu}(x,y;s) = \sum_{i=0}^{\nu-1} q_{m,i}(x,y) \left[i\frac{g(y)}{m} - s\right] \leq 0.$$

So, $K_{20}^{\nu}(x,y;\cdot) \leq 0$, for any $\nu \in \{1, \ldots, m\}$, i.e., $K_{20}(x,y;s) \leq 0$, for $s \in [0, g(y)]$.

By the Mean Value Theorem, one obtains

$$(R_m^x F)(x,y) = F^{(2,0)}(\xi, y) \int_0^{g(y)} K_{20}(x,y;s) ds, \quad \text{for } 0 \leq \xi \leq g(y),$$

with

$$\int_0^{g(y)} K_{20}(x,y;s)ds = \tfrac{1}{2}[x^2 - (Q_m^x e_{20})(x,y)],$$

and using (2.2) we get (2.7). □

Remark 3. Analogous results with the ones in Theorems 3 and 4 could be obtained for the remainder $R_n^y F$ of the formula $F = Q_n^y F + R_n^y F$.

3. Product operators

Let $P_{mn}^1 = Q_m^x Q_n^y$, respectively, $P_{nm}^2 = Q_n^y Q_m^x$ be the products of the operators Q_m^x and Q_n^y.

We have

$$\left(P_{mn}^1 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)\, q_{n,j}\left(i\tfrac{g(y)}{m}, y\right) F\left(i\tfrac{g(y)}{m}, j\frac{f(i\frac{g(y)}{m})}{n}\right),$$

respectively,

$$\left(P_{nm}^2 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}\left(x, j\tfrac{f(x)}{n}\right) q_{n,j}(x,y)\, F\left(i\frac{g(j\frac{f(x)}{n})}{m}, j\tfrac{f(x)}{n}\right).$$

Theorem 5. *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $(P_{mn}^1 F)(V_i) = F(V_i), \quad i = 1,2,3;$
$(P_{mn}^1 F)(\Gamma_3) = F(\Gamma_3),$

(ii) $(P_{nm}^2 F)(V_i) = F(V_i), \quad i = 1,2,3;$
$(P_{nm}^2 F)(\Gamma_3) = F(\Gamma_3),$

Proof. By a straightforward computation, we get the following properties

$$\begin{aligned}
(P_{mn}^1 F)(x,0) &= (Q_m^x F)(x,0),\\
(P_{mn}^1 F)(0,y) &= (Q_n^y F)(0,y),\\
(P_{mn}^1 F)(x,f(x)) &= F(x,f(x)), \qquad x,y \in [0,h]
\end{aligned}$$

and

$$\begin{aligned}
(P_{nm}^2 F)(x,0) &= (Q_m^x F)(x,0),\\
(P_{nm}^2 F)(0,y) &= (Q_n^y F)(0,y),\\
(P_{nm}^2 F)(g(y),y) &= F(g(y),y), \qquad x,y \in [0,h],
\end{aligned}$$

and, taking into account Theorem 1, they imply (i) and (ii). □

We consider the following approximation formula

$$F = P_{mn}^1 F + R_{mn}^{P^1} F,$$

where $R_{mn}^{P^1}$ is the corresponding remainder operator.

Theorem 6. *If $F \in C(\widetilde{T}_h)$ then*

$$\left|(R^{P^1}_{mn}F)(x,y)\right| \le (A_m + B_n - x^2 - y^2 + 1)\omega(F; \tfrac{1}{\sqrt{A_m - x^2}}, \tfrac{1}{\sqrt{B_n - y^2}}),\ \forall (x,y) \in \widetilde{T}_h, \quad (3.1)$$

where

$$\begin{aligned} A_m &= x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x) \\ &\quad -(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)] \\ B_n &= y(1+nb)^{1-n}[S(2,n-2,y+2b,1-y) \\ &\quad -(n-2)bS(2,n-3,y+2b,1-y+\beta)] \end{aligned} \quad (3.2)$$

and $\omega(F;\delta_1,\delta_2)$*, with* $\delta_1 > 0$, $\delta_2 > 0$, *is the bivariate modulus of continuity.*

