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A CERTAIN CLASS OF ANALYTIC FUNCTIONS OF COMPLEX ORDER CONNECTED WITH A q-ANALOGUE OF INTEGRAL OPERATORS

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Abstract. In this paper, we introduce a certain class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ of normalized analytic functions of complex order connected with a q-analogue of integral operators. For this complex-order analytic function class, we determine a sufficient condition in terms of the coefficients, estimates for the coefficients and a maximization theorem concerning the coefficients. Various consequences and applications of our main results are also considered. A brief remark about the demonstrated equivalence of the q-calculus and the so-called (p,q)-calculus is also presented.

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Keywords: analytic functions, univalent functions, complex-order analytic functions, complex-order univalent functions, integral operators, q-numbers and q-factorials, q-derivative operator, maximization theorem, equivalence of the q-calculus and the so-called (p,q)-calculus

1. Introduction, Definitions and Preliminaries

The theory of q-calculus plays an important rôle in many areas of mathematical, physical and engineering sciences. Jackson (see [8] and [9]) was the first to have some applications of the q-calculus and introduced the q-analogue of the classical derivative and integral operators (see also [1]). Let $\mathcal{A}$ denote the class of functions $f(z)$ of the following normalized form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \qquad (z\in\mathbb{U}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}$ given by

$$\mathbb{U}=\{z : z\in\mathbb{C} \quad \text{and} \quad |z|<1\}).$$

We also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of normalized analytic functions which are univalent in $\mathbb{U}$.

For a function $f(z)$ given by (1.1) and the $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (z \in \mathbb{U}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined here by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.3}$$

We use Ω to denote the class of Schwarz functions $w(z)$, which are analytic in U and satisfy the conditions

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \mathbb{U}).$$

We now define the integral operator $\mathcal{K}_{m,n}^{\alpha} : \mathcal{A} \to \mathcal{A}$ for $\alpha > 0$ and $m \geqq 0$ as follows:

$$\mathcal{K}_m^0 f(z) = f(z)$$

and

$$\mathcal{K}_m^{\alpha} f(z) = \frac{(m+1)^{\alpha}}{\Gamma(\alpha)\, z^m} \int_0^z t^{m-1} \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt. \tag{1.4}$$

For $f \in \mathcal{A}$, it can be easily verified that

$$\mathcal{K}_m^{\alpha} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k. \tag{1.5}$$

Next, for $0 < q < 1$, the q-derivative of the function $\mathcal{K}_m^{\alpha} f(z) \in \mathcal{A}$ is defined by

$$D_q \left\{\mathcal{K}_m^{\alpha} f(z)\right\} = \frac{\mathcal{K}_m^{\alpha} f(z) - \mathcal{K}_m^{\alpha} f(qz)}{z(1-q)} \qquad (z \neq 0), \tag{1.6}$$

so that

$$D_q \left\{ z + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k \right\} = 1 + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} [k]_q \; a_k z^{k-1},$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j \qquad \text{and} \qquad [0]_q = 0.$$

Remark 1. The first usage of the above-defined q-derivative operator D_q in Geometric Function Theory was made in 1990 by Ismail *et al.* [7] (see also [1]). Moreover, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a 1989 book-chapter by Srivastava (see, for details, [21]). Several recent developments on various applications of the the q-derivative operator D_q in Geometric Function Theory can be found in (for example) [2, 13, 14, 16, 18, 22–24, 26].

It is easily seen from (1.6) that

$$z\,D_q\left\{\mathcal{K}_m^{\alpha} f(z)\right\} = z + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} [k]_q\; a_k z^k. \tag{1.7}$$

For any non-negative integer n, the q-factorial $[n]_q!$ is given by

$$[n]_q! = \begin{cases} 1 & (k=0) \\ [1]_q\;[2]_q\;[3]_q\;\cdots\;[n]_q & (n \in \mathbb{N}), \end{cases} \tag{1.8}$$

where $\mathbb{N}$ denotes the set positive integers. Also the q-Pochhammer symbol $[\lambda]_{q,n}$ $(\nu \in \mathbb{C})$ is defined by

$$[\nu]_{q,n} = \begin{cases} 1 & (n=0) \\ [\nu]_q\;[\nu+1]_q\;\cdots\;[\nu+n-1]_q & (n \in \mathbb{N}). \end{cases} \tag{1.9}$$

