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REPRESENTATION OF SOLUTIONS OF A TWO-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. In this paper we give a representation formula for the general solution to the following two-dimensional system of difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(a+by_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(a+bx_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0$$

where parameters a, b and initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are real numbers. We also give some theoretical explanations related to the representation.

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1. INTRODUCTION

Solvability of difference equations and system of difference equations has attracted considerable interest recently (see, for example [1–19], and the related references therein).

The following four systems of difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(\pm 1 \pm y_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(\pm 1 \pm x_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0 \tag{1.1}$$

have been studied in [5], where some closed-form formulas for their solutions are given in terms of the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$. The closed-form formulas are given and proved by using the method of induction.

In this work we give an alternative proof in order to explain theoretically the results presented in [5], which were established through a mere application of the induction principle.

Here we consider the following extension of the systems in (1.1)

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(a+by_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(a+bx_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0 \tag{1.2}$$

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where parameters a, b and initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are real numbers.

Our objective is to show that system (1.2) is solvable by finding its closed-form formulas through an analytical approach, and to show that all the closed-form formulas obtained in [5] easily follow from the ones in our present paper.

2. MAIN RESULTS

Assume that $\{x_n, y_n\}_{n\geq -2}$ is a well-defined solution to system (1.2).

Let

$$u_n = x_n y_{n-1}, \quad v_n = y_n x_{n-1}, \tag{2.1}$$

for $n \geq -1$. Then system (1.2) can be written as

$$u_{n+1} = \frac{v_{n-1}}{a+bv_{n-1}}, \quad v_{n+1} = \frac{u_{n-1}}{a+bu_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

To give a closed-form for the well-defined solutions of the system (2.2), we consider the system of two difference equations of first order

$$u_{n+1} = \frac{v_n}{a+bv_n}, \quad v_{n+1} = \frac{u_n}{a+bu_n}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

The system (2.3) can be written as the following equation

$$u_{n+1} = \frac{u_{n-1}}{a^2+b(a+1)u_{n-1}}. \tag{2.4}$$

Let

$$u_n^{(j)} = u_{2n-j}, \quad n \in \mathbb{N}, j \in \{0,1\}. \tag{2.5}$$

Using notation (2.5), we can write (2.4) as

$$u_{n+1}^{(j)} = \frac{u_n^{(j)}}{a^2+b(a+1)u_n^{(j)}}, \tag{2.6}$$

where $j \in \{0,1\}$.

Equation (2.6) can be reduced to the equation :

$$\mathcal{H}_{n+1} = \frac{(a^2+1)\mathcal{H}_n - a^2}{\mathcal{H}_n}, \tag{2.7}$$

by using the change of variable

$$u_n^{(j)} = \frac{1}{b(a+1)}\left(\mathcal{H}_n - a^2\right). \tag{2.8}$$

Now we consider the difference equation (2.7) with the initial value $\mathcal{H}_0$ is non zero real number.

Through an analytical approach, we put

$$\mathcal{H}_n = \frac{k_n}{k_{n-1}}. \tag{2.9}$$

Then equation (2.7) becomes

$$k_{n+1}-(a^2+1)k_n-a^2k_{n-1}=0, \quad n\in\mathbb{N}_0. \tag{2.10}$$

Case $a^2\neq 1$:

Let $\{k_n\}_{n\geq -1}$ be the solution to equation (2.10) such that k_0 and $k_{-1}\in\mathbb{R}$. The zeros of the characteristic polynomial $P(\lambda)=\lambda^2-(a^2+1)\lambda+a^2$ are $\lambda_1=a^2$ and $\lambda_2=1$. Then the general solution to equation (2.10) can be written in the following form

$$k_n=c_1+c_2a^{2n}.$$

Using the initial values k_0 and k_{-1} with some calculations, we get

$$c_1=\frac{k_0-k_{-1}a^2}{1-a^2},$$
$$c_2=\frac{a^2(k_{-1}-k_0)}{1-a^2}.$$

So the general solution of equation (2.10) is

$$k_n=\frac{1}{1-a^2}\left[k_0\left(1-a^{2(n+1)}\right)-a^2k_{-1}\left(1-a^{2n}\right)\right]. \tag{2.11}$$

From all above mentioned we see that the following theorem holds.

