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EXPONENTIAL MONOMIALS ON HYPERGROUP JOINS

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Dedicated to the memory of Prof. Herbert Heyer

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Abstract. Exponential monomials and polynomials are the basic building blocks of spectral synthesis. Recently a systematic study of exponential polynomials has been started on hypergroups. In this paper we join these investigations and describe exponential polynomials on hypergroup joins.

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1. INTRODUCTION

We started the study of basic function classes on hypergroup joins in our paper [14]. The basic function classes we are studying play a fundamental role in spectral analysis and spectral synthesis (see [9, 12]). In this paper we continue this work and describe further basic function classes, called *moment functions* on hypergroup joins (see [1]). In the sequel $\mathbb{C}$ denotes the set of complex numbers. By a *hypergroup* we always mean a locally compact hypergroup. For basics about hypergroups see the monograph [1]. A comprehensive monograph on functional equations on hypergroups is [7].

Let K be a hypergroup with identity e and involution $^\vee$. The non-identically zero continuous function m is called an *exponential* on K if $m : K \to \mathbb{C}$ satisfies $m(x * y) = m(x)m(y)$ for each x, y in K. The description of exponentials on some types of commutative hypergroups can be found in [7]. In [11] (see also [10]) the author defined the concept of exponential monomial on commutative hypergroups: let K be a commutative hypergroup and $f : K \to \mathbb{C}$ a continuous function. We say that f is a *generalized exponential monomial*, if there exists an exponential $m : K \to \mathbb{C}$ and a natural number n such that

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f = 0$$

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holds for every $y_1, y_2, \ldots, y_{n+1}$ in K. We recall that $\Delta_{m;y_1,y_2,\ldots,y_{n+1}}$ is the convolution product

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} = \Pi_{k=1}^{n+1}(\delta_{\check{y}} - m(y)\delta_o),$$

where, in general, δ_x denotes the point mass supported at x in K, and o denotes the identity in K. It is known that if f is nonzero, then the exponential m is unique, and the smallest n with the above property is called the *degree* of f. In this case we say that f is a *generalized m-exponential monomial*, or f is *associated with m*. We call f simply an *exponential monomial*, if the linear space generated by all translates $\delta_{\check{y}} * f$ is finite dimensional. Clearly, any constant multiple of the exponential m is an m-exponential monomial, and if the constant is nonzero, then it has degree 0. Another simple example is that of the m-sine functions: f is called an *m-sine function*, if it satisfies

$$f(x*y) = f(x)m(y) + f(y)m(x)$$

for each x, y in K. If f is nonzero, then it is an m-exponential monomial of degree 1. Linear combinations of generalized exponential monomials are called *generalized exponential polynomials*, and linear combinations of exponential monomials are called *exponential polynomials*. We note that quadratic functions (see [14]) are not necessarily exponential polynomials, but if the hypergroup structure on K arises from a group structure, then every quadratic function is a generalized exponential monomial associated with the exponential identically 1.

Exponential polynomials have fundamental importance in spectral analysis and synthesis. Spectral synthesis on hypergroups has been investigated in the works [7, 9]. For more about spectral analysis and spectral synthesis see the monograph [12]. Characterization of these functions classes and related functional equations on different types of hypergroups have been studied in several papers(see e.g [2, 8]). In this paper we describe generalized exponential classes on hypergroup joins. The hypergroup join construction is a special way to unite two hypergroups, one of them is compact and the other is discrete. Hypergroup joins and their duals have been studied in [17]. Hypergroup joins were generalized by M. Voit [15] and by H. Heyer and S. Kawakami [3], based on exact sequences.