For $\lambda > -1$, we define the operator $\mathcal{N}_{m,q}^{\lambda,\alpha}$ by

$$\mathcal{N}_{m,q}^{\lambda,\alpha} f(z) * \mathcal{M}_{q,\lambda+1}(z) = z\,D_q\left\{\mathcal{K}_m^{\alpha} f(z)\right\}, \tag{1.10}$$

where the function $\mathcal{M}_{q,\lambda+1}(z)$ is given by

$$\mathcal{M}_{q,\lambda+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\lambda+1]_{q,k-1}}{[k-1]_q!}\, z^k.$$

We thus obtain

$$\begin{aligned} \mathcal{N}_{m,q}^{\lambda,\alpha} f(z) &= z + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} \frac{[k]_q\;[k-1]_q!}{[\lambda+1]_{q,k-1}}\, a_k z^k \\ &= z + \sum_{k=2}^{\infty} \frac{[k]_q!}{[\lambda+1]_{q,k-1}} \left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k \end{aligned} \tag{1.11}$$

$$(\alpha > 0;\ \lambda > -1;\ m \geqq 0;\ 0 < q < 1).$$

We can easily verify from (1.11) that

$$[\lambda+1]_q\;\mathcal{N}_{m,q}^{\lambda,\alpha} f(z) = [\lambda]_q\;\mathcal{N}_{m,q}^{\lambda+1,\alpha} f(z) + q^{\lambda}\, z\,D_q\left\{\mathcal{N}_{m,q}^{\lambda+1,\alpha} f(z)\right\}. \tag{1.12}$$

We also note that

$$\lim_{q\to 1-} \mathcal{N}_{m,q}^{\lambda,\alpha} f(z) = I_m^{\lambda,\alpha} f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\lambda+1)_{k-1}} \left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k. \tag{1.13}$$

In the special case when $\alpha = 0$, we have

$$\mathcal{N}_{m,q}^{\lambda,0} f(z) =: \mathfrak{J}_q^{\lambda} f(z).$$

The operator in $\mathfrak{J}_q^{\lambda} f(z)$ was studied by Arif *et al.* [5].

Definition 1. We say that a function $f(z)$ belonging to $\mathcal{A}$ is in the normalized complex-order analytic function class

$$\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M}) \qquad \left(\zeta \in \mathbb{C}^* := \mathbb{C}\setminus\{0\};\ \mathcal{M} > \frac{1}{2}\right)$$

if and only if

$$\left|1-\frac{1}{\zeta}+\frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\zeta\,\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}-\mathcal{M}\right| < \mathcal{M} \tag{1.14}$$

$$(\alpha > 0;\ \lambda > -1;\ m \geqq 0;\ 0 < q < 1;\ z \in \mathbb{U}).$$

By letting $q \to 1-$, it follows from the work of Kulshrestha [11] that

$$g(z) \in \mathcal{H}_{m,q}^{1,0}(1,\mathcal{M}) = F(1,\mathcal{M})$$

if and only if

$$\frac{zg'(z)}{g(z)} = \frac{1+w(z)}{1-mw(z)} \qquad \left(m = 1-\frac{1}{\mathcal{M}};\ \mathcal{M} > \frac{1}{2};\ w(z) \in \Omega\right) \tag{1.15}$$

for $z \in \mathbb{U}$.

It can easily be shown that $f(z) \in \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ if and only if there exists a function

$$g(z) \in \lim_{q\to 1-} \mathcal{H}_{m,q}^{1,0}(1,\mathcal{M}) = F(1,\mathcal{M})$$

such that

$$\mathcal{N}_{m,q}^{\lambda,\alpha}f(z) = z\left[\frac{g(z)}{z}\right]^{\zeta}. \tag{1.16}$$

Thus, from (1.15) and (1.16), it follows that $f(z) \in \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ if and only if

$$\frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)} = \frac{1+[\zeta(1+m)-m]\,w(z)}{1-mw(z)} \tag{1.17}$$

$$\left(m = 1-\frac{1}{\mathcal{M}};\ \mathcal{M} > \frac{1}{2};\ w(z) \in \Omega\right)$$

for $z \in \mathbb{U}$.