Theorem 1. *Let $\{\mathcal{H}_n\}_{n\geq 0}$ be a well-defined solution to the equation (2.7). Then, for $n=2,3,\ldots,$*

$$\mathcal{H}_n=\frac{A(1-a^{2n})-\mathcal{H}_0(1-a^{2(n+1)})}{a^2(1-a^{2(n-1)})-\mathcal{H}_0(1-a^{2n})}. \tag{2.12}$$

Then, from (2.8) we see that

$$u_n^{(j)}=\frac{1}{b(a+1)}\left(\mathcal{H}_n-a^2\right)=\frac{u_0^{(j)}}{a^{2n}+b(a+1)u_0^{(j)}\sum_{r=0}^{n-1}a^{2r}},$$

for each $j\in\{0,1\}$.

Case $a^2=1$:

Then equation (2.7) becomes

$$k_{n+1}-2k_n-k_{n-1}=0, \quad n\in\mathbb{N}_0, \tag{2.13}$$

Let $\{k_n\}_{n\geq -1}$ be the solution to equation (2.13) such that k_0 and $k_{-1}\in\mathbb{R}$. The zero of the characteristic polynomial $P(\lambda)=(\lambda-1)^2$ is $\lambda_1=1$. Then the general solution to equation (2.13) can be written in the following form

$$k_n=c_1+c_2n.$$

Using the initial values k_0 and k_{-1} with some calculations we get

$$c_1 = k_0$$
$$c_2 = k_0 - k_{-1}.$$

So the general solution of equation (2.13) is

$$k_n = k_0(n+1) + k_{-1}n. \tag{2.14}$$

From all above mentioned we see that the following theorem holds.

Theorem 2. *Let $\{\mathcal{H}_n\}_{n\geq 0}$ be a well-defined solution to the equation (2.7). Then, for $n = 2, 3, \dots,$*

$$\mathcal{H}_n = \frac{n - \mathcal{H}_0(n+1)}{(n-1) - \mathcal{H}_0 n}. \tag{2.15}$$

Then, from (2.8) we see that

$$u_n^{(j)} = \frac{u_0^{(j)}}{1 + b(a+1)u_0^{(j)} n} \tag{2.16}$$

where $j \in \{0, 1\}$.

From all above mentioned with using (2.5) we see that the following corollary holds.

Corollary 1. *Let $\{u_n\}_{n\geq -1}$ be a well-defined solution to the equation (2.4). Then*

$$\begin{aligned} &\textbf{\textit{if}}\ a^2 \neq 1: \quad && u_{2n-j} = \frac{u_{-j}}{a^{2n} + b(a+1)u_{-j}\sum_{r=0}^{n-1} a^{2r}}, \\ &\textbf{\textit{if}}\ a^2 = 1: \quad && u_{2n-j} = \frac{u_{-j}}{1 + b(a+1)u_{-j}n}, \end{aligned} \qquad n \in \mathbb{N}_0,$$

where $j \in \{0, 1\}$.

Corollary 2. *Let $\{u_n, v_n\}_{n\geq 0}$ be a well-defined solution to the system (2.3). Then*
if $a^2 \neq 1$:

$$u_{2n} = \frac{u_0}{a^{2n} + b(a+1)u_0\sum_{r=0}^{n-1} a^{2r}}, \qquad u_{2n+1} = \frac{v_0}{a^{2n+1} + bv_0\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right)},$$

$$v_{2n} = \frac{v_0}{a^{2n} + b(a+1)v_0\sum_{r=0}^{n-1} a^{2r}}, \qquad v_{2n+1} = \frac{u_0}{a^{2n+1} + bu_0\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right)}.$$

if $a^2 = 1$:

$$u_{2n} = \frac{u_0}{1 + b(a+1)nu_0}, \qquad u_{2n+1} = \frac{v_0}{a + b((a+1)n+1)v_0},$$

$$v_{2n} = \frac{v_0}{1+b(a+1)nv_0}, \qquad v_{2n+1} = \frac{u_0}{a+b((a+1)n+1)u_0},$$

where $n \in \mathbb{N}_0$.

Proof. Let $\{u_n, v_n\}_{n\geq 0}$ be a solution of system (2.3), so $\{u_n\}_{n\geq -1}$ is a solution of equation (2.6). Then, **if** $a^2 \neq 1$, let

$$u_{2n-1} = \frac{u_{-1}}{a^{2n} + b(a+1)u_{-1}\sum_{r=0}^{n-1} a^{2r}},$$

and

$$v_0 = \frac{u_{-1}}{a+bu_{-1}},$$

so

$$\begin{aligned} u_{2n+1} &= \frac{u_{-1}}{a^{2(n+1)} + b(a+1)\left(\sum_{r=0}^{n} a^{2r}\right)u_{-1}} \\ &= \frac{u_{-1}}{a^{2n+1}(a+bu_{-1}) + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right)u_{-1}} \\ &= \frac{v_0}{a^{2n+1} + b\left(a\left(\sum_{i=0}^{n-1} a^{2r}\right) + \left(\sum_{i=0}^{n} a^{2r}\right)\right)v_0}. \end{aligned}$$