Proof. Using a basic property of the modulus of continuity we have

$$\begin{aligned} \left|(R^{P^1}_{mn}F)(x,y)\right| \le &\left[\tfrac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x-\tfrac{i}{m}g(y)\right|\right. \\ &+\tfrac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y-\tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right| \\ &\left.+\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\right]\omega(F;\delta_1,\delta_2),\quad \forall \delta_1,\delta_2 > 0. \end{aligned}$$

Since

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x-\tfrac{i}{m}g(y)\right| \le \sqrt{(Q^x_m e_{20})(x,y)-x^2},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y-\tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right| \le \sqrt{(Q^y_n e_{02})(x,y)-y^2},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right) = 1,$$

applying (2.2), we get

$$\begin{aligned} &\left|(R^{P^1}_{mn}F)(x,y)\right| \le \left\{\tfrac{1}{\delta_1}[x(1+m\beta)^{1-m}]^{\frac12}\right. \\ &\cdot\{[S(2,m-2,x+2\beta,1-x) \\ &-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]-x^2\}^{\frac12} + \tfrac{1}{\delta_2}[y(1+nb)^{1-n}]^{\frac12} \\ &\cdot\{[S(2,n-2,y+2b,1-y) \\ &-(n-2)bS(2,n-3,y+2b,1-y+\beta)]-y^2\}^{\frac12} \left.+1\right\}\omega(F;\delta_1,\delta_2). \end{aligned}$$

Denoting

$$A_m = x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)$$

$$
\begin{aligned}
&\quad - (m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\\
B_n &= y(1+nb)^{1-n}[S(2,n-2,y+2b,1-y)\\
&\quad - (n-2)bS(2,n-3,y+2b,1-y+\beta)]
\end{aligned}
$$

and, taking $\delta_1 = \frac{1}{\sqrt{A_m - x^2}}$ and $\delta_2 = \frac{1}{\sqrt{B_n - y^2}}$, we get (3.1). $\square$

4. Boolean sum operators

We consider the Boolean sums of the operators Q_m^x and Q_n^y,

$$
\begin{aligned}
S_{mn}^1 &:= Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y,\\
S_{nm}^2 &:= Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x.
\end{aligned}
$$

Theorem 7. *If F is a real-valued function defined on $\widetilde{T}_h$, then*

$$
\begin{aligned}
S_{mn}^1 F \Big|_{\partial \widetilde{T}_h} &= F\Big|_{\partial \widetilde{T}_h},\\
S_{mn}^2 F \Big|_{\partial \widetilde{T}_h} &= F\Big|_{\partial \widetilde{T}_h}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
(Q_m^x Q_n^y F)(x,0) &= (Q_m^x F)(x,0),\\
(Q_n^y Q_m^x F)(0,y) &= (Q_n^y F)(0,y),\\
(Q_m^x F)(x,h-x) &= (Q_n^y F)(x,h-x)\\
&= (P_{mn}^1 F)(x,h-x) = (P_{nm}^2 F)(x,h-x) = F(x,h-x),
\end{aligned}
$$

and, taking into account Theorem 1, the conclusion follows. $\square$

We consider the following approximation formula

$$
F = S_{mn}^1 F + R_{mn}^{S^1} F,
$$

where $R_{mn}^{S^1}$ is the corresponding remainder operator.