By giving specific values to the parameters λ, α and ζ, we obtain the following interesting subclasses:

(i) $\lim_{q\to 1-} \mathcal{H}_{m,q}^{1,0}(\zeta,\mathcal{M}) = F(\zeta,\mathcal{M})$ (see Nasr and Aouf [17]);

(ii) $\lim_{q\to 1-} \mathcal{H}_{m,q}^{1,0}(1,\mathcal{M}) = F(1,\mathcal{M})$ (see Singh and Singh [19]);

(iii) $\lim_{q\to 1-} \mathcal{H}_{m,q}^{1,0}(\cos\lambda e^{-i\lambda},\mathcal{M}) = F_{\lambda,\mathcal{M}}$ $\left(|\lambda| < \frac{\pi}{2}\right)$ (see Kulshrestha [11]);

(iv) $\lim\limits_{q\to 1-} \mathcal{H}_{m,q}^{1,0}((1-\alpha)\cos\lambda e^{-i\lambda}, \infty) = S^{\lambda}(\alpha) \qquad \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \alpha < 1\right)$
(see Libera [12]; see also Chichra [6] and Sižuk [20]);

(v) $\lim\limits_{q\to 1-} \mathcal{H}_{m,q}^{1,0}\big((1-\alpha)\cos\lambda e^{-i\lambda}, \mathcal{M}\big) = F_{\mathcal{M}}(\lambda,\alpha) \qquad \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \alpha < 1\right)$
(see Aouf [3] and Aouf [4]).

We also have the following presumably new function classes:

(i) $\lim\limits_{q\to 1-} \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M}) =: \mathcal{S}_{m}^{\lambda,\alpha}(\zeta,\mathcal{M})$, where

$$\mathcal{S}_{m}^{\lambda,\alpha}(\zeta,\mathcal{M}) := \Bigg\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad \left| 1 - \frac{1}{\zeta} + \frac{z\big(I_m^{\lambda,\alpha} f(z)\big)'}{\zeta\, I_m^{\lambda,\alpha} f(z)} - \mathcal{M} \right| < \mathcal{M}$$
$$\left(M > \frac{1}{2};\ \zeta \in \mathbb{C}^*;\ \alpha > 0;\ \lambda > -1;\ m \geqq 0;\ z \in \mathbb{U}\right) \Bigg\};$$

(ii) $\mathcal{H}_{m,q}^{\lambda,0}(\zeta,\mathcal{M}) =: \mathcal{F}_{q}^{\lambda}(\zeta,\mathcal{M})$, where

$$\mathcal{F}_{q}^{\lambda}(\zeta,\mathcal{M}) := \Bigg\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad \left| 1 - \frac{1}{\zeta} + \frac{z\big(\mathfrak{I}_q^{\lambda} f(z)\big)'}{\zeta\, \mathfrak{I}_q^{\lambda} f(z)} - \mathcal{M} \right| < \mathcal{M}$$
$$\left(\mathcal{M} > \frac{1}{2};\ \zeta \in \mathbb{C}^*;\ \lambda > -1;\ m \geqq 0;\ 0 < q < 1;\ z \in \mathbb{U}\right) \Bigg\}.$$

From the above definitions of the function classes $F(\zeta,\mathcal{M})$ and $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$, we note that

$$f(z) \in \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M}) \quad \Longleftrightarrow \quad \mathcal{N}_{m,q}^{\lambda,\alpha} f(z) \in F(\zeta,\mathcal{M}). \tag{1.18}$$

The purpose of the present paper is to determine a sufficient condition in terms of the coefficients for functions belonging to the normalized complex-order analytic function class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$, estimates for the coefficients and a maximization theorem involving $\left|a_3 - \mu a_2^2\right|$ for the class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ for complex values of the parameter μ.

2. Sufficient Condition for a Function to be in the Class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$

Unless otherwise mentioned, we assume throughout this paper that

$$\alpha > 0,\ \ \lambda > -1,\ \ m \geqq 0,\ \ 0 < q < 1,\ \ \zeta \in \mathbb{C}^*,$$
$$m = 1 - \frac{1}{\mathcal{M}},\ \ \mathcal{M} > \frac{1}{2} \quad \text{and} \quad z \in \mathbb{U}.$$

Theorem 1. *Let the function $f(z)$ be defined by (1.1). Also let the following inequality holds true:*

$$\sum_{k=2}^{\infty}\left\{(k-1)+|\zeta(1+m)+m(k-1)|\right\}|a_k|\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|\zeta(1+m)|. \tag{2.1}$$

Then $f(z)$ belongs to the class normalized complex-order analytic function class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$.