if $a^2 = 1$, let

$$u_{2n-1} = \frac{u_{-1}}{1+b(a+1)nu_{-1}},$$

and

$$v_0 = \frac{u_{-1}}{a+bu_{-1}},$$

so

$$\begin{aligned} u_{2n+1} &= \frac{u_{-1}}{1+b(a+1)(n+1)u_{-1}} = \frac{u_{-1}}{a^2+b(a+1)(n+1)u_{-1}} \\ &= \frac{u_{-1}}{a(a+bu_{-1})+b((a+1)n+1)u_{-1}} = \frac{v_0}{a+b((a+1)n+1)v_0}. \end{aligned}$$

In the same way, after some calculation and use that

$$v_n = \frac{u_{n-1}}{a+bu_{n-1}},$$

we obtain, **if** $a^2 \neq 1$, that

$$v_{2n} = \frac{v_0}{a^{2n} + b(a+1)v_0 \sum_{r=0}^{n-1} a^{2r}}, \qquad v_{2n+1} = \frac{u_0}{a^{2n+1} + bu_0\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right)},$$

and, **if** $a^2 = 1$, that

$$v_{2n} = \frac{v_0}{1 + b(a+1)nv_0}, \qquad v_{2n+1} = \frac{u_0}{a + b((a+1)n+1)u_0}.$$

□

Go back now to the system (2.2), we using an appropriate transformation reducing this system to the system of first-order difference equations (2.3).

The initial values with the smallest indexes are u_{-k} and v_{-k}. By using (2.2) with $n = 0$, we obtain the values of u_1 and v_1 as follows

$$u_1 = \frac{v_{-1}}{a + bv_{-1}}, \qquad v_1 = \frac{u_{-1}}{a + bu_{-1}}.$$

After known the values of u_1 and v_1, by using (2.2) with $n = 2$ we get the values of u_3 and v_3. We have

$$u_3 = \frac{v_1}{a + bv_1}, \qquad v_3 = \frac{u_1}{a + bu_1}.$$

The values of u_3 and v_3, by using (2.2) with $n = 4$, leads us to obtain the values of u_5 and v_5. We have

$$u_5 = \frac{v_3}{a + bv_3}, \qquad v_5 = \frac{u_3}{a + bu_3}.$$

$$u_{2m+1} = \frac{v_{2m-1}}{a + bv_{2m-1}}, \qquad v_{2m+1} = \frac{u_{2m-1}}{a + bu_{2m-1}}.$$

In the same way, it is shown that the initial values u_{-i} and v_{-i}, for a fixed $i \in \{0, 1\}$, determine all the values of the sequences $(u_{2(m+1)-i})_m$ and $(v_{2(m+1)-i})_m$. Also we have

$$u_{2(m+1)-i} = \frac{v_{2m-i}}{a + bv_{2m-i}}, \qquad v_{2(m+1)-i} = \frac{u_{2m-i}}{a + bu_{2m-i}}. \tag{2.17}$$

Let

$$u_n^{(i)} = u_{2n-i}, \qquad v_n^{(i)} = v_{2n-i}. \tag{2.18}$$

Using notation (2.18), we can write (2.2) as

$$u_{n+1}^{(i)} = \frac{v_n^{(i)}}{a + bv_n^{(i)}}, \qquad v_{n+1}^{(i)} = \frac{u_n^{(i)}}{a + bu_n^{(i)}}.$$

From all above mentioned we see that the following theorem holds.

Theorem 3. *Let $\{u_n, v_n\}_{n\geq -1}$ be a well-defined solution to the system (2.2). Then, for $n = 2,3,\ldots,$*
***if** $a^2 \neq 1$:*

$$u_{4n-1} = \frac{u_{-1}}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} u_{-1}}, \qquad v_{4n-1} = \frac{v_{-1}}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} v_{-1}},$$

$$u_{4n} = \frac{u_0}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} u_0}, \qquad v_{4n} = \frac{v_0}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} v_0},$$

$$u_{4n+1} = \frac{v_{-1}}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) v_{-1}}, \qquad v_{4n+1} = \frac{u_{-1}}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) u_{-1}},$$

$$u_{4n+2} = \frac{v_0}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) v_0}, \qquad v_{4n+2} = \frac{u_0}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) u_0}.$$