Another important class is presented by the class of *moment functions*. According to the terminology in [7] we say that the continuous functions $f_n : K \to \mathbb{C}$ ($n = 0, 1, \ldots, N$, N is a natural number) on the hypergroup K form a *(generalized) moment function sequence* of order N if f_0 is non-identically zero and

$$f_n(x*y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) f_{n-k}(y) \tag{1.1}$$

holds for each x, y in K and for $k = 0, 1, \ldots, N$(see e.g [4–6]). Clearly, f_0 is an exponential and we say that the sequence $(f_n)_{n\in\mathbb{N}}$ is *associated with the exponential* f_0. For the sake of simplicity we omit the adjective "generalized". The function f_k in this sequence is called a *moment function* of order k. Hence moment functions of order 0 are exactly the exponentials and moment functions of order 1 associated with the exponential $m = f_0$ are exactly the m-sine functions. In [13] generalized moment functions on hypergroup joins were characterized and described. We shall see below that generalized moment functions are exponential monomials.

2. Hypergroup join

The definition of hypergroup join can be found in [1], p. 59. Here we recall the construction. Let $(C,*)$ be a compact hypergroup with normalized Haar measure ω_C and $(D,\cdot)$ a discrete hypergroup with $C \cap D = \{e\}$, the identity of both hypergroups. The *hypergroup join* $C \vee D$ is the set $C \cup D$ with the unique topology for which both C and D are closed subspaces. Involution on $C \cup D$ is defined in the way that its restriction to C and to D, respectively, coincides with the involution on C and on D, respectively. Convolution on $C \vee D$ is defined in the following way:

(1) For x,y in C the convolution of δ_x and δ_y is $\delta_x * \delta_y$.
(2) For x,y in D and $x \neq \check{y}$ the convolution of δ_x and δ_y is $\delta_x \cdot \delta_y$.
(3) For x in C and $y \neq e$ in D the convolution of δ_x and δ_y and also the convolution of δ_y and δ_x is δ_y.
(4) For $y \neq e$ in D we have the unique representation

$$\delta_y \cdot \delta_{\check{y}} = \sum_{w \in D} c_w \delta_w$$

with some complex numbers c_w for w in D. Then the convolution of δ_y and $\delta_{\check{y}}$ and also the convolution of $\delta_{\check{y}}$ and δ_y is

$$c_e \omega_C + \sum_{w \in D, w \neq e} c_w \delta_w = \delta_y \cdot \delta_{\check{y}} + c_e(\omega_C - \delta_e).$$

For the sake of simplicity, by virtue of 1. above, we denote the convolution in $C \vee D$ with $*$, too. We note that commutativity is not assumed in C nor in D. In fact, $C \vee D$ is commutative if and only if C and D is commutative. Clearly, C is a compact subhypergroup of $C \vee D$, D is a discrete subset of $C \vee D$, but D is not necessarily a subhypergroup of $C \vee D$. For further information about hypergroup joins and their applications see [1].

In what follows we shall always assume that $D \neq \{e\}$ and we denote $D \setminus \{e\}$ with D_e. It follows that D_e is nonempty.

3. Moment functions as exponential monomials

A special case of exponential monomials is presented by the generalized moment functions as it is shown in the following theorem.

Theorem 1. *Let K be a commutative hypergroup and $(\varphi_n)_{n \in \mathbb{N}}$ a generalized moment function sequence. Then φ_n is an exponential monomial of degree at most n for each n.*

Proof. By definition, the sequence satisfies

$$\varphi_n(x*y)=\sum_{k=0}^{n}\varphi_k(x)\varphi_{n-k}(y)$$

for each x,y in K ($n=0,1,\dots$). We prove the statement by induction on n and it is obvious for $n=0$. Clearly $m=\varphi_0$ is an exponential. Suppose that $n\geq 1$ and we have proved our statement for $k=0,1,\dots,n-1$. Now we prove it for $k=n$. Let $y_1,y_2,\dots,y_{n+1}$ be arbitrary in K. We have