Proof. Suppose that the inequality (2.1) holds true. Then we find for $z\in\mathbb{U}$ that

$$\begin{aligned}
&\left|z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right|-\left|\zeta(1+m)\,\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)+m\left\{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right\}\right|\\
&=\left|\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right|\\
&\quad-\left|\zeta(1+m)\left\{z+\sum_{k=2}^{\infty}\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right\}\right.\\
&\quad\left.+m\left\{\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right\}\right|\\
&=\left|\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right|\\
&\quad-\left|\zeta(1+m)z+\sum_{k=2}^{\infty}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right|\\
&\leqq\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|a_k|\,r^k\\
&\quad-\left\{|\zeta(1+m)|\,r-\sum_{k=2}^{\infty}\{|\zeta(1+m)|+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|a_k|\,r^k\right\}\\
&=\sum_{k=2}^{\infty}\{(k-1)+|\zeta(1+m)|+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|a_k|\,r^k-|\zeta(1+m)|\,r.
\end{aligned}$$

Letting $r\to 1-$ in the above equation, we get

$$\begin{aligned}
&\left|z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right|-\left|\zeta(1+m)\,\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)+m\left\{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right\}\right|\\
&\leqq\sum_{k=2}^{\infty}\{(k-1)+|\zeta(1+m)|+m(k-1)\}\,|a_k|\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}-|\zeta(1+m)|\leqq 0,
\end{aligned}$$

where we have made use of the assertion (2.1) of Theorem 1. Consequently, we obtain

$$\left| \frac{\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)} - 1}{\zeta(1+m) + m\left(\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)} - 1\right)} \right| < 1 \qquad (z \in \mathbb{U}).$$

If we now set

$$w(z) = \frac{\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)} - 1}{\zeta(1+m) + m\left(\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)} - 1\right)},$$

then $w(0) = 0$, $w(z)$ is analytic in the open unit disk $\mathbb{U}$ and

$$|w(z)| < 1 \qquad (z \in \mathbb{U}).$$

Hence we have

$$\frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)} = \frac{1 + [\zeta(1+m) - m]\, w(z)}{1 - m w(z)},$$

which shows that the function $f(z)$ belongs to the class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta, \mathcal{M})$. □

In the limit when $q \to 1-$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the function $f(z)$ be defined by (1.1). Also let the following inequality holds true*:

$$\sum_{k=2}^{\infty} \{(k-1) + |\zeta(1+m) + m(k-1)|\}\, |a_k| \cdot \frac{k!}{(\lambda+1)_{k-1}} \left(\frac{m+1}{m+k}\right)^{\alpha} \leqq |\zeta(1+m)|. \tag{2.2}$$

Then the function $f(z)$ belongs to the class $\mathcal{S}_{m}^{\lambda,\alpha}(\zeta, \mathcal{M})$.

If we set $\alpha = 0$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let the function $f(z)$ be defined by (1.1). Also let the following inequality holds true*:

$$\sum_{k=2}^{\infty} \{(k-1) + |\zeta(1+m) + m(k-1)|\}\, |a_k|\, \frac{[k]_q!}{[\lambda+1]_{q,k-1}} \leqq |\zeta(1+m)|. \tag{2.3}$$

Then the function $f(z)$ belongs to the class $\mathcal{F}_{q}^{\lambda}(\zeta, \mathcal{M})$.

3. Coefficient Estimates

In this section, we first state and prove the following result.