***if** $a^2 = 1$:*

$$u_{4n-1} = \frac{u_{-1}}{1 + b(a+1)nu_{-1}}, \qquad v_{4n-1} = \frac{v_{-1}}{1 + b(a+1)nv_{-1}},$$

$$u_{4n} = \frac{u_0}{1 + b(a+1)nu_0}, \qquad v_{4n} = \frac{v_0}{1 + b(a+1)nv_0},$$

$$u_{4n+1} = \frac{v_{-1}}{a + b((a+1)n+1)v_{-1}}, \qquad v_{4n+1} = \frac{u_{-1}}{a + b((a+1)n+1)u_{-1}},$$

$$u_{4n+2} = \frac{v_0}{a + b((a+1)n+1)v_0}, \qquad v_{4n+2} = \frac{u_0}{a + b((a+1)n+1)u_0}.$$

where $n \in \mathbb{N}_0$.

From (2.1) we have

$$x_n = \frac{u_n}{y_{n-1}}, \tag{2.19}$$

$$y_n = \frac{v_n}{x_{n-1}}. \tag{2.20}$$

Using (2.20) in (2.19), we obtain

$$x_{4n} = \frac{u_{4n} u_{4n-2}}{v_{4n-1} v_{4n-3}} x_{4n-4}. \tag{2.21}$$

Using (2.19) in (2.20), we obtain

$$y_{4n} = \frac{v_{4n} v_{4n-2}}{u_{4n-1} u_{4n-3}} y_{4n-4}. \tag{2.22}$$

For $n\in\mathbb{N}$, multiplying the equalities which are obtained form (2.21) and (2.22) from 1 to n, respectively, it follows that

$$x_{4n}=x_0\prod_{i=0}^{n-1}\left(\frac{u_{4i}u_{4i-2}}{v_{4i-1}v_{4i-3}}\right),\tag{2.23}$$

$$y_{4n}=y_0\prod_{i=0}^{n-1}\left(\frac{v_{4i}v_{4i-2}}{u_{4i-1}u_{4i-3}}\right).\tag{2.24}$$

Using the equalities (2.23) and (2.24) in (2.19) and (2.20), we obtain

$$x_{4n-1}=\frac{v_{6n}}{y_{6n}}=\frac{v_{4n}}{y_0}\prod_{i=0}^{n-1}\left(\frac{u_{4i-1}u_{4i-3}}{v_{4i}v_{4i-2}}\right).\tag{2.25}$$

We have

$$y_{4n-1}=\frac{u_{4n}}{x_{4n}}=\frac{u_{4n}}{x_0}\prod_{i=0}^{n-1}\left(\frac{v_{4i-1}v_{4i-3}}{u_{4i}u_{4i-2}}\right).\tag{2.26}$$

Using the equalities (2.25) and (2.26) in (2.19) and (2.20), we obtain

$$x_{4n-2}=\frac{v_{4n-1}}{y_{4n-1}}=x_0\frac{v_{4n-1}}{u_{4n}}\prod_{i=0}^{n-1}\left(\frac{u_{4i}u_{4i-2}}{v_{4i-1}v_{4i-3}}\right),\tag{2.27}$$

and

$$y_{4n-2}=\frac{u_{4n-1}}{x_{4n-1}}=y_0\frac{u_{4n-1}}{v_{4n}}\prod_{i=0}^{n-1}\left(\frac{v_{4i}v_{4i-2}}{u_{4i-1}u_{4i-3}}\right).\tag{2.28}$$

Using the equalities (2.27) and (2.28) in (2.19) and (2.20), we obtain

$$x_{4n+1}=\frac{u_{4n+1}}{y_{4n}}=\frac{u_{4n+1}}{y_0}\prod_{i=0}^{n-1}\left(\frac{u_{4i-1}u_{4i-3}}{v_{4i}v_{4i-2}}\right),\tag{2.29}$$

and

$$y_{4n+1}=\frac{v_{4n+1}}{x_{4n}}=\frac{v_{4n-1}}{x_0}\prod_{i=0}^{n-1}\left(\frac{v_{4i-1}v_{4i-3}}{u_{4i}u_{4i-2}}\right).\tag{2.30}$$