$$\begin{aligned}
\Delta_{m;y_1,y_2,\dots,y_{n+1}}\varphi_n(x)&=\Delta_{m;y_1,y_2,\dots,y_n}(\varphi_n(x*y_{n+1})-m(y_{n+1})\varphi_n(x))\\
&=\Delta_{m;y_1,y_2,\dots,y_n}\Big(\sum_{k=0}^{n}\binom{n}{k}\varphi_k(x)\varphi_{n-k}(y_{n+1})\Big)-m(y_{n+1})\Delta_{m;y_1,y_2,\dots,y_n}\varphi_n(x)\\
&=\Delta_{m;y_1,y_2,\dots,y_n}\varphi_n(x).m(y_{n+1})-m(y_{n+1})\Delta_{m;y_1,y_2,\dots,y_{n+1}}\varphi_n(x)=0,
\end{aligned}$$

which proves the statement. □

Theorem 2. *Let K be a commutative hypergroup and $\Phi: K\times\mathbb{C}^n\to\mathbb{C}$ an exponential family for K. Let N be a nonnegative integer and $1\leq i\leq n$ an integer. Then for every polynomial $P:\mathbb{C}^n\to\mathbb{C}$ of degree N the function $x\mapsto P(\partial_\lambda)\Phi(x,\lambda)$ is an exponential monomial of degree at most N.*

Proof. The proof can be found in [10, Theorem 3]. □

4. Exponential monomials on compact hypergroups

Theorem 3. *On a compact commutative hypergroup every nonzero generalized exponential monomial is of degree zero.*

Proof. We prove the statement by induction on the degree of the generalized m-exponential monomial $f\neq 0$, and it is obvious if $\deg f=0$. First we note that every nonzero generalized m-exponential polynomial of degree 0 is a constant multiple of m. Indeed, we have

$$0=\Delta_{m;y}*f(x)=f(x*y)-m(y)f(x)$$

for each x,y in K. Interchanging x and y we obtain $f(x)m(y)=f(y)m(x)$, hence the substitution $y=o$ gives $f(x)=f(o)m(x)$.

We introduce the function $g(x)=f(x)-f(o)m(x)$. Then, by the obvious property

$$\Delta_{m;y}*g=\Delta_{m;y}*f,$$

it follows that g is a generalized m-exponential monomial of degree at most n. In addition, $g(o)=0$.

We assume that we have proved that every nonzero generalized m-exponential polynomial of degree at most $n-1$ is of degree zero, and now we prove it for $\deg g\leq n$. By assumption,

$$\Delta_{m;y_1,y_2,\dots,y_n}*(\Delta_{m;y}*g)(x)=\Delta_{m;y,y_1,y_2,\dots,y_n}*g(x)=0,$$

that is, $\Delta_{m;y} * g$ is a generalized m-exponential polynomial of degree at most $n-1$, hence $\Delta_{m;y} * g = c(y)m(x)$ holds for each y in K, where $c : K \to \mathbb{C}$ is some continuous function. In other words, we have

$$g(x * y) = m(y)g(x) + c(y)m(x)$$

for each x, y in K. Putting $x = o$ we have $g = c$, hence g is an m-sine function, and by the results in [16], g is identically zero. It follows that $f = f(o)m$ and our theorem is proved. $\square$

In [16] M. Voit proved that on commutative compact hypergroups every m-sine function is zero. It is not known if this statement is true on non-commutative compact hypergroups. The following theorem shows that it is true at least for 1-sine functions, that is, for *additive functions*. In fact, we prove a stronger statement.

Theorem 4. *On a compact hypergroup every generalized* 1*-exponential monomial is constant.*

Proof. Let $f : K \to \mathbb{C}$ be a generalized 1-exponential monomial on the compact hypergroup K. If $\deg f = 0$, then we have $f(x * y) - f(x) = 0$, hence with $x = o$ it follows $f(y) = f(o)$ for each y in K.