Theorem 2. *Let the function $f(z)$ given by (1.1) be in the normalized complex-order analytic function class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$.*
(a) *If*

$$2m(k-1)\Re(\zeta) > (k-1)^2(1-m) - |\zeta|^2(1+m),$$

let

$$\mathcal{G} = \left[\frac{2m(k-1)\Re(\zeta)}{(k-1)^2(1-m) - |\zeta|^2(1+m)}\right] \qquad (k = 2,3,4,\cdots,j-1),$$

where $\mathcal{N} = [\mathcal{G}]$ (the Gaussian symbol) and $[\mathcal{G}]$ is the greatest integer not greater than $\mathcal{G}$. Then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\left(\dfrac{m+1}{m+j}\right)^{\alpha}(j-1)!}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)| \tag{3.1}$$

$$(j = 2,3,4,\cdots,\mathcal{N}+2)$$

and

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\,(j-1)\left(\dfrac{m+1}{m+j}\right)^{\alpha}(\mathcal{N}+1)!}\cdot\prod_{k=2}^{\mathcal{N}+3}|\zeta(1+m)+m(k-2)|. \tag{3.2}$$

$$(j > \mathcal{N}+2)$$

(b) *If*

$$2m(k-1)\Re(\zeta) \leqq (k-1)^2(1-m) - |\zeta|^2(1+m),$$

then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}(1+m)|\zeta|}{\left(\dfrac{m+1}{m+j}\right)^{\alpha}[j]_q!\,(j-1)} \qquad (j \geqq 2). \tag{3.3}$$

The inequalities (3.1) and (3.3) are sharp.

Proof. Let us assume that $f(z) \in \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$. Then we find from (1.16) that

$$\sum_{k=2}^{\infty}(k-1)\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_k z^k$$
$$= \left\{\zeta(1+m)z + \sum_{k=2}^{\infty}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_k z^k\right\}w(z), \tag{3.4}$$

which is equivalent to

$$\sum_{k=2}^{j}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_k z^k+\sum_{k=j+1}^{\infty}c_k z^k$$
$$=\left\{\zeta(1+m)z+\sum_{k=2}^{j-1}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_k z^k\right\}w(z),$$

where the coefficients c_j are some complex numbers and the series $\sum_{k=j+1}^{\infty}c_k z^k$ converges when $z\in\mathbb{U}$. Then, since

$$|w(z)|<1\qquad(z\in\mathbb{U}),$$

we have

$$\left|\sum_{k=2}^{j}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_k z^k+\sum_{k=j+1}^{\infty}c_k z^k\right|$$
$$\leqq\left|\zeta(1+m)z+\sum_{k=2}^{j-1}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_k z^k\right|. \tag{3.5}$$

Squaring both sides of (3.5), we get

$$\sum_{k=2}^{j}(k-1)^2\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}|a_k|^2r^{2k}+\sum_{k=j+1}^{\infty}|c_k|^2\,r^{2k}$$
$$\leqq\left\{(1+m)^2|\zeta|^2r^2+\sum_{k=2}^{j-1}|\zeta(1+m)+m(k-1)|^2\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\right.$$
$$\left.\cdot\left(\frac{m+1}{m+k}\right)^{2\alpha}|a_k|^2\,r^{2k}\right\}.$$

We now let $r\to 1-$. Then, on some simplification, we obtain

$$(j-1)^2|a_j|^2\left(\frac{[j]_q!}{[\lambda+1]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}$$
$$\leqq(1+m)^2|\zeta|^2+\sum_{k=2}^{j-1}\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}$$
$$\cdot|a_k|^2\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}. \tag{3.6}$$

The following two cases arise:

(a) Let

$$2m(k-1)\Re(\zeta)>(k-1)^2(1-m)-|\zeta|^2(1+m).$$

Suppose also that $j \leqq \mathcal{N}+2$. Then, for $j=2$, the equation (3.6) gives

$$|a_2| \leqq \frac{(1+m)\,[\lambda+1]_{q,1}\,|\zeta|}{[2]_q!\left(\dfrac{m+1}{m+2}\right)^{\alpha}},$$

which yields (3.1) for $j=2$. We establish the assertion (3.1) by appealing to the principle of mathematical induction. Suppose (3.1) is valid for $k=2,3,4,\cdots,j-1$. Then, clearly, it follows from (3.6) that

$$\begin{aligned}
&(j-1)^2\,|a_j|^2\left(\frac{[j]_q!}{[\lambda+1]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}\\
&\leqq (1+m)^2\,|\zeta|^2+\sum_{k=2}^{j-1}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}\\
&\quad\cdot\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}\\
&\quad\cdot\frac{\left([\lambda+1]_{q,k-1}\right)^2}{\left([k]_q!\right)^2\left(\dfrac{m+1}{m+k}\right)^{2\alpha}((k-1)!)^2}\prod_{p=2}^{k}|\zeta(1+m)+m(p-2)|^2\\
&=\frac{1}{((j-2)!)^2}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)|^2.
\end{aligned}$$

We thus find that

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\left(\dfrac{m+1}{m+j}\right)^{\alpha}(j-1)!}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)|,$$

which completes the proof of the assertion (3.1) of Theorem 2.