Using Theorem (3) we get
if $a^2\neq 1$:

$$u_{4n-3}=\frac{v_{-1}}{a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)v_{-1}},\quad v_{4n-3}=\frac{u_{-1}}{a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)u_{-1}},$$

$$u_{4n-2}=\frac{v_0}{a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)v_0},\quad v_{4n-2}=\frac{u_0}{a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)u_0},$$

$$u_{4n-1} = \frac{u_{-1}}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} u_{-1}}, \qquad v_{4n-1} = \frac{v_{-1}}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} v_{-1}},$$

$$u_{4n} = \frac{u_0}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} u_0}, \qquad v_{4n} = \frac{v_0}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} v_0}.$$

if $a^2 = 1$:

$$u_{4n-3} = \frac{v_{-1}}{a + b((a+1)n - a)v_{-1}}, \qquad v_{4n-3} = \frac{u_{-1}}{a + b((a+1)n - a)u_{-1}},$$

$$u_{4n-2} = \frac{v_0}{a + b((a+1)n - a)v_0}, \qquad v_{4n-2} = \frac{u_0}{a + b((a+1)n - a)u_0},$$

$$u_{4n-1} = \frac{u_{-1}}{1 + b(a+1)nu_{-1}}, \qquad v_{4n-1} = \frac{v_{-1}}{1 + b(a+1)nv_{-1}},$$

$$u_{4n} = \frac{u_0}{1 + b(a+1)nu_0}, \qquad v_{4n} = \frac{v_0}{1 + b(a+1)nv_0}.$$

From all above mentioned and

$$u_{-1} = x_{-1}y_{-2}, \quad u_0 = x_0 y_{-1}, \quad v_{-1} = y_{-1}x_{-2}, \quad v_0 = y_0 x_{-1}. \tag{2.31}$$

we see that the following result holds.

Theorem 4. *Let* $\{x_n, y_n\}_{n \geq -2}$ *be a well-defined solution to the system* (1.2)*. Then, for* $n = 0, 1, 2, 3, \ldots,$
if $a^2 \neq 1$:

$$x_{4n-2} = \prod_{i=0}^{n-1} \left(\frac{\left(a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} y_{-1}x_{-2}\right)\left(a^{2n-1} + b\left(a\sum_{r=0}^{n-2} a^{2r} + \sum_{r=0}^{n-1} a^{2r}\right) x_{-1}y_{-2}\right)}{\left(a^{2n} + b(a+1)\sum_{r=0}^{i-1} a^{2r} x_0 y_{-1}\right)\left(a^{2i-1} + b\left(a\sum_{r=0}^{i-2} a^{2r} + \sum_{r=0}^{i-1} a^{2r}\right) y_0 x_{-1}\right)} \right)$$

$$\times \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}} \left(\frac{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} x_0 y_{-1}}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} y_{-1}x_{-2}} \right),$$

$$x_{4n-1} = \prod_{i=0}^{n-1} \left(\frac{\left(a^{2n} + b(a+1)\sum_{r=0}^{i-1} a^{2r} y_0 x_{-1}\right)\left(a^{2n-1} + b\left(a\sum_{r=0}^{i-2} a^{2r} + \sum_{r=0}^{i-1} a^{2r}\right) x_0 y_{-1}\right)}{\left(a^{2n} + b(a+1)\sum_{r=0}^{i-1} a^{2r} x_{-1}y_{-2}\right)\left(a^{2n-1} + b\left(a\sum_{r=0}^{i-2} a^{2r} + \sum_{r=0}^{i-1} a^{2r}\right) y_{-1}x_{-2}\right)} \right)$$

$$\times \frac{x_{-1}y_{-2}^{n}x_{-2}^{n}}{x_0^{n}y_0^{n}}\left(\frac{1}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_0x_{-1}}\right),$$

$$x_{4n}=\frac{x_0^{n+1}y_0^{n}}{y_{-2}^{n}x_{-2}^{n}}$$

$$\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-1}x_{-2}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)x_{-1}y_{-2}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_0y_{-1}\right)\left(a^{2i-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_0x_{-1}\right)}\right),$$

$$x_{4n+1}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}y_0x_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)x_0y_{-1}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-1}y_{-2}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-1}x_{-2}\right)}\right)$$

$$\times \frac{y_{-1}y_{-2}^{n}x_{-2}^{n+1}}{x_0^{n}y_0^{n+1}}\left(\frac{1}{a^{2n+1}+b\left(a\sum_{r=0}^{n-1}a^{2r}+\sum_{r=0}^{n}a^{2r}\right)y_{-1}x_{-2}}\right),$$

$$y_{4n-2}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-2}x_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)x_{-2}y_{-1}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-1}y_0\right)\left(a^{2i-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-1}x_0\right)}\right)$$

$$\times \frac{x_0^{n}y_0^{n}}{y_{-2}^{n-1}x_{-2}^{n}}\left(\frac{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}x_{-1}y_0}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-2}x_{-1}}\right),$$