Assume that we have proved our statement for $\deg f \leq n-1$ and now we let $\deg f = n \geq 1$. We have for $y, y_1, y_2, \ldots, y_n$ in K

$$0 = \Delta_{1;y_1,y_2,\ldots,y_n,y} * f(x) = \Delta_{1;y_1,y_2,\ldots,y_n} * \big(\Delta_{1;y} * f\big)(x) = 0,$$

hence, by assumption, $\Delta_{1;y} * f$ is a constant:

$$\Delta_{1;y} * f(x) = f(x * y) - f(x) = c(y)$$

holds for each x, y in K, where $c : K \to \mathbb{C}$ is a continuous function. We have

$$f(x * y) = f(x) + c(y)$$

whenever x, y is in K. Using associativity we have

$$f(x * y * z) = f(x * y) + c(z) = f(x) + c(y) + c(z),$$

and

$$f(x * y * z) = f(x) + c(y * z),$$

hence we infer $c(y * z) = c(y) + c(z)$ for each y, z in K. As K is compact and c is continuous, the range of c is compact in $\mathbb{C}$. If there is an x_0 such that $c(x_0) \neq 0$, then $c(n \cdot x_0) = c(x_0 * x_0 * \ldots x_0) = n \cdot c(x_0)$, which implies that the range of c is unbounded, a contradiction. Hence $c \equiv 0$ and f is constant. $\square$

In [14] we proved that at least the integral of every m-sine function is zero on any compact hypergroup. Now we prove the analogous result for generalized m-exponential functions on compact hypergroups. We note that, clearly, on a compact hypergroup the integral of a constant is the constant itself, as the integral always refers to the integral with respect to the unique normalized Haar measure.

Theorem 5. *On a compact hypergroup the integral of every non-constant generalized m-exponential monomial is zero.*

Proof. Let K be a compact hypergroup with normalized Haar measure dx. We can show easily, by induction on n, that for any continuous function $g: K \to \mathbb{C}$ and exponential m on K we have

$$\int_K \Delta_{m;y_1,y_2,\ldots,y_{n+1}} * g(x)\,dx = \Pi_{k=1}^{n+1}\big(1-m(y_k)\big)\int_K g(x)\,dx$$

for each $y_1, y_2, \ldots, y_{n+1}$ in K. Indeed, th statement is true for $n = 0$.

Clearly, this equation implies the statement. Indeed, if g is a generalized m-exponential monomial of degree at most n, then the left side is zero for each $y_1, y_2, \ldots, y_{n+1}$ in K. Hence the right side is zero, too, consequently if $\int_K g(x)\,dx \neq 0$, then $m \equiv 1$, that is g is a generalized 1-exponential monomial, and, by the previous theorem, it is constant, which proves our statement. ☐

We note that generalized 1-exponential polynomials are called *generalized polynomials*.

5. Exponential monomials on hypergroup joins

Here we recall the theorem about the description of exponentials on hypergroup joins (see [14, Theorem 1]).

Theorem 6. *Let C, D be as above. The continuous function $m: C \cup D \to \mathbb{C}$ is an exponential on the hypergroup join $C \vee D$ if and only if one of the following possibilities holds:*

i) $m|_C \neq 1$ is an exponential on C and $m|_{D_e}$ is identically zero;
ii) $m|_C$ is identically 1 and $m|_D$ is an exponential on D.

Our main theorem follows.

Theorem 7. *Let C be a compact hypergroup and D a discrete commutative hypergroup. Then the continuous function $f: C \cup D \to \mathbb{C}$ is a generalized exponential monomial of degree at most n on the hypergroup join $C \vee D$ if and only if any of the following cases holds:*

i) $f\big|_C$ is a generalized exponential monomial of degree at most n associated with an exponential $m_C \neq 1$ on C, and $f\big|_{D_e}$ is zero.
ii) $f\big|_C$ is constant, and $f\big|_D$ is a generalized exponential monomial of degree at most n on D.

Proof. In this proof we shall denote the convolution on C and on $C \vee D$ by $x * y$, and on D by $x \cdot y$, further ω_C denotes the unique normalized Haar measure on the compact hypergroup C. We note that $f\big|_C$ is always a generalized exponential monomial of degree at most n on C, as the convolution on $C \vee D$ coincides with the convolution on C and the restriction of any exponential on $C \vee D$ to C is an exponential on C.