We next suppose that $j > \mathcal{N}+2$. Then (3.6) gives

$$\begin{aligned}
&(j-1)^2\,|a_j|^2\left(\frac{[j,q]!}{[\lambda+1]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}\\
&\leqq (1+m)^2\,|\zeta|^2+\sum_{k=2}^{\mathcal{N}+2}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}\\
&\quad\cdot\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}|a_k|^2\\
&\quad+\sum_{k=\mathcal{N}+3}^{j-1}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}
\end{aligned}$$

$$\cdot\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}|a_k|^2$$
$$\leqq (1+m)^2|\zeta|^2+\sum_{k=2}^{\mathcal{N}+2}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}$$
$$\cdot\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}|a_k|^2.$$

Upon substituting the above-derived upper estimates for $a_2,a_3,\cdots,a_{\mathcal{N}+2}$ if we simplify the resulting equations, we obtain the assertion (3.2) of Theorem 2.

(b) If we let

$$2m(k-1)\Re(\zeta)\leqq (k-1)^2(1-m)-|\zeta|^2(1+m),$$

then it follows from (3.6) that

$$\left(\frac{[j]_q!}{[\lambda+1,]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}(j-1)^2|a_j|^2\leqq(1+m)^2|\zeta|^2 \qquad (j\geqq 2),$$

which proves the assertion (3.3) of Theorem 2. □

Taking $q\to 1-$ in Theorem 2, we obtain the following corollary.

Corollary 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}_m^{\lambda,\alpha}(\zeta,\mathcal{M})$.*

(a) *If*

$$2m(k-1)\Re(\zeta)>(k-1)^2(1-m)-|\zeta|^2(1+m),$$

let

$$\mathcal{G}=\left[\frac{2m(k-1)\Re(\zeta)}{(k-1)^2(1-m)-|\zeta|^2(1+m)}\right] \qquad (k=2,3,4,\cdots,j-1),$$

where $\mathcal{N}=[\mathcal{G}]$ (the Gaussian symbol) and $[\mathcal{G}]$ is the greatest integer not greater than $\mathcal{G}$. Then

$$|a_j|\leqq\frac{(\lambda+1)_{k-1}}{\left(\frac{m+1}{m+j}\right)^{\alpha}k!(j-1)!}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)| \tag{3.7}$$
$$(j=2,3,4,\cdots,\mathcal{N}+2)$$

and

$$|a_j|\leqq\frac{(\lambda+1)_{k-1}}{(j-1)k!\left(\frac{m+1}{m+j}\right)^{\alpha}(\mathcal{N}+1)!}\prod_{k=2}^{\mathcal{N}+3}|\zeta(1+m)+m(k-2)| \tag{3.8}$$
$$(j>\mathcal{N}+2).$$

(b) *If*

$$2m(k-1)\Re(\zeta) \leqq (k-1)^2(1-m) - |\zeta|^2(1+m),$$

then

$$|a_j| \leqq \frac{(\lambda+1)_{k-1}(1+m)|\zeta|}{\left(\frac{m+1}{m+j}\right)^{\alpha} k!(j-1)} \qquad (j \geqq 2). \tag{3.9}$$

The inequalities (3.7) and (3.9) are sharp.

If we set $\alpha = 0$ in Theorem 2, then we obtain the following corollary.