$$y_{4n-1}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}y_{-1}x_0\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)x_{-1}y_0\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-2}y_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-2}x_{-1}\right)}\right)$$

$$\times \frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_0^{n}y_0^{n}}\left(\frac{1}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-1}x_0}\right),$$

$$y_{4n}=\frac{x_0^n y_0^{n+1}}{y_{-2}^n x_{-2}^n}$$
$$\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-2}x_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)x_{-2}y_{-1}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-1}y_0\right)\left(a^{2i-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-1}x_0\right)}\right),$$

$$y_{4n+1}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}y_{-1}x_0\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)x_{-1}y_0\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-2}y_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-2}x_{-1}\right)}\right)$$
$$\times\frac{x_{-1}y_{-2}^{n+1}x_{-2}^n}{x_0^{n+1}y_0^n}\left(\frac{1}{a^{2n+1}+b\left(a\sum_{r=0}^{n-1}a^{2r}+\sum_{r=0}^{n}a^{2r}\right)y_{-2}x_{-1}}\right).$$

if $a^2=1$:

$$x_{4n-2}=\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_{-2})\,(a+b((a+1)i-a)x_{-1}y_{-2})}{(1+b(a+1)ix_0y_{-1})\,(a+b((a+1)i-a)y_0x_{-1})}\right).$$
$$\times\frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\left(\frac{1+b(a+1)nx_0y_{-1}}{1+b(a+1)nx_{-2}y_{-1}}\right),$$
$$x_{4n-1}=\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_0x_{-1})\,(a+b((a+1)i-a)x_0y_{-1})}{(1+b(a+1)ix_{-1}y_{-2})\,(a+b((a+1)i-a)y_{-1}x_{-2})}\right)$$
$$\times\frac{x_{-1}y_{-2}^n x_{-2}^n}{x_0^n y_0^n}\left(\frac{1}{1+b(a+1)ny_0x_{-1}}\right),$$
$$x_{4n}=\frac{x_0^{n+1}y_0^n}{y_{-2}^n x_{-2}^n}\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_{-2})\,(a+b((a+1)i-a)x_{-1}y_{-2})}{(1+b(a+1)ix_0y_{-1})\,(a+b((a+1)i-a)y_0x_{-1})}\right),$$
$$x_{4n+1}=\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_0x_{-1})\,(a+b((a+1)i-a)x_0y_{-1})}{(1+b(a+1)ix_{-1}y_{-2})\,(a+b((a+1)i-a)y_{-1}x_{-2})}\right)$$
$$\times\frac{y_{-1}}{x_0^n y_0^{n+1}}\left(\frac{y_{-2}^n x_{-2}^{n+1}}{a+b((a+1)n+1)y_{-1}x_{-2}}\right),$$
$$y_{4n-2}=\frac{x_0^n y_0^n}{y_{-2}^{n-1}x_{-2}^n}\left(\frac{1+b(a+1)nx_{-1}y_0}{1+b(a+1)ny_{-2}x_{-1}}\right)$$

$$\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-2}x_{-1})\,(a+b((a+1)i-a)x_{-2}y_{-1})}{(1+b(a+1)ix_{-1}y_{0})\,(a+b((a+1)i-a)y_{-1}x_{0})}\right),$$

$$y_{4n-1}=\frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_{0}^{n}y_{0}^{n}}\left(\frac{1}{1+b(a+1)ny_{-1}x_{0}}\right)$$

$$\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_{0})\,(a+b((a+1)i-a)x_{-1}y_{0})}{(1+b(a+1)ix_{-2}y_{-1})\,(a+b((a+1)i-a)y_{-2}x_{-1})}\right),$$

$$y_{4n}=\frac{x_{0}^{n}y_{0}^{n+1}}{y_{-2}^{n}x_{-2}^{n}}\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-2}x_{-1})\,(a+b((a+1)i-a)x_{-2}y_{-1})}{(1+b(a+1)ix_{-1}y_{0})\,(a+b((a+1)i-a)y_{-1}x_{0})}\right),$$

$$y_{4n+1}=\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_{0})\,(a+b((a+1)i-a)x_{-1}y_{0})}{(1+b(a+1)ix_{-2}y_{-1})\,(a+b((a+1)i-a)y_{-2}x_{-1})}\right)$$

$$\times\frac{x_{-1}}{x_{0}^{n+1}y_{0}^{n}}\left(\frac{y_{-2}^{n+1}x_{-2}^{n}}{a+b((a+1)n+1)y_{-2}x_{-1}}\right).$$

3. Some applications

As some applications we show how are obtained closed-form formulas for solutions to the systems in (1.1), which were presented in [5].