Let f be a generalized m-exponential monomial of degree at most n on $C \vee D$. By the previous theorem, we have two possibilities for m. In the first case $m\big|_C$ is an exponential on C, and $m\big|_{D_e} \equiv 0$. Then clearly, $f\big|_C$ is a generalized $m\big|_C$-exponential monomial of degree at most n on C. On the other hand, let $y \neq e$ be in D, then we have for each x in D_e:

$$0 = \Delta_{m;y}^{n+1} * f(x) = f(x),$$

hence f is identically zero on D_e, which is case $i)$ above.

In the second case $m\big|_C$ is identically 1, and $m\big|_D$ is an exponential on D. Clearly, $f\big|_C$ is a generalized 1-exponential monomial on C, hence it is constant, by Theorem 4. We claim that $f\big|_D$ is a generalized $m\big|_D$-exponential monomial of degree at most n on D. We have to prove the equality

$$\Delta_{m|_D;y_1,y_2,\ldots,y_{n+1}} \cdot f(x) = 0$$

whenever $x, y_1, y_2, \ldots, y_{n+1}$ are in D. Here Δ is formed using the convolution in D which is the same as in $C \vee D$ if there is no pair among the elements $x, y_1, y_2, \ldots, y_{n+1}$ which are involutive to each other. But if there are such pairs $z, \check{z}$, then we have

$$f(z * \check{z}) = f(z \cdot \check{z}) + c_e\Big(\int_C f(t)\, d\omega_C(t) - f(e)\Big) = f(z \cdot \check{z}),$$

as $f\big|_C$ is constant, that is $f\big|_C = f(e)$, which implies our statement.

For the converse we suppose first that $i)$ holds, and $f\big|_C$ is a generalized m_C-exponential monomial of degree at most n on C, where $m_C \neq 1$ is an exponential on C, further $f\big|_{D_e}$ is zero. First we note, that $f\big|_C$ is non-constant, as $m_C \neq 1$. We define $m : C \cup D \to \mathbb{C}$ as $m(x) = m_C(x)$ for x in C, and $m(x) = 0$ for x in D_e. Then, by Theorem 1 in [14], m is an exponential on $C \vee D$. We show that f is a generalized m-exponential monomial of degree at most n on K.

If $x, y_1, y_2, \ldots, y_{n+1}$ are in C, then clearly we have

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x) = \Delta_{m_C;y_1,y_2,\ldots,y_{n+1}} * f\big|_C(x) = 0.$$

Let y be in D and x in C. Then $f(x*y)$ is either $f(x)$ or $f(y) = 0$ depending on if $y = e$ or $y \neq e$, by the definition of the convolution on K. Hence

$$\Delta_{m;y} * f(x) = \begin{cases} f(x) - f(x) = 0 & \text{if } y = e \\ f(y) - m(y)f(y) = 0 & \text{if } y \neq e. \end{cases}$$

It follows that

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x) = 0$$

if x is in C and at least one of the y_i's is in D.

Now let x, y be in D, $x \neq \check{y}$. Then we have

$$\Delta_{m;y} * f(x) = f(x*y) - m(y)f(x) = \begin{cases} f(x*y) = f(x \cdot y) = 0 & \text{if } y \neq e \\ 0 & \text{if } y = e. \end{cases}$$

The first part follows form the fact that if $x \neq \check{y}$, then e is not in the support of $x \cdot y$, hence $f(x \cdot y) = \sum_{w \in D_e} c_w f(w) = 0$, as f vanishes on D_e. On the other hand, if $x = \check{y}$, then we have

$$\Delta_{m;y} * f(x) = f(\check{y} * y) - m(y) f(\check{y}) = \begin{cases} f(\check{y} * y) & \text{if } \ y \neq e \\ 0 & \text{if } \ y = e. \end{cases}$$