Corollary 4. *Let the function $f(z)$ be defined by (1.1) be in the class $\mathcal{F}_q^{\lambda}(\zeta, \mathcal{M})$.*
(a) *If*

$$2m(k-1)\Re(\zeta) > (k-1)^2(1-m) - |\zeta|^2(1+m),$$

let

$$\mathcal{G} = \left[\frac{2m(k-1)\Re(\zeta)}{(k-1)^2(1-m) - |\zeta|^2(1+m)}\right] \qquad (k = 2, 3, 4, \cdots, j-1),$$

where $\mathcal{N} = [\mathcal{G}]$ (the Gaussian symbol) and $[\mathcal{G}]$ is the greatest integer not greater than $\mathcal{G}$. Then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\,(j-1)!} \prod_{k=2}^{j} |\zeta(1+m) + m(k-2)| \tag{3.10}$$

$$(j = 2, 3, 4, \cdots, \mathcal{N}+2)$$

and

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!(j-1)(\mathcal{N}+1)!} \prod_{k=2}^{\mathcal{N}+3} |\zeta(1+m) + m(k-2)| \tag{3.11}$$

$$(j > \mathcal{N}+2).$$

(b) *If*

$$2m(k-1)\Re(\zeta) \leqq (k-1)^2(1-m) - |\zeta|^2(1+m),$$

then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}(1+m)|\zeta|}{[j]_q!\,(j-1)} \qquad (j \geqq 2). \tag{3.12}$$

The inequalities (3.10) and (3.12) are sharp.

4. Maximization of $\left|a_3 - \mu a_2^2\right|$

In this section, we shall need the following lemma in our discussion.

Lemma 1 ([10]). *Let*

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega.$$

If μ *is any complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leqq \max\{1, |\mu|\} \tag{4.1}$$

for any complex number μ. *Equality in (4.1) may be attained with the functions* $w(z) = z^2$ *and* $w(z) = z$ *for* $|\mu| < 1$ *and* $|\mu| \geqq 1$, *respectively.*

We now state and prove our main result in this section.

Theorem 3. *Let the function* $f(z)$ *defined by (1.1) be in the normalized complex-order analytic function class* $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta, \mathcal{M})$. *Suppose also that* μ *is any complex number. Then*

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{|\zeta(1+m)|}{2\dfrac{[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}} \max\{1, |\delta|\}, \tag{4.2}$$

where

$$\delta = \frac{2\dfrac{[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha} \mu\zeta(1+m)}{\left(\dfrac{[2]_q!}{[\lambda+1]_q}\right)^2 \left(\dfrac{m+1}{m+2}\right)^{2\alpha}} - [\zeta(1+m)+m]. \tag{4.3}$$

The result is sharp.

Proof. Since $f(z) \in \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta, \mathcal{M})$, we have

$$\begin{aligned}
w(z) &= \frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)\right)' - \mathcal{N}_{m,q}^{\lambda,\alpha} f(z)}{[\zeta(1+m)-m]\, \mathcal{N}_{m,q}^{\lambda,\alpha} f(z) + mz\left(\mathcal{N}_{m,q}^{\lambda,\alpha} f(z)\right)'} \\
&= \frac{\sum\limits_{k=2}^{\infty} (k-1) \dfrac{[3]_q!}{[\lambda+1]_{q,2}} \left(\dfrac{m+1}{m+3}\right)^{\alpha} a_k z^{k-1}}{\zeta(1+m) + \sum\limits_{k=2}^{\infty} [\zeta(1+m) + m(k-1)] \dfrac{[k]_q!}{[\lambda+1]_{q,k-1}} \left(\dfrac{m+1}{m+k}\right)^{\alpha} a_k z^{k-1}} \\
&= \frac{\sum\limits_{k=2}^{\infty} (k-1) \dfrac{[k]_q!}{[\lambda+1]_{q,k-1}} \left(\dfrac{m+1}{m+k}\right)^{\alpha} a_k z^{k-1}}{\zeta(1+m)}
\end{aligned}$$

$$\cdot\left(1+\frac{\sum\limits_{k=2}^{\infty}[\zeta(1+m)+m(k-1)]\dfrac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\dfrac{m+1}{m+k}\right)^{\alpha}a_k z^{k-1}}{\zeta(1+m)}\right)^{-1}. \tag{4.4}$$

We now compare the coefficients of z and z^2 on both sides of the last equation (4.4). We thus obtain

$$a_2=\frac{\zeta(1+m)\,[\lambda+1]_q\,c_1}{[2]_q!\left(\dfrac{m+1}{m+2}\right)^{\alpha}} \tag{4.5}$$

and

$$a_3=\frac{\zeta(1+m)}{\dfrac{2\,[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}}\left\{c_2+[\zeta(1+m)+m]\,c_1^2\right\}. \tag{4.6}$$