First result proved in [5] is the following.

Corollary 3. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1}=\frac{y_{n-1}x_{n-2}}{y_{n}(1+y_{n-1}x_{n-2})},\quad y_{n+1}=\frac{x_{n-1}y_{n-2}}{x_{n}(1+x_{n-1}y_{n-2})},\quad n\in\mathbb{N}_0. \tag{3.1}$$

Then

$$x_{4n-2}=\frac{x_{0}^{n}y_{0}^{n}}{y_{-2}^{n}x_{-2}^{n-1}}\left(\frac{1+2nx_{0}y_{-1}}{1+2nx_{-2}y_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_{-2})\,(1+(2i-1)x_{-1}y_{-2})}{(1+2ix_{0}y_{-1})\,(1+(2i-a)y_{0}x_{-1})}\right),$$

$$x_{4n-1}=\frac{x_{-1}y_{-2}^{n}x_{-2}^{n}}{x_{0}^{n}y_{0}^{n}}\left(\frac{1}{1+2ny_{0}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{0}x_{-1})\,(1+(2i-a)x_{0}y_{-1})}{(1+2ix_{-1}y_{-2})\,(1+(2i-1)y_{-1}x_{-2})}\right),$$

$$x_{4n}=\frac{x_{0}^{n+1}y_{0}^{n}}{y_{-2}^{n}x_{-2}^{n}}\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_{-2})\,(1+(2i-a)x_{-1}y_{-2})}{(1+2ix_{0}y_{-1})\,(1+(2i-a)y_{0}x_{-1})}\right),$$

$$x_{4n+1}=\frac{y_{-1}}{x_{0}^{n}y_{0}^{n+1}}\left(\frac{y_{-2}^{n}x_{-2}^{n+1}}{1+(2n+1)y_{-1}x_{-2}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{0}x_{-1})\,(1+((2i-1)x_{0}y_{-1})}{(1+2ix_{-1}y_{-2})\,(1+(2i-1)y_{-1}x_{-2})}\right),$$

and

$$y_{4n-2}=\frac{x_{0}^{n}y_{0}^{n}}{y_{-2}^{n-1}x_{-2}^{n}}\left(\frac{1+2nx_{-1}y_{0}}{1+2ny_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-2}x_{-1})\,(1+(2i-1)x_{-2}y_{-1})}{(1+2ix_{-1}y_{0})\,(1+(2i-a)y_{-1}x_{0})}\right),$$

$$y_{4n-1} = \frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_0^n y_0^n}\left(\frac{1}{1+2iy_{-1}x_0}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_0)(1+(2i-1)x_{-1}y_0)}{(1+2ix_{-2}y_{-1})(1+(2i-1)y_{-2}x_{-1})}\right),$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}}{y_{-2}^n x_{-2}^n}\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-2}x_{-1})(1+(2i-a)x_{-2}y_{-1})}{(1+2ix_{-1}y_0)(1+(2i-1)y_{-1}x_0)}\right),$$

$$y_{4n+1} = \frac{x_{-1}}{x_0^{n+1}y_0^n}\left(\frac{y_{-2}^{n+1}x_{-2}^n}{1+(2n+1)y_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_0)(1+(2i-1)x_{-1}y_0)}{(1+2ix_{-2}y_{-1})(1+(2i-1)y_{-2}x_{-1})}\right).$$

Proof. System (3.1) is obtained from system (1.2) with $a = b = 1$, so by using Theorem (4) corollary (3) follows. □

The following corollary is Theorem 2.2 in [5].

Corollary 4. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(1-y_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(1-x_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0. \tag{3.2}$$

Then

$$x_{4n-2} = \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\left(\frac{1-2nx_0y_{-1}}{1-2nx_{-2}y_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_{-2})(1-(2i-1)x_{-1}y_{-2})}{(1-2ix_0y_{-1})(1-(2i-1)y_0x_{-1})}\right),$$

$$x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n}{x_0^n y_0^n}\left(\frac{1}{1-2ny_0x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_0x_{-1})(1-(2i-1)x_0y_{-1})}{(1-2ix_{-1}y_{-2})(1-(2i-1)y_{-1}x_{-2})}\right),$$

$$x_{4n} = \frac{x_0^{n+1}y_0^n}{y_{-2}^n x_{-2}^n}\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_{-2})(1-(2i-1)x_{-1}y_{-2})}{(1-2ix_0y_{-1})(1-(2i-1)y_0x_{-1})}\right),$$