Theorem 8. *If $F \in C(\widetilde{T}_h)$ then*

$$
\begin{aligned}
&\left|(R_{mn}^{S^1} F)(x,y)\right| \le \qquad (4.1)\\
&\le (1 + A_m - x^2)\omega\big(F(\cdot,y); \tfrac{1}{\sqrt{A_m - x^2}}\big) + (1 + B_n - y^2)\omega\big(F(x,\cdot); \tfrac{1}{\sqrt{B_n - y^2}}\big)\\
&+ (A_m + B_n - x^2 - y^2 + 1)\omega\big(F; \tfrac{1}{\sqrt{A_m - x^2}}, \tfrac{1}{\sqrt{B_n - y^2}}\big),
\end{aligned}
$$

with A_m and B_n given in (3.2).

Proof. The identity

$$F - S^1_{mn}F = (F - Q^x_m F) + (F - Q^y_n F) - (F - P^1_{mn}F)$$

implies that

$$\left|(R^{S^1}_{mn}F)(x,y)\right| \leq \left|(R^x_m F)(x,y)\right| + \left|(R^y_n F)(x,y)\right| + \left|(R^{P^1}_{mn}F)(x,y)\right|,$$

and, applying Theorems 3 and 6, we get (4.1). □

5. Numerical examples

We consider the function:

$$\text{Gentle:} \quad F(x,y) = \tfrac{1}{3}\exp[-\tfrac{81}{16}\left((x-0.5)^2 + (y-0.5)^2\right)],$$

generally used in the literature, (see, e.g., [11]). In Figure 2 we plot the graphs of F, $Q^x_m F$, $Q^y_n F$, $P^1_{mn}F$, $S^1_{mn}F$, on $\tilde{T}_h$, considering $h = 1, m = 5$, $n = 6$, $\beta = 1$ and we can see the good approximation properties.

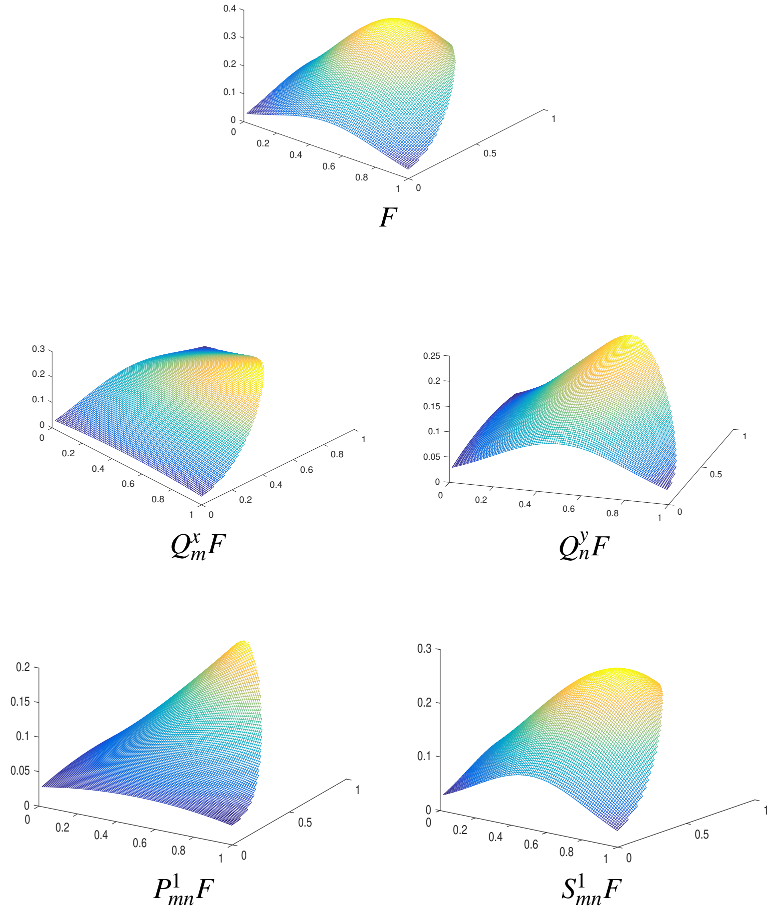

FIGURE 2. The Cheney-Sharma approximants for $\tilde{T}_h$.

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