Hence

$$a_3-\mu a_2^2=\frac{\zeta(1+m)}{\dfrac{2\,[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}}\left\{c_2-\phi c_1^2\right\}, \tag{4.7}$$

where

$$\phi=\frac{\dfrac{2\,[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}\mu\zeta(1+m)}{\left(\dfrac{[2]_q!}{[\lambda+1]_q}\right)^{2}\left(\dfrac{m+1}{m+2}\right)^{2\alpha}}-[\zeta(1+m)+m]. \tag{4.8}$$

Taking the modulus on both sides of (4.7), we have

$$\left|a_3-\mu a_2^2\right|\leqq\left|\frac{\zeta(1+m)}{\dfrac{2\,[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}}\right|\cdot\left|c_2-\phi c_1^2\right|. \tag{4.9}$$

Now, by using the above lemma in (4.9), we have

$$\left|a_3-\mu a_2^2\right|\leqq\left|\frac{\zeta(1+m)}{\dfrac{2\,[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}}\right|\max\{1,|\phi|\},$$

where ϕ is given by (4.8).

Finally, the assertion (4.2) of Theorem 3 is sharp in view of the fact that the assertion (4.1) of the above lemma is known to be sharp. □

5. Concluding Remarks and Observations

In our present investigation, we have introduced and systematically studied the general class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ of normalized analytic functions of complex order, which are connected with a q-analogue of integral operators. For this complex-order analytic function class, we have successfully determined a sufficient condition in terms of the coefficients and the estimates for the coefficients and a maximization theorem concerning the coefficients. Our main results are stated and proved as theorems (see Theorems 1, 2 and 3). Various interesting consequences and applications of our main results are stated as corollaries.

In conclusion, it seems to worthwhile to reiterate the now well-understood fact that the results for the q-calculus, which we have considered in this presentation for $0 < q < 1$, can easily be translated into the corresponding results for the so-called (p,q)-calculus (with $0 < q < p \leqq 1$) by applying some obviously trivial parametric and argument variations, the additional parameter p being redundant. As a matter of fact, the so-called (p,q)-number $[n]_{p,q}$ is given (for $0 < q < p \leqq 1$) by

$$[n]_{(p,q)} := \begin{cases} \dfrac{p^n - q^n}{p-q} & (n \in \{1,2,3,\cdots\}) \\ 0 & (n=0) \end{cases} \tag{5.1}$$

$$=: p^{n-1}\,[n]_{\frac{q}{p}}, \tag{5.2}$$

where, for the classical q-number $[n]_q$, we have (see also Section 1 above)

$$[n]_q := \frac{1-q^n}{1-q} \tag{5.3}$$

$$= p^{1-n}\left(\frac{p^n-(pq)^n}{p-(pq)}\right)$$

$$= p^{1-n}\,[n]_{(p,pq)}. \tag{5.4}$$

Furthermore, the so-called (p,q)-derivative or the so-called (p,q)-difference of a suitable function $f(z)$ is denoted by $(D_{p,q}\,f)(z)$ and defined, in a given subset of $\mathbb{C}$, by

$$(D_{p,q}\,f)(z) = \begin{cases} \dfrac{f(pz)-f(qz)}{(p-q)\,z} & (z \in \mathbb{C}\setminus\{0\};\ 0 < q < p \leqq 1) \\ f'(0) & (z=0;\ 0<q<p \leqq 1), \end{cases} \tag{5.5}$$

so that, clearly, we have the following connection with the familiar q-derivative $(D_q f)(z)$ used in (1.6):

$$(D_{p,q}\,f)(z) = \left(D_{\frac{q}{p}}\,f\right)(pz) \quad \text{and} \quad (D_q\,f)(z) = (D_{p,pq}\,f)\left(\frac{z}{p}\right) \tag{5.6}$$

$$(z \in \mathbb{C};\ 0 < q < p \leqq 1).$$

Remarkably, therefore, any claimed extensions of at least some investigations involving the classical q-calculus to the corresponding obviously straightforward investigations involving the (p,q)-calculus are somewhat inconsequential. The interested reader will find a recent investigation [25] which is intended here to provide an illustration of such transitions from the classical q-calculus to the (p,q)-calculus.

Further investigations on the applications of the q-calculus to meromorphic univalent and meromorphic multivalent functions along the lines of a recent work [15] may be worthy of consideration.

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