$$x_{4n+1} = \frac{y_{-1}}{x_0^n y_0^{n+1}}\left(\frac{y_{-2}^n x_{-2}^{n+1}}{1-(2n+1)y_{-1}x_{-2}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_0x_{-1})(1-(2i-1)x_0y_{-1})}{(1-ix_{-1}y_{-2})(1-(2i-1)y_{-1}x_{-2})}\right),$$

$$y_{4n-2} = \frac{x_0^n y_0^n}{y_{-2}^{n-1}x_{-2}^n}\left(\frac{1-2nx_{-1}y_0}{1-2ny_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-2}x_{-1})(1-(2i-1)x_{-2}y_{-1})}{(1-2ix_{-1}y_0)(1-(2i-1)y_{-1}x_0)}\right),$$

$$y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n}{x_0^n y_0^n}\left(\frac{1}{1-2ny_{-1}x_0}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_0)(1-(2i-1)x_{-1}y_0)}{(1-2ix_{-2}y_{-1})(1-(2i-1)y_{-2}x_{-1})}\right),$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}}{y_{-2}^n x_{-2}^n}\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-2}x_{-1})(1-(2i-1)x_{-2}y_{-1})}{(1-2ix_{-1}y_0)(1-(2i-1)y_{-1}x_0)}\right),$$

$$y_{4n+1} = \frac{x_{-1}}{x_0^{n+1}y_0^n}\left(\frac{y_{-2}^{n+1}x_{-2}^n}{1-(2n+1)y_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_0)(1-(2i-1)x_{-1}y_0)}{(1-2ix_{-2}y_{-1})(1-(2i-1)y_{-2}x_{-1})}\right).$$

Proof. System (3.2) is obtained from system (1.2) with $a = 1$ and $b = -1$, so by using Theorem (4) corollary (4) follows. □

The following corollary is Theorem 5.3 in [5].

Corollary 5. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(-1+y_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(-1+x_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0. \tag{3.3}$$

Then

$$x_{4n-2} = \frac{x_0^n y_0^n (-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(-1+y_0x_{-1})^n}, \qquad y_{4n-2} = \frac{y_0^n x_0^n (-1+y_{-1}x_{-2})^n}{x_{-2}^n y_{-2}^{n-1}(-1+x_0y_{-1})^n},$$

$$x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (-1+x_0y_{-1})^n}{x_0^n y_0^n (-1+y_{-1}x_{-2})^n}, \qquad y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n (-1+y_0y_{-1})^n}{y_0^n x_0^n (-1+x_{-1}y_{-2})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n (-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (-1+y_0x_{-1})^n}, \qquad y_{4n} = \frac{y_0^{n+1} x_0^n (-1+y_{-1}x_{-2})^n}{x_{-2}^n y_{-2}^n (-1+x_0y_{-1})^n},$$

$$x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1} (-1+x_0y_{-1})^n}{x_0^n y_0^{n+1} (-1+y_{-1}x_{-2})^{n+1}}, \qquad y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1} (-1+y_0x_{-1})^n}{y_0^n x_0^{n+1} (-1+x_{-1}y_{-2})^{n+1}}.$$

Proof. System (3.3) is obtained from system (1.2) with $a = 1$ and $b = -1$, so by using Theorem (4) corollary (5) follows. □

The following corollary is Theorem 5.4 in [5].

Corollary 6. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(-1-y_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(-1-x_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

Then

$$x_{4n-2} = \frac{x_0^n y_0^n (-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(-1-y_0x_{-1})^n}, \qquad y_{4n-2} = \frac{y_0^n x_0^n (-1-y_{-1}x_{-2})^n}{x_{-2}^n y_{-2}^{n-1}(-1-x_0y_{-1})^n},$$

$$x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (-1-x_0y_{-1})^n}{x_0^n y_0^n (-1-y_{-1}x_{-2})^n}, \qquad y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n (-1-y_0y_{-1})^n}{y_0^n x_0^n (-1-x_{-1}y_{-2})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n (-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (-1-y_0x_{-1})^n}, \qquad y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n (-1-y_0y_{-1})^n}{y_0^n x_0^n (-1-x_{-1}y_{-2})^n},$$

$$x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1} (-1-x_0y_{-1})^n}{x_0^n y_0^{n+1} (-1-y_{-1}x_{-2})^{n+1}}, \qquad y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1} (-1-y_0x_{-1})^n}{y_0^n x_0^{n+1} (-1-x_{-1}y_{-2})^{n+1}}.$$

Proof. System (3.4) is obtained from system (1.2) with $a = b = -1$, so by using Theorem (4) corollary (6) follows.

□

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