We recall that, by the definition of the convolution on $C \vee D$, we have

$$\delta_{\check{y}} * \delta_y = \delta_y \cdot \delta_{\check{y}} + c_e(\omega_C - \delta_e),$$

where c_e is the coefficient of δ_e in the expansion

$$\delta_y \cdot \delta_{\check{y}} = \sum_{w \in D} c_w \delta_w$$

on the hypergroup D. It follows

$$\begin{aligned} f(\check{y} * y) &= f(\check{y} \cdot y) + c_e\Big(\int_C f(t)\, d\omega_C(t) - f(e)\Big) \\ &= \sum_{w \in D} c_w f(w) - c_e f(e) + c_e \int_C f(t)\, d\omega_C(t) \\ &= \sum_{w \in D, w \neq e} c_w f(w) + c_e \int_C f(t)\, d\omega_C(t) \\ &= c_e \int_C f(t)\, d\omega_C(t), \end{aligned}$$

as f vanishes on D_e. Since f is non-constant on C, this integral is zero, by Theorem 5.

We conclude that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f(x) = 0$$

holds for each $x, y_1, y_2, \dots, y_{n+1}$ in $C \cup D$, hence f is a generalized m-exponential monomial on $C \vee D$ of degree at most n.

Now we assume that $ii)$ holds, that is, $f\big|_C$ is constant, and $f\big|_D$ is a generalized m_D-exponential monomial of degree at most n on D with some exponential m_D on D. Now we define $m : C \cup D \to \mathbb{C}$ as $m(x) = 1$ for x in C and $m(x) = m_D(x)$ for x in D. Then m_D is an exponential on $C \vee D$, by Theorem 6. We claim that f is a generalized m-exponential monomial of degree at most n on $C \vee D$.

As $f\big|_C$ is constant and $m\big|_C = 1$, we clearly have

$$\Delta_{m;y} * f(x) = 0$$

for each x, y in C. It follows that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f(x) = 0$$

holds for each $x, y_1, y_2, \dots, y_{n+1}$ in C.

For x in C and y in D we have

$$\Delta_{m;y} * f(x) = f(x*y) - m(y)f(x) = f(y) - f(e)m_D(y) = \Delta_{m_D;y} \cdot f(e).$$

By iteration, we have

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x) = \Delta_{m_D;y_1,y_2,\ldots,y_{n+1}} \cdot f(e) = 0,$$

for each x in C and $y_1, y_2, \ldots, y_{n+1}$ in D, as, by assumption, $f\big|_D$ is a generalized m_D-exponential monomial of degree at most n on D. We note that on the right side Δ is formed by using convolution in D.

For x in D and y in C we have

$$\Delta_{m;y} * f(x) = f(x*y) - m(y)f(x) = f(x) - f(x) = 0,$$

hence by iteration, we have again

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x) = 0$$

whenever x is in D and there is a y_i which is in C.

If x, y are in D and $x \neq \check{y}$, then

$$\Delta_{m;y} * f(x) = f(x*y) - m(y)f(x) = f(x \cdot y) - m_D(y)f(x).$$

On the other hand, if $x = \check{y}$, then

$$\begin{aligned}\Delta_{m;\check{y}} * f(y) &= f(y*\check{y}) - m_D(\check{y})f(y)\\ &= f(y\cdot\check{y}) + c_e\Big(\int_C f(t)\,d\omega_C(t) - f(e)\Big) - m_D(\check{y})f(y)\\ &= f(y\cdot\check{y}) - m_D(\check{y})f(y),\end{aligned}$$

as $f(t) = f(e)$ for t in C. By iteration, we conclude, that if $x, y_1, y_2, \ldots, y_{n+1}$ are in D then

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x) = \Delta_{m_D;y_1,y_2,\ldots,y_{n+1}} \cdot f(x) = 0,$$

where Δ on the right side formed by using the convolution in D, and, by assumption, $f\big|_D$ is a generalized m_D-exponential monomial of degree at most n on D. The proof is complete. $\square$

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